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Chaos and bifurcation of a nonlinear discrete prey-predator system

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Received 12 May 2012; *Accepted 20 June 2012*; *Published online 1 September 2012*

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Abstract

The discrete-time Prey-predator system obtained by two dimensional map was studied in present study. The fixed points and their stability were analyzed. Bifurcation diagram has been obtained for selected range of different parameters. As some parameters varied, the model exhibited chaos as a long time behavior. Lyapunov exponents and fractal dimension of the chaotic attractor of our map were also calculated. Complex dynamics such as cycles and chaos were observed.

Keywords prey-predator system; chaotic behavior; Lyapunov exponents; fractal dimension.

1 Introduction

In 1926, the Italian mathematician Vito Volterra (Voltera, 1962) first proposed a simple differential equations model to describe the population dynamics of two interacting species, a predator and its prey. He hoped to explain the observed increase in predator fish and corresponding decrease in prey fish in the Adriatic during world war. The model is

$$\begin{aligned} \frac{dx}{dt} &= ax(1-x) - bxy, \\ \frac{dy}{dt} &= -cy + dxy, \end{aligned} \tag{1}$$

where $x(t)$ be the population density of prey, $y(t)$ be the population density of predator at time t and *a, b, c, d* are all positive parameters. Here *a* represents the natural growth rate of the prey in the absence of predators; *b* represents the effect of predation on the prey; *c* represents the natural death rate of the predator in the absence of prey; *d* represents the efficiency and propagation rate of the of predator in the presence of prey. The model (1) is known as Lotka-Volterra model since the same equations were also derived by Alfred Lotka (Lotka, 1925), a chemist, from autocatalysis in chemical reaction.

The dynamics of a prey-predator differential equations model is studied by many authors see (Hainzl, 1988; He, 1996; Murray, 1998). Another possible way to understand a prey-predator model is by using discrete models. Actually these models are more reasonable than the continuous time models when populations have non-overlapping generations see May (May, 1975). Discrete-time models can give rise to more efficient computational models for numerical simulations and it exhibits more plentiful dynamical behaviors than a continuous-time model of the same type.

In ecology, predator-prey or host-parasite models can be formulated as discrete time models. Jing et al. are applied the Euler scheme method on some differential equation models. They studied the dynamics of corresponding discrete models obtained by Euler method in Jing et al. (2002), Jing et al. (2004), and Jing and Yang (2006). Also complex behavior of predator-prey system obtained by Euler method examined in Liu and Xiao. Chaotic dynamics of a discrete prey-predator model with Holling type II studied in Agiza et al. (2009). The modification of Beddington Free Lawton model of parasite host system dynamics was investigated in Ivanchikov and Nedorezov (2011). Also, Ivanchikov and Nedorezov (2012) studied modification of May model of parasite-host system dynamics. Elsadany (2012) discussed dynamical complexities in a discrete-time food chain. Danca et al. (1997) studied discrete prey-predator model, when the prey grows logistically. They showed that it contains stable fixed points, cycles, bifurcations and chaos. They did not consider the natural death rate of the predators. To make more realistic, we consider natural death rate in the predator equation which was not studied in Danca et al (1997). In this work we study the discrete prey-predator model which the natural death of the predator in the absence of prey included.

The paper is organized as follows, the time evaluation of a dynamic discrete prey predator model is described, and the system modeled by a two dimensional map. The existence and local stability of fixed points of non-invertable two dimensional map are analyzed in section 2. Chaotic behavior under some change of control parameters of the model is investigated by numerical simulation in section 3. Also Lyapunov exponents and fractal dimension of the strange attractor of our map is measured numerically. Finally, section 4 concludes of this paper.

2 Model

The dynamics of the two-species system consisting of one prey and one predator is governed by the following system of difference equations

$$\begin{cases} x_{n+1} = x_n f(x_n, y_n) \\ y_{n+1} = y_n g(x_n, y_n), \end{cases} \tag{2}$$

where $f(x_n, y_n)$ and $g(x_n, y_n)$ satisfy condition $\frac{\partial f}{\partial y_n} \le 0,\ \frac{\partial g}{\partial x_n} \ge 0$. We take

$f(x, y) = a(1-x) - by$ and $g(x, y) = -cy + dx.$ So the discrete prey-predator model is the system

$$H: \begin{cases} x_{n+1} = ax_n(1-x_n) - bx_n y_n \\ y_{n+1} = -cy_n + dx_n y. \end{cases} \tag{3}$$

