Article

Dynamical complexities in a discrete-time food chain

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Abstract
In this paper, a discrete-time food chain characterized by three species is modeled by a system of three nonlinear difference equations. The existence and local stability of the equilibrium points of the discrete dynamical system are analyzed. It is shown that for some parameter values the interior equilibrium point loses its stability through a discrete Hopf bifurcation. Basic properties of the discrete system are analyzed by means of phase portraits, bifurcation diagrams and Lyapunov exponents. We have varied the result through numerical calculation.

Keywords discrete food chain; discrete Hopf bifurcation; strange attractor; Lyapunov exponents.

1 Introduction

Lotka-Volterra model describes interactions between two species in an ecosystem, a predator and a prey. The model was developed independently by Lotka (1925) and Volterra (1962). After them, more realistic prey-predator models were introduced by Holling suggesting three kinds of functional responses for different species to model the phenomena of predation (Holling, 1965). After that, the dynamics of a Lotka-Volterra model was studied by many authors (May, 1974; Kasarinoff and van der Deiesch, 1978; Kuang, 1988; Danca et al., 1997; Zhu et al., 2002; Jing and Yang, 2006; Elabbasy et al., 2009). The research dealing with interspecific interactions has mainly focused on continuous prey-predator models of two variables, where dynamics include only stable equilibrium or limit cycles (Kasarinoff and van der Deiesch, 1978; Kuang, 1988; Zhu et al., 2002).

Ecological food webs or food chains typically, contain several layers, where the consumers which eat from the bottom resource layer are the prey of another predator. The prey-predator models mentioned above can easily be extended with a "top predator" that lives on the predator population. Doing so, one obtains a food chain of three species (Freedman and Waltman, 1977; Freedman and So, 1985). It was reported that the three species continuous time models have more complicated patterns. These models form dissipative dynamical systems which can posses three distinct dynamical possibilities like stable focus, limit cycle and chaos. The last two decades of research demonstrated very complex dynamics which can arise in continuous time food chain of three or more species (Hastings and Powell, 1991; Klebanoff and Hastings, 1994; McCann and Yodzis, 1994; Kuznetsov and Rinaldi, 1996; Deng, 2001; El-Owaidy et al., 2001; Letellier and Aziz-Alaoui, 2002; Aziz-Alsoui, 2002; Chauvet, 2002).

For example, Hastings and Powell (1991), Klebanoff and Hastings (1994), demonstrated the occurrence of chaotic dynamics in a simple continuous three species food chain model in which both consumer species have Holling type II functional response.

Many authors have studied food chain models concentrate on continuous case. On the other side, sometimes, it could be desirable to replace the set of continuous-time differential equations by a set of difference equations for which the time is a discrete variable. Also, discrete time models governed by difference equations are more appropriate than continuous ones when the interactions of species are non-overlapping generations (Ivanchikov and Nedorezov, 2011, 2012). Discrete time systems can also provide efficient computational models of continuous ones of numerical simulations (Liu and Xu, 2003).

2 Discrete-time Food Chain

Discrete-time dynamical systems are appropriate for expressing dynamics of population densities of species whose generations do not overlap (Kon, 2001). Such species are found in temperate regions, because of their seasonal environments. Most of univoltine insects have non-overlapping generations see Kon (2001) and Murray (1998).

The discrete-time three species food chain model is studied analytically as well as numerically. We now consider a food chain of three interacting species, each with non-overlapping generations, which affect each other's population dynamics. This food chain is describes the insects group of three fully different insects. These insects are a lowest-level prey x is preyed upon by a mid-level species y, which, in turn, is preyed by a top-level predator z. The proposed model to study such ecosystems can be described by the following system of nonlinear difference equations in non-dimensional form,

$$T: \begin{cases} x_{n+1} = ax_n(1-x_n) - bx_n y_n, \\ y_{n+1} = cx_n y_n - dy_n z_n, \\ z_{n+1} = ry_n z_n, \end{cases} \qquad (1)$$

In the absence of predation, prey grow logistically (x), a Holling type I for predator (y), and Holling type I for top predator (z). The parameter a is the intrinsic rate of growth of the prey x; b is the per capita searching efficiency of the predator y; c is the per-prey x clutch of the predator y; d is the per capita searching efficiency of the predator z and r is the per-prey y clutch of the predator z. For all these parameters, we are assuming only positive values. The map given by system (1) is a noninvertable one of the space. The study of the dynamical properties of the above map allows us to get information of the long-run behavior of food chain populations. Starting from given initial condition (x_0, y_0, z_0), the iteration of system (1) is uniquely determined a trajectory of the states of population output in the following form

$$(x_n, y_n, z_n) = T^n(x_0, y_0, z_0), \quad where \quad n = 0,1,2,...$$

3 Equilibrium Points and Local Stability

In order to study the qualitative behavior of the solutions of the nonlinear difference equations (1), we define the equilibrium points of the dynamic system as a nonnegative fixed point of the map (1), i.e. the solutions of the following nonlinear algebraic system

$$\begin{cases} x = ax(1-x) - bxy, \\ y = cxy - dyz, \\ z = ryz. \end{cases} \quad (2)$$

The system (2) has four equilibria:

(i) $E_0 = (0,0,0)$ is the origin, all species are extinct.

