Article

Analytical treatment of system of KdV equations by Homotopy Perturbation Method (HPM) and Homotopy Analysis Method (HAM)

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Abstract

In this article the Homotopy Perturbation Method (HPM) and Homotopy Analysis Method (HAM) are applied to obtain analytic approximate solution to three system of nonlinear wave equations, namely two component evolutionary system of a homogeneous KdV Equations of order three (system-I) as well as (system-II) and the generalized coupled Hirota Satsuma KdV (System-III).

Keywords Homotopy Perturbation Method (HPM); Homotopy Analysis Method (HAM); Systems of KdV equations.

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1 Introduction

In different field the nonlinear phenomena is very important and it played a tremendous role, especially in the field of applied mathematics, engineering and physics etc. Now for mechanism of physical model described by differential equations, different types of effective methods have been discovered for helping the engineers, scientist and physicist to know about the problem and its application, because in most cases it is still difficult to obtain the exact solution.

Like other nonlinear analytic technique Homotopy Perturbation Method (HPM) and Homotopy Analysis Method (HAM) are two well known methods for obtaining the analytic approximate solutions to differential equations. In He (1999), the Homotopy Perturbation Method (HPM) was first presented. The Homotopy Perturbation Method (HPM) applied by many authors (Shakil et al, 2013; Siddiqui et al., 2014; Wahab et al., 2013; Wahab et al., 2014), to find the solution of nonlinear problems in the field of science and engineering. This method have the ability to solve linear and nonlinear problems (Alquran and Muhammad, 2011; Hemeda, 2012). Homotopy Perturbation Method (HPM) provides an opportunity that is no requirement of small parameter like perturbation methods, in the equations. Homotopy Perturbation Method (HPM) also applicable to different types of equations like Volterra's equation, integro equation, nonlinear oscillator equation, bifurcation equation, nonlinear wave equation etc (Hemeda, 2012). In most cases Homotopy Perturbation Method (HPM) provides a very rapid and fast convergence (Hemeda, 2012). Thus a method which have the ability to solve different types of nonlinear equations is known as Homotopy Perturbation Method (HPM).

Liao has presented another analytic approximation in 1992 (Liao, 1992). This method is based on a interesting property called homotopy, a fundamental concept of differential geometry and topology (Taiwo et al., 2012). Homotopy Analysis Method (HAM) is an analytic approximated method through which we can find the solution of nonlinear problem. Since perturbation techniques are often non valid in case of strong non linearity, but Homotopy Analysis Method (HAM) is valid in non-linearity case (Liao, 2003). By using one interesting property of homotopy, the non-linear problem can be changed into an infinite number of linear problems, no matter comes from small or large parameter. If a non-linear problem has even a single solution, then through this method namely Homotopy Analysis Method (HAM), there exist an infinite number of disjoint solution expression whose the region of convergence and rate of convergence dependent on an axillary parameter (Liao, 2003).

The purpose of this paper is to present analytic approximate solution to system of KdV equations by using Homotopy Perturbation Method (HPM) and Homotopy Analysis Method (HAM).

2 Analysis of Homotopy Perturbation Method (HPM)

Let us consider a general nonlinear differential equation of the form,

$$L(x)+\mathrm{N}(x)=g(z), z\in\Xi \qquad (1)$$

subject to the boundary condition,

$$\mathrm{K}\left(x,\frac{\partial x}{\partial n}\right)=0, z\in\Theta \qquad (2)$$

In equation (1)-(2), $"L"$ is the linear operator. $"N"$ is the nonlinear operator. $"\mathrm{K}"$ is defined to be the boundary operator. Boundary of the domain $"\Xi"$ is $"\Theta"$. The known function is define to be the function $"g(z)"$. Now by Homotopy Analysis Method (HAM) constructing a homotopy, such that,

$\phi(r,q):\Xi\times[0,1]\rightarrow R$,

which satisfies,

$$H(\phi,\sigma)=(1-\sigma)\left[L(\phi)-L(x_0)\right]+\sigma\left[L(\phi)+\mathrm{N}(\phi)-g(z)\right]=0 \qquad (3)$$