The map (3) is a non-invertable map of the plane. The study of the dynamical properties of the map (3) allows us to have information long-run behavior of prey-predator populations. Starting from given initial condition (x_0, y_0), the iteration of Eq. (3) uniquely determines a trajectory of the states of population output

$$(x_n, y_n) = H^n(x_0, y_0, z_0), \quad where \quad n = 0,1,2,...$$

3 Fixed Points and Local Stability

We now study the existence of fixed points of the system (3). Particularly we are interested in the non-negative or interior fixed point. To begin with we list all possible fixed points

(i) $E_0 = (0,0)$ is trivial fixed point.

(ii) $E_1 = (\frac{a-1}{a},0)$ is the axial fixed point in the absence of predator (y =0).

(iii) $E_2 = (x^*, y^*)$ is the interior fixed point, where

$$x^* = \frac{1+c}{d} \;\; and \;\; y^* = \frac{a}{b}(1-\frac{1+c}{d}) - \frac{1}{b}. \tag{4}$$

For the interior fixed point $E_2 = (x^*, y^*)$ to be positive we need $d > \frac{a(1+c)}{a-1}$ and $a > 1$.

3.1 Dynamic behavior of the model

In this subsection, we investigate the local behavior of the model (3) around each fixed point. The local stability analysis of the model (3) can be studied by computing the variation matrix corresponding to each fixed point. The variation matrix of the model at state variable is given by

$$J(x_n, y_n) = \begin{bmatrix} a(1-2x_n) - by_n & -bx_n \\ dy_n & -c + dx_n \end{bmatrix}. \tag{5}$$

The determinant of the Jacobian $J(x_n, y_n)$ is $Det = a(1\text{-}2x_n)(dx_n - c) + bcy_n$. Hence the model (3) is dissipative dynamical system when

$$|a(1 - 2x_n)(dx_n - c) + bcy_n| < 1.$$

Proposition 1 *The fixed point* E_0 *locally asymptotically stable if* $a, c < 1$, *otherwise unstable fixed point.*

Proof. In order to prove this result, we estimate the eigenvalues of Jacobian matrix J at E_0. The Jacobian matrix for E_0 is

$$J(E_0) = \begin{bmatrix} a & 0 \\ 0 & -c \end{bmatrix}.$$

Hence the eigenvalues of $J(E_0)$ are $\lambda_1 = a$ and $\lambda_2 = -c$ .Thus E_0 for the values *c*. Otherwise E_0 is unstable fixed point.

Proposition 2 *The fixed point* E_1 *locally asymptotically stable if* $1 < a < 3$ *and* $0 < c < 1$. *Otherwise unstable fixed point.*

Proof. One can see that The Jacobian matrix for E_1 is given by

$$J(E_1) = \begin{bmatrix} 2-a & \frac{b(1-a)}{a} \\ 0 & -c \end{bmatrix}.$$

The eigenvalues of the matrix are $\lambda_1 = 2 - a$ and $\lambda_2 = -c$. Hence E_1 is locally asymptotically stable when $1 < a < 3$ and $0 < c < 1$, and unstable when $a > 3$ or $c > 1$.

The fixed point E_1 can through flip bifurcation when parameters vary and a center manifold of Eq. (3) at E_1 is y = 0 which is restricted to the center manifold of logistic model (Murray, 1998). In this case the predator becomes extinction and the prey pass through period doubling bifurcation to chaos in the sense of Li-Yorke by varying bifurcation parameter a.

3.2 Local stability and dynamic behavior around interior fixed point E_2

We now investigate the local stability and bifurcations of interior fixed point E_2. The Jacobian matrix (5) at E_2 has the form

$$J(E_2) = \begin{bmatrix} \dfrac{1 - a(1+c)}{d} & \dfrac{-b(1+c)}{d} \\ \dfrac{ad}{b}(1 - \dfrac{1+c}{d}) - \dfrac{d}{b} & 1 \end{bmatrix}. \tag{6}$$

Its characteristic equation is

$$P(\lambda) = \lambda^2 - Tr\lambda + Det = 0 \tag{7}$$

where Tr is the trace and Det is the determinant of the Jacobian matrix $J(E_2)$ defines in Eq. (6),