(ii) $E_1 = (\frac{a-1}{a},0,0)$ is the axial fixed point in the absence of mid-level species $y = 0$ and top-level predator $z = 0$ exists for $a > 1$.

(iii) $E_2 = (\frac{1}{c}, \frac{a}{b}(1-\frac{1}{c}) - \frac{1}{b}, 0)$ is the axial fixed point in the absence of top-level predator $z = 0$ exists for $a > \frac{c}{c-1}$.

(iv) The interior (positive) fixed point $E_3 = (x^*, y^*, z^*)$, where

$$x^* = \frac{r(a-1)-b}{ar}, \quad y^* = \frac{1}{r}, \quad z^* = \frac{r(ac-a-c)-cb}{adr}, \quad (3)$$

all species coexist.

It is obvious that E_0, E_1, E_2 are boundary equilibrium points and E_3 is the interior equilibrium. The dynamical behavior of the fixed points of the three-dimensional system (1) can be studied by computation of the eigenvalues of Jacobian matrix of (1). The Jacobian matrix J at the state variables (x, y, z) has the form

$$J(x,y,z) = \begin{bmatrix} a(1-2x)-by & -bx & 0 \\ cy & cx-dz & -dy \\ 0 & rz & ry \end{bmatrix}. \quad (4)$$

The determinant of the Jacobian is

$$Det(J(x,y,z)) = acrxy - 2acrx^2 y.$$

The system (1) is said to be dissipative (Wiggins, 1990) if

$$|xy(1-x)| < \frac{1}{acr}.$$

In order to study the stability of the fixed points of the model, we first give the following Theorem.

Theorem (Elaydi, 1996). Let

$$P(\lambda) = \lambda^3 + A_1\lambda^2 + A_2\lambda + A_3 = 0 \quad (5)$$

be the characteristic equation for a matrix defined by Eq. (4).

The following statements are true:

(a) *If every root of Eq. (5) has absolute value less than one, then the fixed point of system (1) is locally asymptotically stable and fixed point is called a sink.*

(b) *If at least one of the roots of Eq. (5) has absolute value grater than one, then the fixed point of system (1) is unstable and fixed point is called a saddle.*

(c) *If every root of Eq. (5) has absolute value grater than one, then the fixed point of system (1) is a source.*

(d) *The fixed point of system (1) is called hyperbolic if no root of Eq. (5) has absolute value equal one. If there exists a root of Eq. (5) with absolute value equal to one, then the fixed point is called non-hyperbolic.*

Lemma 1 *The boundary equilibrium point E_0 of the system (1) is a stable fixed point when $a<1$ and unstable otherwise.*

Proof. By linearizing system (1) at E_0, one obtains the Jacobian

$$J(E_0)=\begin{bmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The eigenvalues of matrix $J(E_0)$ are

$$\lambda_1 = a, \quad \lambda_{2,3} = 0,$$

with eigenvectors, respectively given by $d_0^{(1)} = (1,0,0)$, $d_0^{(2)} = (0,1,0)$, and $d_0^{(3)} = (0,0,1)$. It results $\lambda_{2,3} < 1$ and $\lambda_1 < 1$ whenever $a<1$. Hence the equilibrium point E_0 is a stable. Also the equilibrium point E_0 unstable if $a>1$. Moreover E_0 is called non-hyperbolic point when $a=1$.

Lemma 2 *When $a>1$ there are at least three different topological types of $E_1 = (\frac{a-1}{a},0,0)$ for all permissible values of parameters*

(i) If $1<a<3$ and $\frac{c}{c-1} < a < \frac{c}{c+1}$ then E_1 is a sink point**;**

(ii) If $a>3$ or $a > \frac{c}{c+1}$ then E_1 is a saddle point;

(iii) If $a=3$ or $a = \frac{c}{c+1}$ then E_1 is a non-hyperbolic point.