Equation (3), becomes

$$H(\phi,\sigma)=L(\phi)-L(x_0)+\sigma L(x_0)+\sigma\left[N(\phi)-g(z)\right]=0 \qquad (4)$$

where $"z\in\Theta"$ and $"\sigma\in[0,1]"$ is known to be the embedding parameter, $"x_0"$ is define to be the, so called initial approximation, must satisfies the boundary condition. Now setting $"\sigma=0"$ and $"\sigma=1"$ in equation (4), then

$$H(\phi,0)=L(\phi)-L(x_0)=0 \qquad (5)$$

$$H(\phi,1)=L(\phi)+\mathrm{N}(\phi)-g(z)=0 \qquad (6)$$

Equation (5) and equation (6), are called homotopic equations and the value of $"\sigma"$ changing from

"0" to unity is called deformation in topology (Hemeda, 2012). Now according to Homotopy Perturbation Method (HPM), the basic assumption of the for the solution of equation (3)-(4) the basic assumption is that, the "σ" can be expressed in a power series, such that,

$$\phi(x,t;\sigma)=v_0(x,t)+\sigma v_1(x,t)+... \tag{7}$$

and thus the analytical approximate solution for equation (1), can be derived through Homotopy Perturbation Method (HPM) by setting "$\sigma=1$" in equation (7), which becomes,

$$x=\lim_{q\to 1}\phi(x,t;\sigma)=v_0+v_1+v_2+... \tag{8}$$

which will be the required approximated solution of the given nonlinear problem derived by Homotopy Perturbation Method (HPM).

3 Analysis of Homotopy Analysis Method (HAM)

Homotopy Analysis Method (HAM) is a straight forward and very simple method. This method was presented by means of homotopy (Hilton, 1953; Liao, 2003), which is a fundamental concept of topology.

Consider a differential equation, such that

$$\mathrm{N}\left[\gamma(x,t)\right]=0 \tag{9}$$

In equation (9), "N" is non-linear operator and "$\gamma(x,t)$" is the unknown function. Now the generalization of the traditional homotopy by Homotopy Perturbation Method (HPM) presented by Liao (2003), construct a new type of homotopy called deformation equation of zero-order, such that,

$$(1-\sigma)L\left[\Psi(x,t;\sigma)-\gamma_0(x,t)\right]=\sigma\eta\mathrm{H}(x,t)\mathrm{N}\left[\Psi(x,t;\sigma)\right] \tag{10}$$

In equation (10), "$\gamma_0(x,t)$" is known to be the initial approximation of the given unknown function that is,"$\gamma(x,t)$". "$\Psi(x,t;\sigma)$" is a function, which is not known. "σ" is the embedding parameter, "η" and "$H(x,t)$"are the non-zero auxiliary parameter and non-zero auxiliary function respectively, "N" is the operator called non-linear and "L" is the auxiliary operator called linear operator. In this method it is very important that we can easily and with great freedom chose the auxiliary materials (Hemeda, 2012).

Now if $\sigma=0$ and $\sigma=1$, then equation (10), becomes,

$$\Psi(x,t;0)=\gamma_0(x,t) \text{ and } \Psi(x,t;1)=\gamma(x,t) \tag{11}$$

Thus equation (11), shows that the variation of the embedding parameter varies from zero to unity make the solution "$\Psi(x,t;\sigma)$"from the initial approximation to the exact solution. The variation of these kind is called deformation in the manner of topology (Liao, 2003).

According to Homotopy Analysis Method (HAM), expending "Ψ" in a power series with respect to "σ". Such that,

$$\begin{aligned}\Psi(x,t;\sigma)&=\gamma_0+\sigma\gamma_1+\sigma^2\gamma_2+...\\ \Psi(x,t;\sigma)&=\gamma_0+\sum_{m=0}^{\infty}\gamma_m\sigma^m\end{aligned} \tag{12}$$