$Tr = \dfrac{2 - a(1+c)}{d}$ and $Det = a(1+c)(1 - \dfrac{2+c}{d}) - c.$

If the eigenvalues of the Jacobian matrix of fixed point E_2 are inside the unit circle of the complex plane, interior fixed point E_2 is local stability. Using Jury's conditions (Elaydi, 1996), we have necessary and sufficient condition for local stability of interior fixed point which are the necessary and sufficient condition for $|\lambda_i| < 1,\ i = 1,2$.

$$(i)\ \ P(1) = 1 - Tr + Det > 0$$
$$(ii)\ P(-1) = 1 + Tr + Det > 0$$
$$(iii)\ |P(0)| = |Det| < 1.$$

Using formulas of Tr and Det, we find that inequality (i) is equivalent to

$$a(1 - \frac{1+c}{d}) > 1,$$

which implies that

$$d > \frac{a(1+c)}{a-1}. \tag{8}$$

The second inequality (ii) is equivalent to

$$d > \frac{a(1+c)(3+c)}{a(1+c) - c + 3}. \tag{9}$$

The third inequality (iii) is equivalent to

$$d < \frac{a(2+c)}{a-1}. \tag{10}$$

For the interior fixed point E_2 the roots of Eq. (7) are:

$$\lambda_{1,2} = (1-\frac{a(1+c)}{2d}) \pm \sqrt{(1+\frac{a(1+c)}{2d})^2 - a(1+c)(1-\frac{c}{d})+c}. \tag{11}$$

Both eigenvalues are real for λ_R and $\left|\lambda_{1,2}\right| < 1$ if $(1+\frac{a(1+c)}{2d})^2 > a(1+c)(1-\frac{c}{d})+c$ and $\frac{a(1+c)}{d}+1<a$, which implies

$$d \in (\frac{a(1+c)}{a-1}, \frac{a(1+c)(3+c)}{a(1+c)-c+3}], \qquad a>1. \tag{12}$$

The eigenvalues $\lambda_{1,2}$ become complex λ_C and are inside the unit circle in the complex plane if $(1+\frac{a(1+c)}{2d})^2 < a(1+c)(1-\frac{c}{d})+c$ and $\frac{a(2+c)}{d}+1<a$, which implies

$$d \in (\frac{a(1+c)(3+c)}{a(1+c)-c+3}, \frac{a(2+c)}{a-1}], \qquad a>1. \tag{13}$$

The conditions (12) and (13) determine the domains of the values of parameters a, c and d for which E_2 fixed point is stable. Under certain conditions, it can be obtained that system (3) undergoes a discrete Hopf bifurcation at E_2. From the above analysis, we have the following proposition:

Proposition 3 *If* $(1+\frac{a(1+c)}{2d})^2 < a(1+c)(1-\frac{c}{d})+c$ and $d = \frac{a(2+c)}{a-1}$, the system (3) undergoes a discrete Hopf bifurcation at E_2. Moreover an attracting invariant closed curve bifurcates from the fixed point for $d > \frac{a(2+c)}{a-1}$.

4 Numerical Simulations

In this section we present some results of numerical simulations and discuss their implications. We provide some numerical evidence for the qualitative dynamic behavior of the map (3). We use the bifurcation diagrams, phase portraits, sensitive dependence on initial conditions, Lyapunov exponents and fractal dimension to illustrating the above analytic results and the dynamics behavior of map (3) as the parameters varying. The bifurcation diagrams are considered in three cases:

(1) Fixing parameters $b=3.4,\ c=0.2,\ d=3.5$ and varying a.

(2) Fixing parameters $b=2,\ c=0.2,\ d=3.5$ and varying a.

(3) Fixing parameters $b=1,\ c=0.2,\ d=3.5$ and varying a.

The case (1) bifurcation diagram of map (3) in ($a-xy$) plane is showing the dynamical behavior of the prey-predator systems as a varying and fixing parameters b $=3.4, c=0.2, d=3.5$. From Fig. 1 one can see that the orbit with initial conditions*(0.1, 0.2)* approaches to the stable fixed point E_2 for $a< 2.6923$, and a discrete Hopf bifurcation occurs at $a=2.6923$. As a increases, the interior fixed point becomes unstable through a discrete Hopf bifurcation and the behavior of prey-predator model becomes chaotic. It means for a large values of natural growth rate of the prey species *a*, the system converge always to complex dynamics. If the case of $a \geq 2.6923$ one observes a discrete Hopf bifurcation occurs and complex dynamic behavior begin to appear for $2.6923 \leq a \leq 5$.