Proof. From Eq. (4), the Jacobian of system (1), linearized at E_1, is

$$J(E_1)=\begin{bmatrix} 2-a & \frac{-b(a-1)}{a} & 0 \\ 0 & \frac{c(a-1)}{a} & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

whose eigenvalues are $\lambda_1 = 2 - a,\ \ \lambda_2 = \dfrac{ca - c}{a}$ *and* $\lambda_3 = 0$. It is clear that, $|\lambda_1| < 1$ *and* $|\lambda_2| < 1$ when $1 < a < 3$ and $\dfrac{c}{c-1} < a < \dfrac{c}{c+1}$. Hence the equilibrium point E_1 is a sink if $1 < a < 3$ and $\dfrac{c}{c-1} < a < \dfrac{c}{c+1}$, E_1 is a saddle point if $a > 3$ or $a > \dfrac{c}{c+1}$ and E_1 is a non-hyperbolic if $a = 3$ or $a = \dfrac{c}{c+1}$.■

Lemma 3 when $a > \dfrac{c}{c-1}$ the equilibrium point E_2 for system (1) exists and

(i) If $(\dfrac{a}{2c} + 1)^2 > a$ then the equilibrium point E_2 is asymptotically stable when the following conditions holds $\dfrac{c(r-b)}{rb(c-1)} < a < \dfrac{c(r+b)}{rb(c-1)}$ and $\dfrac{a}{a-1} < c < \dfrac{a}{2(\sqrt{a}-1)}$, E_2 unstable otherwise.

(ii) If $(\dfrac{a}{2c} + 1)^2 < a$ then the equilibrium point E_2 is asymptotically stable when the following conditions holds $\dfrac{c(r-b)}{rb(c-1)} < a < \dfrac{c(r+b)}{rb(c-1)}$ and $\dfrac{a}{2(\sqrt{a}-1)} < c < \dfrac{2a}{a-1}$, E_2 unstable otherwise.

Proof. The Jacobian matrix $J(E_2)$ at the equilibrium point

$E_2 = (\dfrac{1}{c}, \dfrac{a}{b}(1 - \dfrac{1}{c}) - \dfrac{1}{b}, 0)$ is given by

$$J(E_2) = \begin{bmatrix} 1 - \dfrac{a}{c} & \dfrac{-b}{c} & 0 \\ \dfrac{ac}{b}(1 - \dfrac{1}{c}) - \dfrac{c}{b} & 1 & -\dfrac{ad}{b}(1 - \dfrac{1}{c}) + \dfrac{d}{b} \\ 0 & 0 & ar(1 - \dfrac{1}{c}) - \dfrac{r}{b} \end{bmatrix}$$

which has eigenvalues given by

$$\lambda_1 = \frac{arb(c-1)}{cb}, \qquad \lambda_{2,3} = (1 - \frac{a}{2c}) \pm \sqrt{(\frac{a}{c} + 2)^2 - 4a}.$$

It is easy to see that eigenvalues $\lambda_{2,3}$ are real when $(\dfrac{a}{2c} + 1)^2 > a$ and all eigenvalues are inside the unit circle ($|\lambda_{1,2,3}| < 1$) when

$$\frac{c(r-b)}{rb(c-1)} < a < \frac{c(r+b)}{rb(c-1)} \text{ and } \frac{a}{a-1} < c < \frac{a}{2(\sqrt{a}-1)} \qquad (6)$$

So, the equilibrium point E_2 is asymptotically stable when Eq. (6) is satisfied while it is unstable otherwise.

Also one can see that the eigenvalues $\lambda_{2,3}$ are complex when $(\frac{a}{2c}+1)^2 < a$ and all eigenvalues are inside the unit circle ($|\lambda_{1,2,3}| < 1$) when

$$\frac{c(r-b)}{rb(c-1)} < a < \frac{c(r+b)}{rb(c-1)} \text{ and } \frac{a}{2(\sqrt{a}-1)} < c < \frac{2a}{a-1} \qquad (7)$$

Hence if $(\frac{a}{2c}+1)^2 < a$ the equilibrium point E_2 is stable if the condition (7) holds and unstable otherwise.■

Here we discuss the stability of the equilibrium point E_3.

Lemma 4 *The interior equilibrium point* $E_3(x^*, y^*, z^*)$ *of the system (1) exists if and only if* $r > \frac{b}{a-1}$ *and* $cb < r(ca-c-a)$, *where* x^*, y^* *and* z^* *are given by Eq. (3).*

We consider the stability properties of E_3. If we linearized the system (1) about E_3, then the Jacobian matrix of the system (1) at E_3 is given by

$$J(E_3) = \begin{bmatrix} \frac{r(2-a)}{a}+\frac{b}{r} & \frac{b^2-rb(a-1)}{ar} & 0 \\ \frac{c}{r} & 1 & -\frac{d}{r} \\ 0 & \frac{r(ca-c-a)}{ad} & 1 \end{bmatrix} \qquad (8)$$

The necessary and sufficient conditions for E_3 to be asymptotically stable is that all roots of characteristic equation (5) have magnitudes less than one. We have

$$\begin{aligned} A_1 &= \frac{(2r^3a + a^2br - r^3a^2 + 2a^2r^2)}{a^2r^2} \\ A_2 &= \frac{(ab^2c + ar^2c + 2arbc + 2r^3a^2 - a^2r^2c - 4r^3a - a^2bcr - 2a^2br)}{a^2r^2} \\ A_3 &= \frac{(r^2acb + 2a^2brc + 3r^3ac - 2arbc - 2r^3c - 2r^2cb - r^3a^2c - 2ab^2c)}{a^2r^2} \end{aligned} \qquad (9)$$