In equation (12), "$\gamma_m(x,t)=\dfrac{\partial^m\Psi(x,t;\sigma)}{m!\partial\sigma^m}$" at "$\sigma=0$". Now if the auxiliary elements that is, the auxiliary linear-operator, auxiliary parameter, the initial-approximation and the auxiliary function are chosen through proper way, then the above series that is series in equation (12), is convergent at "$\sigma=1$". Thus the solution of the original non-linear problem becomes,

$$\gamma=\gamma_0+\sum_{m=1}^{\infty}\gamma_m\sigma^m \tag{13}$$

which is one-solution of the original nonlinear problem. Now according to the fundamental theorem of Homotopy Analysis Method (HAM), consider a vector, such that,

$$\vec{\gamma}=\left[\gamma_0,\gamma_1,...,\gamma_n\right] \tag{14}$$

Then the deformation equation of order "m" is given by

$$L\left[\gamma_m-\lambda_m\gamma_{m-1}\right]=\eta.\mathrm{H}\left(x,t\right).\mathrm{E}_m\left(\overrightarrow{\gamma_{m-1}}\right) \tag{15}$$

In equation (15),

$$\lambda_m=\begin{cases}0, m\le 1\\ 1, m>1\end{cases} \quad \text{and} \quad \mathrm{E}_m\left(\overrightarrow{\gamma_{m-1}}\right)=\frac{\partial^{m-1}\mathrm{N}\Psi\left(x,t;\sigma\right)}{\left(m-1\right)!\partial\sigma^{m-1}} \tag{16}$$

Now if "$\eta=-1$" and "$H\left(x,t\right)=1$" in equation (10), then it become homotopy constructed in Homotopy Perturbation Method (HPM), which shows that Homotopy Perturbation Method (HPM) is a specified case of Homotopy Analysis Method (HAM). The genialized homotopy only not depend on the parameter "σ", but it also dependent on"η" and"$\mathrm{H}\left(x,t\right)$" called auxiliary parameter and auxiliary function respectively. Thus the generalized homotopy give us a family of approximation series whose the region of convergence depend upon on "η" and "$\mathrm{H}\left(x,t\right)$". Also the generalized homotopy provide us a straight forward way to control and adjust the convergence region and rate of approximation series (Liao, 2003). Homotopy Analysis Method (HAM) is more general and valid for non-linear and linear Differential equations in many types.

This is a very brief introduction, for details, we refer to Liao (2003), Hemeda (2012), Zedan and El Adrous (2012).

4 Applications

Taking the following systems of nonlinear KdV equations.

System-I

Consider the first system is a system of KdV equations of order three, such that

$$\frac{\partial\alpha}{\partial t}=\frac{\partial^3\alpha}{\partial x^3}+\alpha\frac{\partial\alpha}{\partial x}-\beta\frac{\partial\beta}{\partial x}, \tag{17}$$

$$\frac{\partial\beta}{\partial t}=-2\frac{\partial^3\beta}{\partial x^3}-\alpha\frac{\partial\beta}{\partial x}, \tag{18}$$

subject to,

$$\alpha_0=3-6\tanh^2\left(\frac{x}{2}\right), \tag{19}$$

$$\beta_0 = -3i(2)^{\frac{1}{2}}\tanh^2\left(\frac{x}{2}\right). \tag{20}$$

The closed form solution given in Alquran and Muhammad (2011), is as,

$$\alpha_{closed} = 3 - 6\tanh^2\left(\frac{x+t}{2}\right),$$

$$\beta_{closed} = -3i\sqrt{2}\tanh^2\left(\frac{x+t}{2}\right).$$

Homotopy Perturbation Method (HPM) Solution

Through Homotopy Perturbation Method (HPM) the approximate solution of the system-I is derived as,

$$\alpha_{HPM}(x,t) = \begin{Bmatrix} 3 - \tanh^2\left(\frac{x}{2}\right) - 6t\left[3 - 6\tanh^2\left(\frac{x}{2}\right)\right]\tanh\left(\frac{x}{2}\right)\sec h^2\left(\frac{x}{2}\right) + 24\sec h^4\left(\frac{x}{2}\right) \\ \tanh\left(\frac{x}{2}\right) - 24t\tanh^3\left(\frac{x}{2}\right)\sec h^2\left(\frac{x}{2}\right) + \ldots \end{Bmatrix}$$

$$\beta_{HPM}(x,t) = \begin{Bmatrix} 9\sqrt{2}it\sec h^2\left(\frac{x}{2}\right)\tanh\left(\frac{x}{2}\right) - 24\sqrt{2}it\sec h^4\left(\frac{x}{2}\right)\tanh\left(\frac{x}{2}\right) - 12\sqrt{2}it \\ \sec h^2\left(\frac{x}{2}\right)\tanh^3\left(\frac{x}{2}\right) + \ldots \end{Bmatrix}$$