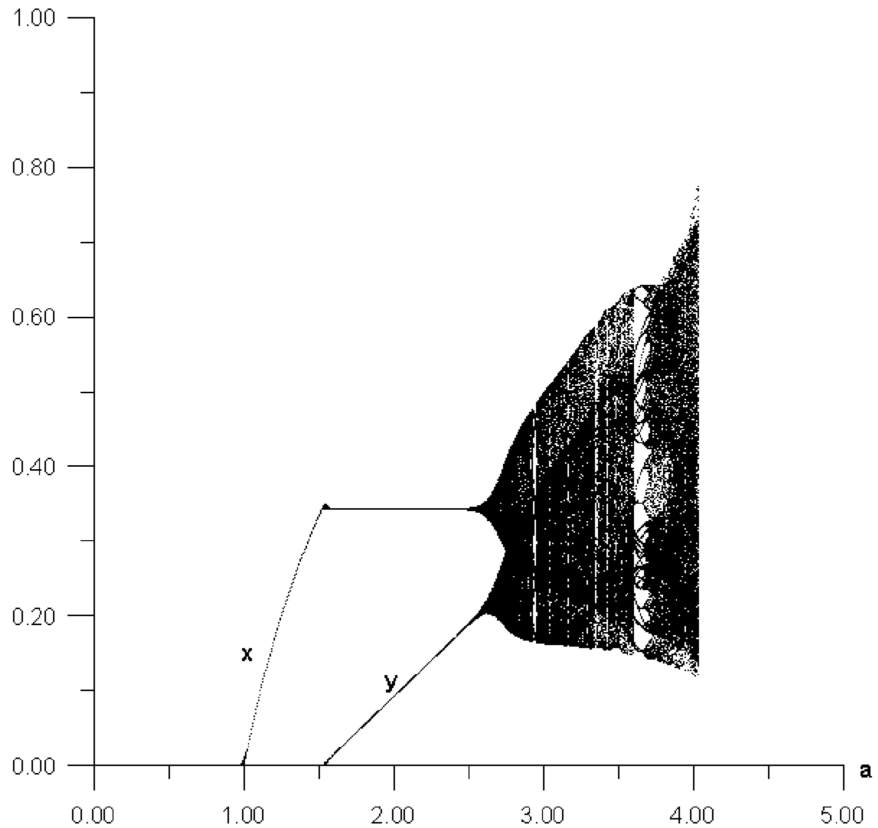

Fig. 1. Bifurcation diagram of model (3) with respect to a. Other parameters are b=3.4, c=0.2, d=3.5.

A bifurcation diagram with respect to a in Fig. 2, while other parameters are fixed as follows $b=2, c=0.2$ and $d=3.5$. Also bifurcation diagram with respect to a plotted in Fig. 3 other parameters are fixed as follows $b=1, c=0.2$ and $d=3.5$. From Figs. (1-3) the changes of parameter a effect of stability of the systems and occurs of a discrete Hopf bifurcation point. But, changes of parameter b effect only of output system (i.e. range of outputs which means x values and y values) not effect on stability or occurs of a discrete Hopf bifurcation. We show the graph of a strange attractor for the parameter constellation *(a, b, c, c, d)=(4, 1, 0.2, 3.5)* in Fig. 4, which exhibits a fractal structure.

In order to analyze the parameter sets for which aperiodic behavior occur, one can compute the maximal Lyapunov exponent depend on a. For example, if the maximal Lyapunov exponent is positive, one has evidence for chaos. Moreover by comparing the standard bifurcation diagram, one obtains a better understanding of the particular properties of the dynamical behavior of system. In order to study local stability of the interior point E_2 with natural growth parameter of prey species *a*, one can compute maximal Lyapunov exponent as function in a. Fig. 5 displays the related maximal Lyapunov exponent of system (3) as function in a. From Fig. 5, one can easily determine the degree of the local stability for different values of $a \in (2.692, 4)$. At values of $a > 2.692$ maximal Lyapunov exponent is positive. A positive value of maximal Lyapunov exponent implies sensitive dependence to initial conditions for the system (3). From maximal Lyapunov exponent diagram it is easy to determine the parameters sets for which the system converges to cycles , quasi-periodic and chaotic behavior. Beyond that its even possible to differentiate between cycles of every higher order and aperiodic behavior of the system (3) see Fig. 5.