Following the *Jury's criterion* (see for instance [26]), the roots of (5) satisfy $|\lambda_i| < 1, \quad i = 1,2,3$ if and only if

$$\begin{aligned}&3 + A_1 - A_2 - 2A_3 > 0,\\&1 - A_2 + A_3(A_1 - A_3) > 0,\\&1 - A_1 + A_2 - A_3 > 0.\end{aligned} \qquad (10)$$

If the stability conditions (10) are satisfied it implies that the equilibrium point E_3 is stable. Obviously, the interior equilibrium point E_3 is stable in the regions defined by Eqs. (10), otherwise unstable.

4 Numerical Simulations

As we are dealing with a nonlinear 3-dimensional map, and since the theoretical tools to prove the existence of chaotic behavior in 3-dimensions are still very poor, so we present some numerical simulation results to verify the existence and bifurcations of periodic solutions emerging from discrete Hopf bifurcation and to show the interesting and complexity of some dynamical behaviors in discrete time food chain.

Without loss of generality, we fix the parameters and assume that $b = 3.7$, $c = 3$, $d = 3.5$, $r = 3.8$ and assume that a vary. We consider the following initial conditions $(x_0, y_0, z_0) = (0.3,0.2,0.1)$ situated in the basin of attraction of the fixed point E_3 and we start to study the dynamic behavior of the system (1) when the parameter a is varied in the interval [2.8, 4.3].

The phase portraits are considered in the following cases:

Fig. 1 shows that the fixed point E_3 is a stable attractor at *a=2.95*.

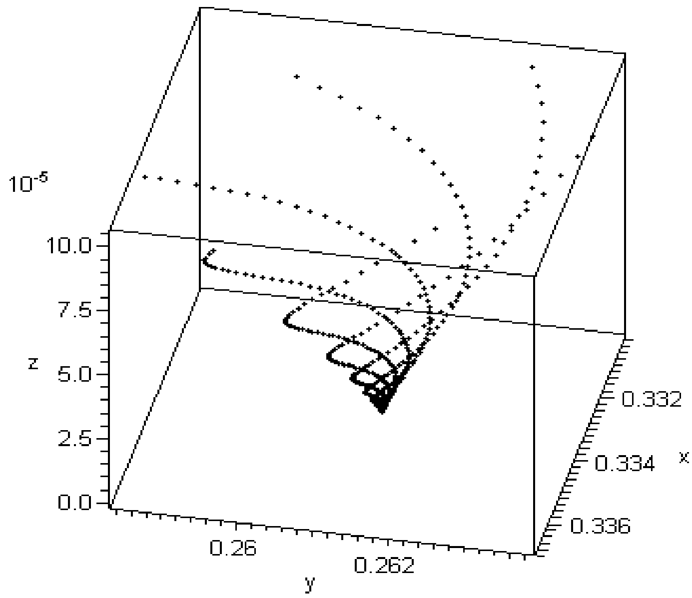


Fig. 1 Stable fixed point for system (1) at a = 2.95.

The behavior of the system (1) at *a=2.98* before a discrete Hopf bifurcation is shown in Fig. 2. On the other side, Fig. 3 demonstrates the behavior of the system (1) after a discrete Hopf bifurcation when *a=3.01*. From Figs 2 and 3, we deduce that the fixed point E_3 loses its stability through a discrete Hopf bifurcation, when the parameter a varies from *2.98* to *3.01*.

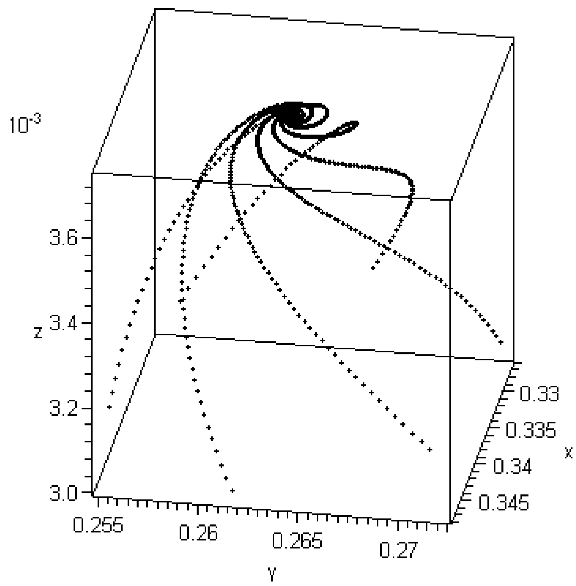


Fig. 2 Stable fixed point for system (1) before a discrete Hopf bifurcation at a = 2.98.