Solution by Homotopy Analysis Method (HAM)

To solve system-I by Homotopy Analysis Method (HAM), and keeping in the view the given conditions, we define a linear-operator as, $"L = \frac{\partial}{\partial t}"$ and the inverse of linear operator is define as, $"L^{-1} = \int_0^t ()dt"$. Now by definition of Homotopy Analysis Method (HAM), the deformation equation of order-zero for the given system of non-linear partial differential equations becomes,

$$(1-\sigma)L\left[\Psi_1(x,t;\sigma) - \alpha_0\right] = \sigma.\eta.\mathrm{H}(x,t).\mathrm{N}\Psi_1(x,t;\sigma), \tag{21}$$

$$(1-\sigma)L\left[\Psi_2(x,t;\sigma) - \beta_0\right] = \sigma.\eta.\mathrm{H}(x,t).\mathrm{N}\Psi_2(x,t;\sigma), \tag{22}$$

where $"\alpha_0, \beta_0"$ are the given initial approximation define as,

$$\alpha_0 = \alpha(x,0) = 3 - 6\tanh^2\left(\frac{x}{2}\right), \tag{23}$$

$$\beta_0 = \beta(x,0) = -3i(2)^{\frac{1}{2}}\tanh^2\left(\frac{x}{2}\right). \tag{24}$$

Now since $"\sigma"$ is the embedding parameter, so the deformation process gives us,

$$\Psi_1(x,t;0) = \alpha_0, \tag{25}$$

$$\Psi_2(x,t;0)=\beta_0, \tag{26}$$

$$\Psi_1(x,t;1)=\alpha, \tag{27}$$

$$\Psi_2(x,t;1)=\beta. \tag{28}$$

Now suppose that, the solution of the original equation can be expressed in the power of the embedding parameter, such that,

$$\Psi_1(x,t;\sigma)=\alpha_0+\sum_{n=1}^{\infty}\alpha_n\sigma^n, \tag{29}$$

$$\Psi_2(x,t;\sigma)=\beta_0+\sum_{n=1}^{\infty}\beta_n\sigma^n. \tag{30}$$

In equation (29)-(30), $\alpha_n=\frac{1}{n!}\frac{\partial^n\Psi_1(x,t;\sigma)}{\partial\sigma^n}$ and $\beta_n=\frac{1}{n!}\frac{\partial^n\Psi_2(x,t;\sigma)}{\partial\sigma^n}$ at "$\sigma=0$" and exist for "$n\geq 1$" also converges at "$\sigma=1$". Then the solution of the given problem becomes,

$$\alpha=\alpha_0+\sum_{n=1}^{\infty}\alpha_n, \tag{31}$$

$$\beta=\beta_0+\sum_{n=1}^{\infty}\beta_n. \tag{32}$$

Now the fundamental theorem of Homotopy Analysis Method (HAM) provide us, that the deformation equation of nth-order for the given system of nonlinear PDEs becomes,

$$L[\alpha_n-\lambda_n\alpha_{n-1}]=\eta\mathrm{H}(x,t)\mathrm{E}_n^{\alpha}\left(\overrightarrow{\alpha_{n-1}}\right), \tag{33}$$

$$L[\beta_n-\lambda_n\beta_{n-1}]=\eta\mathrm{H}(x,t)\mathrm{E}_n^{\beta}\left(\overrightarrow{\beta_{n-1}}\right). \tag{34}$$

Applying the inverse operator that is, "$L^{-1}=\int_0^t(.)\,dt$", on equation (33)-(34), we obtain,

$$\alpha_n=\lambda_n\alpha_{n-1}+\int_0^t\eta\mathrm{H}(x,t)\mathrm{E}_n^{\alpha}\left(\overrightarrow{\alpha_{n-1}}\right)dt, \tag{35}$$

$$\beta_n=\lambda_n\beta_{n-1}+\int_0^t\eta\mathrm{H}(x,t)\mathrm{E}_n^{\beta}\left(\overrightarrow{\beta_{n-1}}\right)dt. \tag{36}$$