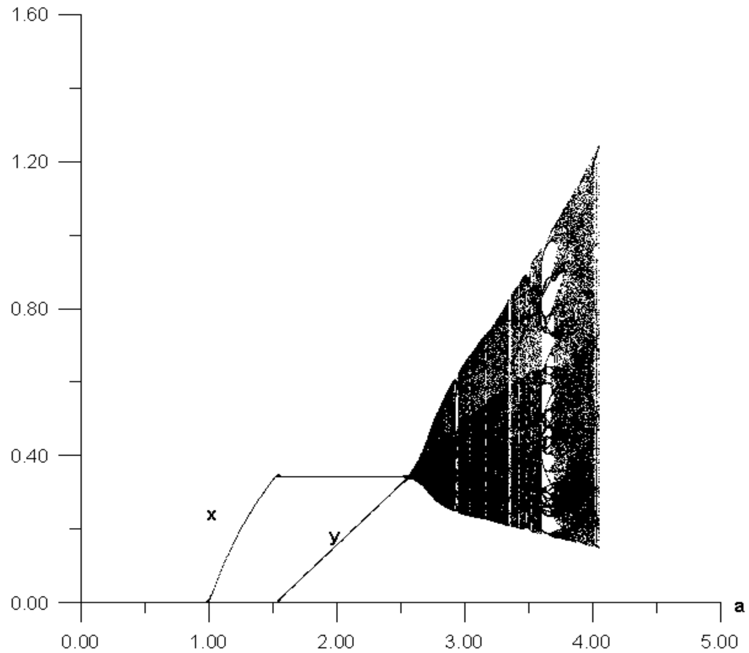

Fig. 2. Bifurcation diagram of model (3) with respect to a. Other parameters are b=2, c=0.2, d=3.5.

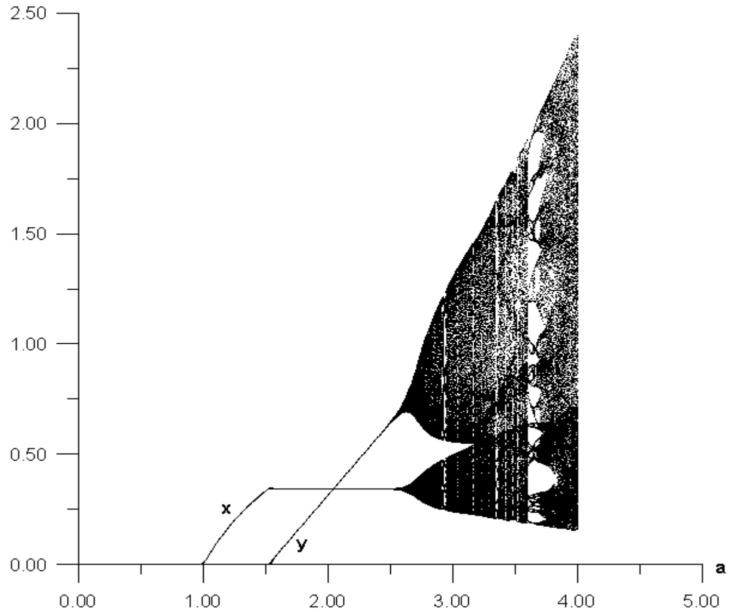

Fig. 3. Bifurcation diagram of model (3) with respect to a. Other parameters are b=1, c=0.2, d=3.5.