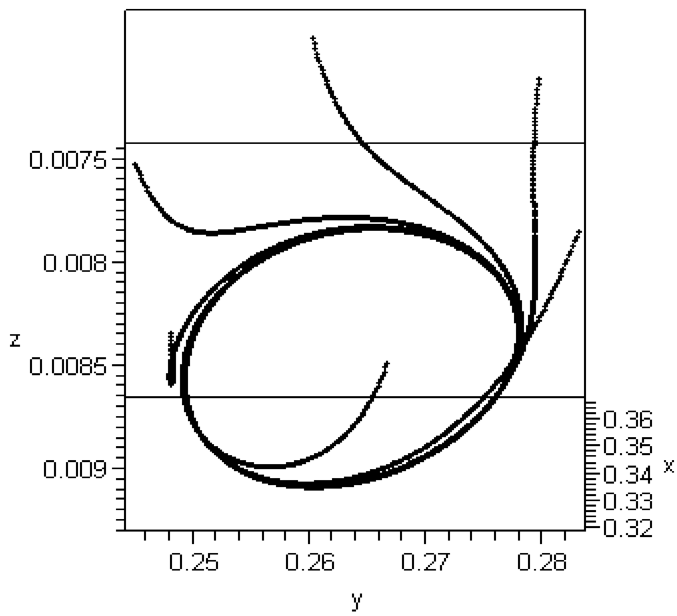


Fig. 3 The fixed point for system (1) after a discrete Hopf bifurcation at a = 3.01.

Increasing the control parameter a forward (a =3.1) leads to make the fixed point E_3 unstable and the creation of an invariant closed curve around the fixed point, see Fig. 4.

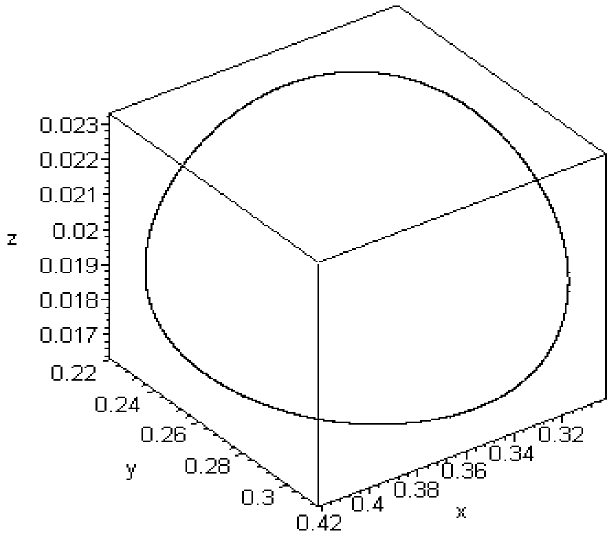


Fig. 4 The first attracting invariant cycle for system (1) around the fixed point at a = 3.1.

In Fig. 5 the equilibrium point E_3 becomes stable again at a *=3.6*.
The second invariant cycle for system (1) which exists for a *=3.89*, is shown in Fig. 6.

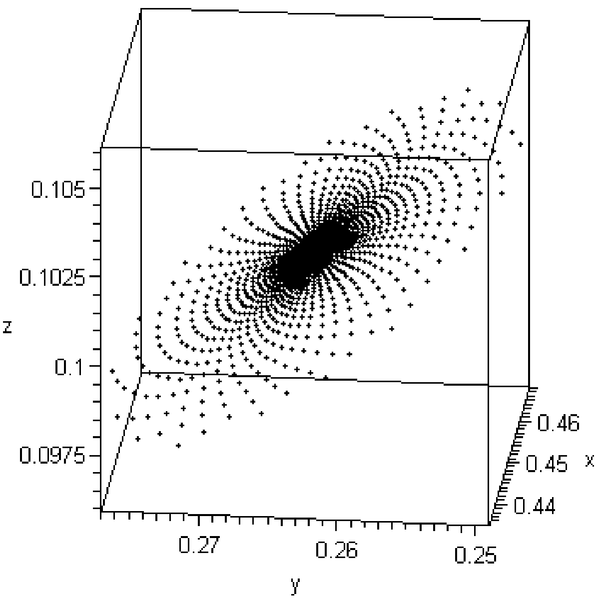


Fig. 5 System (1) back to stable fixed point at a = 3.6.

Fig. 7 showed the breakdown of the second invariant cycle which exists for a =*3.904*.

Fig. 8 represents a set of 17 closed curves brought about by a discrete Hopf bifurcation of the 17^{th} iterate of the system (1), obtained for a =*3.973*.