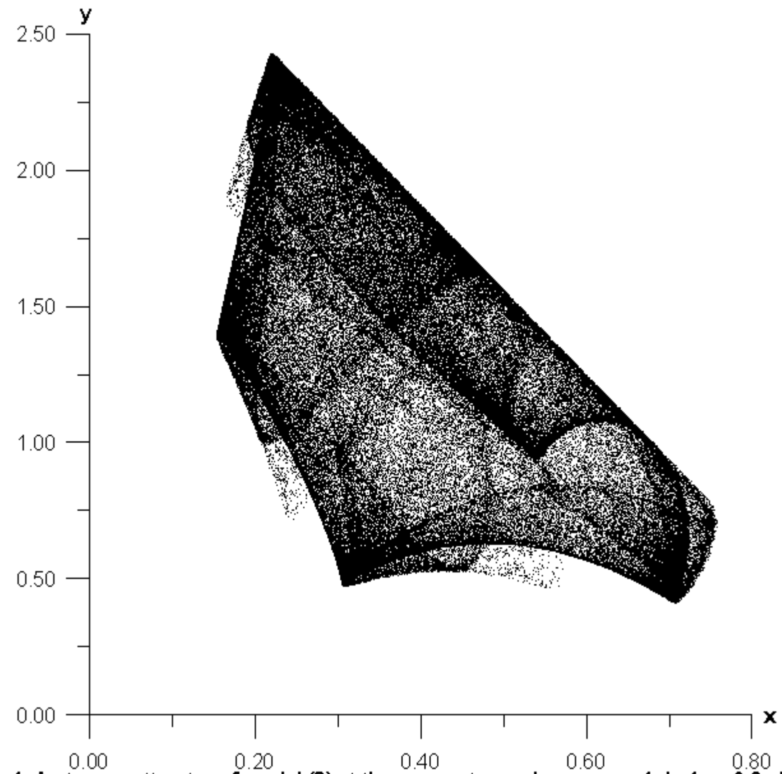

Fig. 4. A strange attractor of model (3) at the parameters values are a=4, b=1, c=0.2, d=3.5.

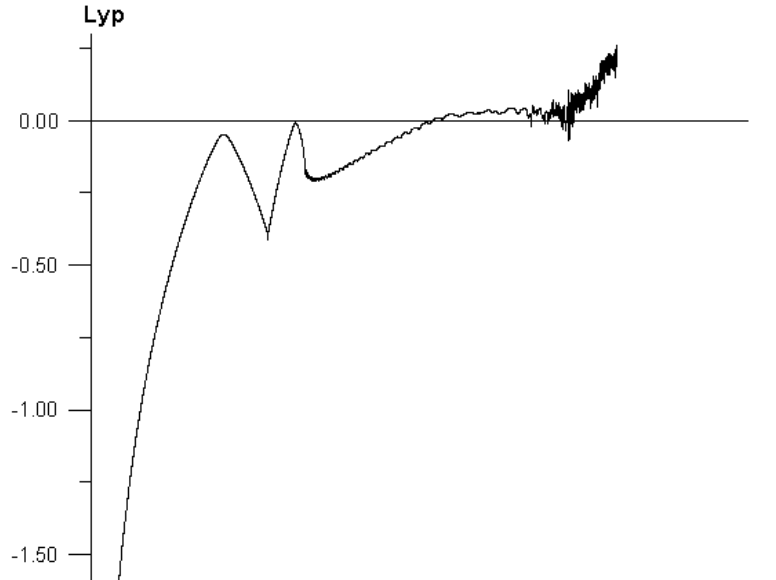

Fig. 5. Maximal Lyapunov exponent of model (3) with respect to a. Other parameters values are b=1, c=0.2, d=3.5.

The bifurcation diagram in *(d−xy)* plane for *a =3.5 ,b =1*and *c =0.2* in Fig. 6 The interior fixed point E_2 of the system (3) loses its stability at *d =3.08* on account of the norm of its corresponding eigenvalues grater than 1, and there is an invariant circle when *d> 3.08*. Also the maximal Lyapunov exponent is plotted in Fig. 6. Bifurcation diagram in ***(c − xy)*** plane for *a =3.5,b =1* and *d =3.5* with initial values *(0.2, 0.1)* in Fig. 7. From Fig. 7, we see that there are a discrete Hopf bifurcation phenomenon's.