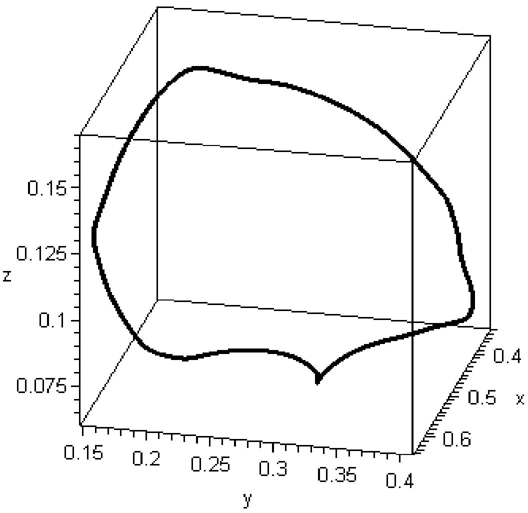


Fig. 6 The second invariant cycle for system (1) at a = 3.89.

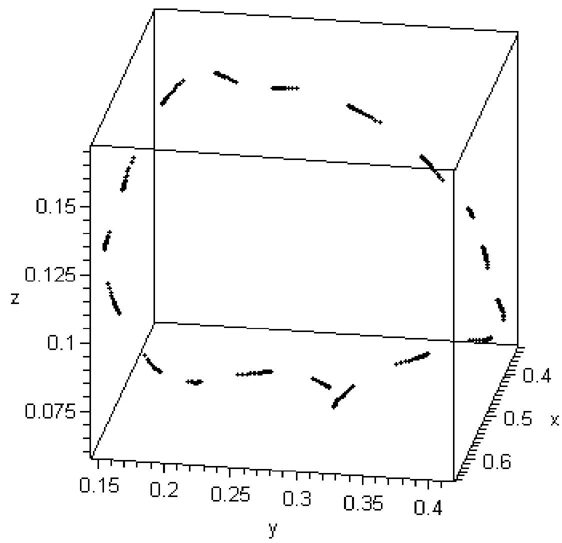


Fig. 7 The breakdown of the second invariant cycle for system (1) at a = 3.904.

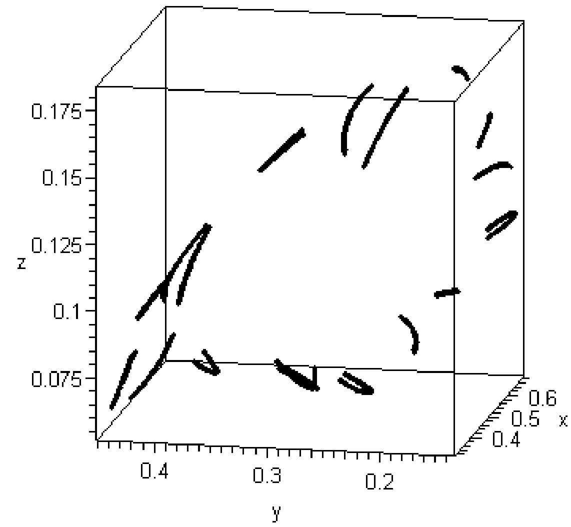


Fig. 8 The existence of multiple invariant cycle for system (1) at a = 3.973.

Further increase in *a* value gives a sequence of discrete Hopf bifurcations and then a chaotic attractor arises.

The strange attractor is produced by the breaking of the invariant circles and the appearance of the seventeen chaotic regions changes as they are linked into a single attractor at a =*3.985* see Fig. 9.

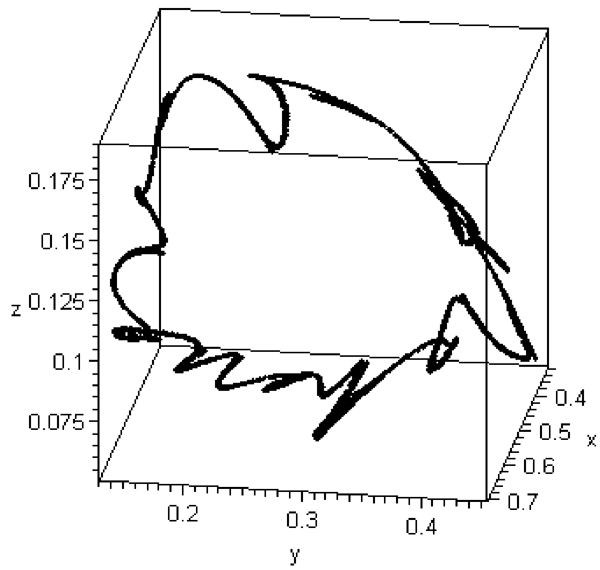
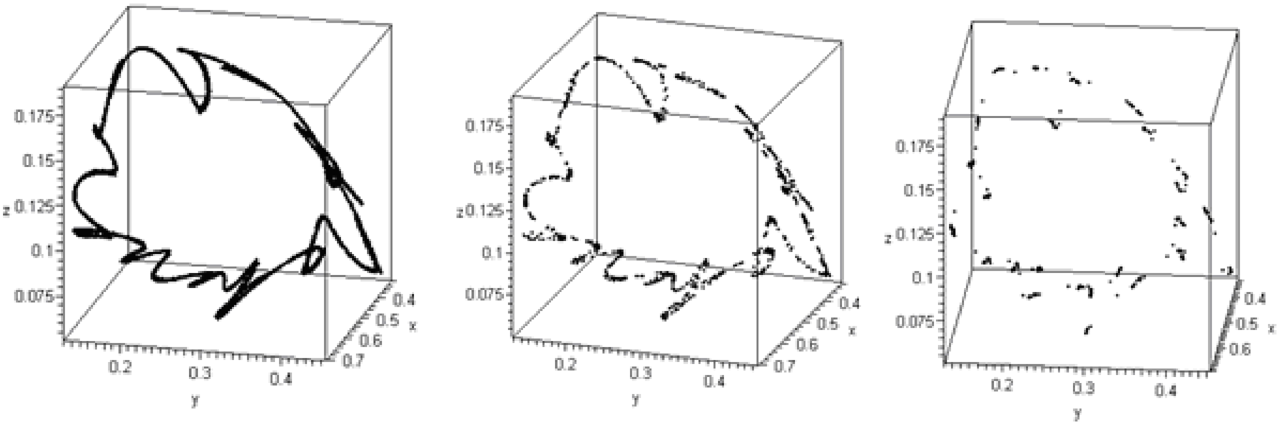