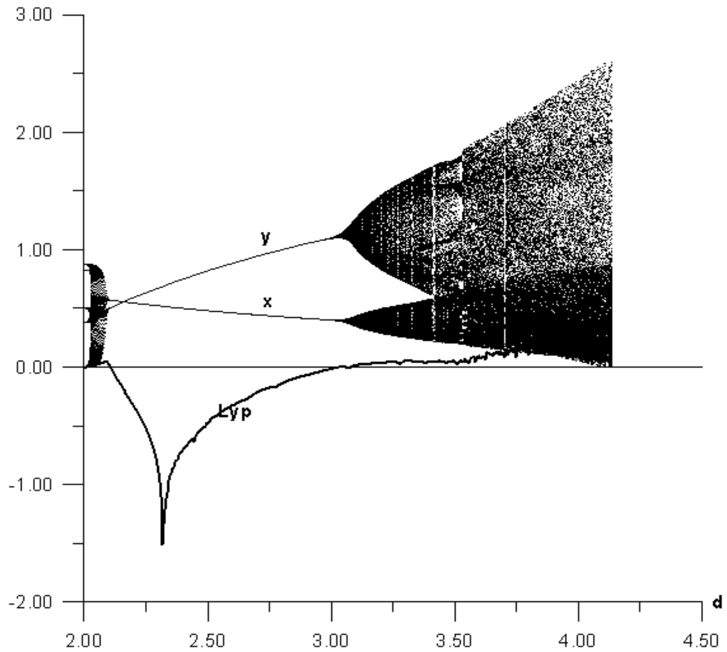

Fig. 6. Bifurcation diagram of model (3) with respect to d. Also maximal Lyapunov exponent plotted. Other parameters values are a=3.5, b=1, c=0.2.

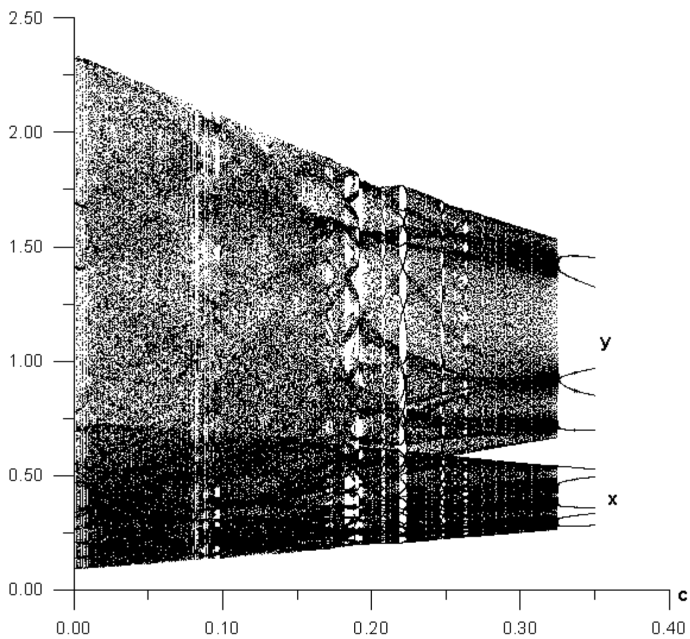

Fig. 7. Bifurcation diagram of model (3) with respect to c. Other parameters are a=3.5, b=1, d=3.5.

4.1 Sensitive dependence on initial conditions

The sensitivity to initial conditions is a characteristic of chaos. In order to demonstrate the sensitivity to initial conditions of system (3), we compute two orbits with initial points (x_0, y_0) and $(x_0 + 0.0001, y_0)$, respectively. The compositional results are shown in Fig. 8 and Fig.9. From the these figures it is clear that, at the beginning, the time series are indistinguishable; but after a number of iterations, the difference between them builds up rapidly.

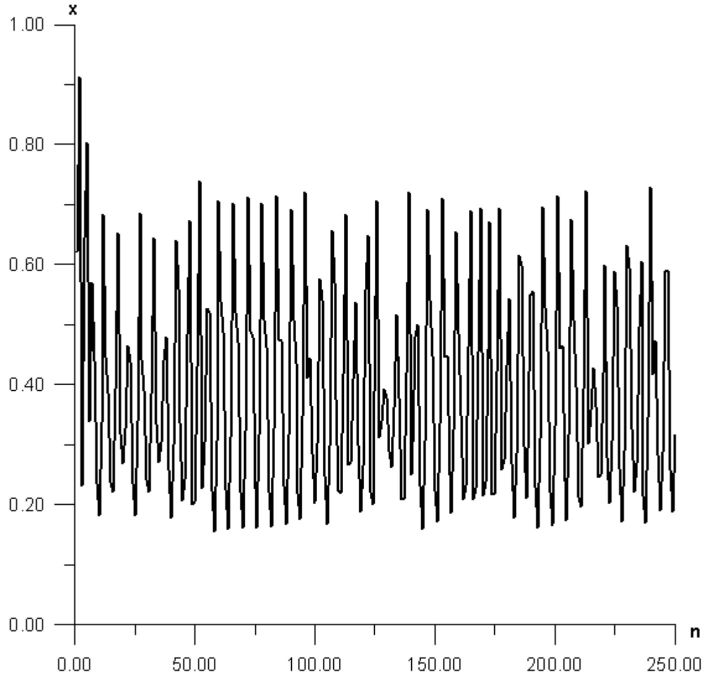

Fig. 8. Time series x_n corresponding to the initial density (0.2,0.1).