Fig. 9 Strange attractor for the system (1) at a =3.985.

The breakdown and disappears of a strange attractor for three successive values of control parameter a control parameter *a* which belongs to [3.99, 3.995] is illustrated in Fig. 10.

Figs 11 and 12 represent strange attractors for the system (1) with a =*3.997* and a =*4.1*, respectively. These attractors exhibit fractal structure.



(10.1) (10.2) (10.3)

Fig. 10 The breakdown and disappears of a strange attractor for three successive values of control parameter a; (10.1) for a = 3.99, (10.2) for a = 3.994 and (10.3) for a = 3.995.

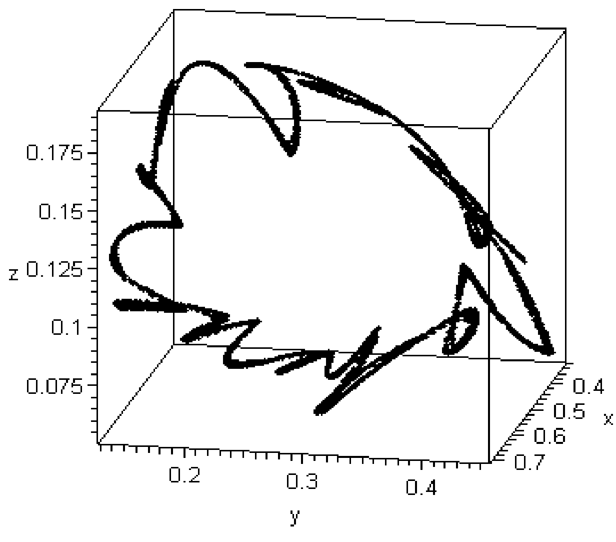


Fig. 11 Strange attractor for the system (1) at a = 3.997.

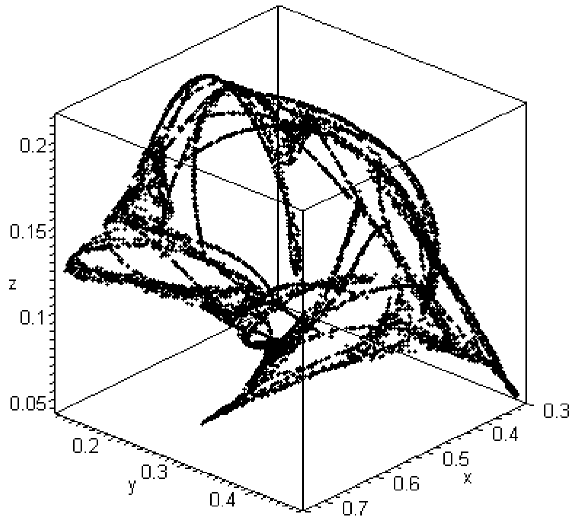


Fig. 12 Strange attractor for the system (1) at a = 4.1.

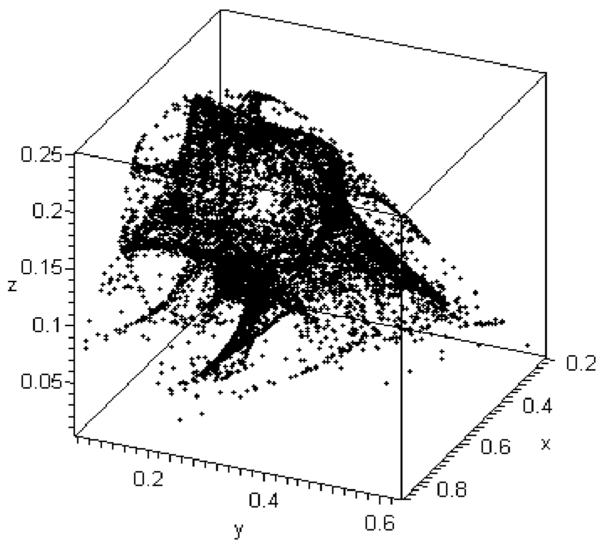


Fig. 13 Full developed chaos for the system (1) at a = 4.26.

In addition, the system (1) occurs a full developed chaos when a *=4.26*, Fig. 13. For a *>4.26* the broadening chaotic attractor disappears and this means that the phase trajectories of the system become infinite.

Lyapunov exponents measure the exponential rates of convergence or divergence, in time, of adjacent trajectories in phase space. So, the Lyapunov exponents are one of the most important tools for understanding chaotic behavior. A positive Lyapunov exponent is characteristic of chaos while zero and negative values of the exponent signify a marginally stable or quasiperiodic orbit and periodic orbit, respectively. For our system (1), to analyze the parameter sets for which aperiodic behavior occurs, one can compute the maximal

Lyapunov exponent depend on parameter a . For example, if the maximal Lyapunov exponent is positive, one has evidence for chaos. Moreover, by comparing the standard bifurcation diagram in parameter a , one obtains a better understanding of the particular properties of our system. In order to study the relations between the local stability of the interior fixed point E_3 compute the maximal Lyapunov exponents for intrinsic rate of growth of the prey x .

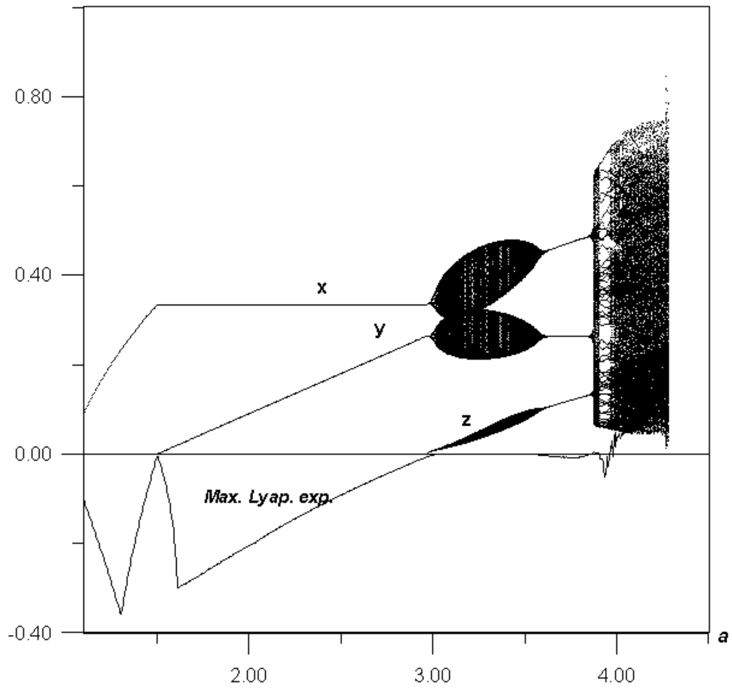


Fig. 14 Bifurcation diagram for the system (1) versus a and corresponding maximal Lyapunov exponent.

In Fig. 14, the bifurcation diagram for system (1) is plotted as a function of the control parameter a for $1.1 \leqslant a \leqslant 4.5$ and the maximal Lyapunov exponent is plotted. From this figure it is clear that the fixed point is stable for $a < 3.01$, and loses its stability through the discrete Hopf bifurcation. As parameter a increases the behavior of this model is becomes complicated, including many a discrete Hopf bifurcations and crises. System (1) goes back to stable fixed point when $a = 3.6$. Again the fixed point loses its stability through a discrete Hopf bifurcations at $a = 3.87$. For $a > 3.87$ a second invariant cycle appears enclosing the fixed point E_3, and its radius becomes larger with respect to the growth of a . As a increases the periodic orbits become irregular and reach chaotic state at last, as one can see form Fig. 14. In addition, there are many complex dynamics including the chaotic bands and from maximal Lyapunov exponent one can see that as a increases chaotic dynamics appear.

5 Conclusions

In this paper, the behavior of the food chain as a discrete dynamical system in R^3 is investigated. Basic properties of the system have been analyzed by means of phase portraits, Bifurcation diagrams and maximal Lyapunov exponent. Under certain parametric conditions, the interior fixed point enters a discrete Hopf bifurcation phenomenon. This paper may be useful to ecologist who works in the biological control for insects.

Although, we focused our attention, only, on the varying control parameter a and as a result the studied system exhibited many complex dynamics, the system is sensitive to more than one parameter. So, a more detailed analysis for this system focus on the bifurcation analysis by using Normal form theorem and center manifold theory will be provided in the near future (Elsadany, in preparation).

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