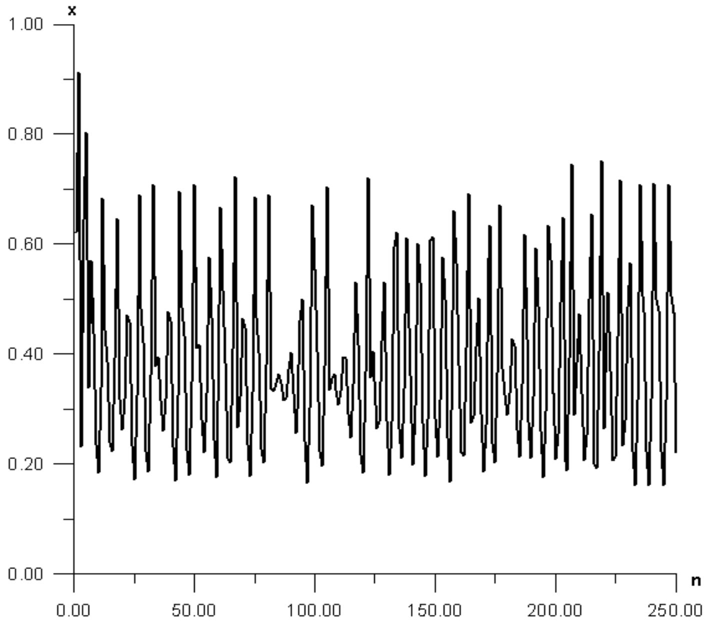

Fig. 9. Time series x_n corresponding to the initial density (0.20001,0.1).

In addition, Fig. 8 and Fig. 9 show sensitive dependence on initial conditions, *x*-coordinates of the two orbits, for system (3), is plotted against the time with the parameter constellation *(a, b, c, d)=(4, 1, 0.2, 3.5).* The *x*-coordinates of initial conditions differ by *0.0001* and the other coordinates are kept equal.

4.2 Fractal dimension of the map (3)

Strange attractors are typically characterized by fractal dimensions. We examine the important characteristic of neighboring chaotic orbits to see how rapidly they separate each other. The Lyapunov dimension (Kaplan and Yorke, 1979; Cartwright, 1999) is defined as follows:

$$d_L = j + \frac{\sum_{i=1}^{j} \Lambda_i}{\left|\Lambda_j\right|},$$

with $\Lambda_1, \Lambda_2, ..., \Lambda_n$,where *j* is the largest integer such that $\sum_{i=1}^{j} \Lambda_i \geq 0$ and $\sum_{i=1}^{j+1} \Lambda_i < 0$. In our system of the two-dimensional map has the Lyapunov dimension, which is given by

$$d_L = 2 + \frac{\Lambda_1}{\left|\Lambda_2\right|}, \quad \Lambda_1 > 0 > \Lambda_2$$

By the definition of the Lyapunov dimension and with help of the computer simulation one can show that the Lyapunov dimension is dimension of the strange attractor of system (3). At the parameters values *(a, b, c, d)= (4, 1, 0.2, 3.5),* two Lyapunov exponents exists and are $\Lambda_1 \approx 0.15137$ and $\Lambda_2 \approx -0.28494$. Therefore the map (3) exhibits a fractal structure and its attractor has the fractal dimension $d_L \approx 1 + \frac{0.15137}{0.28494} \approx 1.5312$, which is chaotic behavior.

5 Conclusions

In this paper, we analyzed the complex dynamics of a nonlinear discrete-time prey-predator system. The stability of fixed points, bifurcation and chaotic behavior are investigated in this system. The influence of the main parameters (such as the natural growth rate of prey; c the natural death rate of the predator; d the efficiency and propagation rate of the predator) on the local stability is studied. We observed that under certain parametric conditions the interior fixed point enters a discrete Hopf bifurcation phenomenon. We showed that the system exhibits a huge variety of complicated dynamical behavior, including many forms chaotic behavior.

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