Starting with the initial approximation,

$$\alpha_0(x,t)=3-6\tanh^2\left(x/2\right), \tag{37}$$

$$\beta_0(x,t)=-3i\sqrt{2}\tanh^2\left(x/2\right). \tag{38}$$

Now substituting $"n=1"$ and also for simplicity using $"\mathrm{H}(x,t)=1"$ in equation (35)-(36), we get,

$$\alpha_1(x,t)=\begin{Bmatrix} 6t\eta\left(3-6\tanh^2\left(\frac{x}{2}\right)\right)\tanh\left(\frac{x}{2}\right)\sec h^2\left(\frac{x}{2}\right)-24t\eta\sec h^4\left(\frac{x}{2}\right) \\ \tanh\left(\frac{x}{2}\right)+24t\eta\tanh^3\left(\frac{x}{2}\right)\sec h^2\left(\frac{x}{2}\right) \end{Bmatrix}, \tag{39}$$

$$\beta_1(x,t)=\begin{Bmatrix} -9\sqrt{2}i\eta t\sec h^2\left(\frac{x}{2}\right)\tanh\left(\frac{x}{2}\right)-24\sqrt{2}i\eta t\sec h^4\left(\frac{x}{2}\right) \\ \tanh\left(\frac{x}{2}\right)-12\sqrt{2}i\eta t\sec h^2\left(\frac{x}{2}\right)\tanh^3\left(\frac{x}{2}\right) \end{Bmatrix}. \tag{40}$$

Similarly for obtaining $\alpha_2,\beta_2,\alpha_3,\beta_3,$ and so on, using $"n=2,3,4,..."$ in equation (35)-(36). Now to obtain the analytic approximate solution of the given system of KdV equation by Homotopy Analysis Method (HAM), since

$$\alpha(x,t)=\alpha_0(x,t)+\alpha_1(x,t)+\alpha_2(x,t)+... \tag{41}$$

$$\beta(x,t)=\beta_0(x,t)+\beta_1(x,t)+\beta_2(x,t)+... \tag{42}$$

Using the initial gauss, $"\alpha_1,\beta_1"$ and also the 2nd components that is $"\alpha_2,\beta_2"$ obtained through Maple package given in appendix, in equation (41)-(42), we will get the solution by Homotopy Analysis Method (HAM). After the derivation of the solution by Homotopy Analysis Method (HAM), if we use the non-zero auxiliary parameter that is, $"\eta=-1"$ in the obtained solution by Homotopy Analysis Method (HAM), it will give us the Homotopy Perturbation Method (HPM) solution, which shows that Homotopy Perturbation Method (HPM) is a specified case of Homotopy Analysis Method (HAM).

System-II

In this system, we consider two component evolutionary system of *KdV* equation of order three, such that,

$$\frac{\partial\alpha}{\partial t}-\frac{\partial^3\alpha}{\partial x^3}-2\beta\frac{\partial\alpha}{\partial x}-\alpha\frac{\partial\beta}{\partial x}=0, \tag{43}$$

$$\frac{\partial\beta}{\partial t}-\alpha\frac{\partial\alpha}{\partial x}=0, \tag{44}$$

initial condition,

$$\alpha_0=-\tanh\left(\frac{x}{(3)^{\frac{1}{2}}}\right), \tag{45}$$

$$\beta_0=-\frac{1}{6}-\frac{1}{2}\tanh^2\left(\frac{x}{(3)^{\frac{1}{2}}}\right). \tag{46}$$

The closed form solution given in Alquran and Muhammad (2011), that is

$$\alpha_{closed} = -\tanh\left(\frac{x-t}{(3)^{\frac{1}{2}}}\right),$$

$$\beta_{closed} = -\frac{1}{6} - \frac{1}{2}\tanh^2\left(\frac{x-t}{(3)^{\frac{1}{2}}}\right).$$

Homotopy Perturbation solution

The approximated analytic solution obtained by Homotopy Perturbation Method (HPM) is as,

$$\alpha_{HPM} = \left\{\begin{aligned} & -\tanh\left(\frac{x}{(3)^{\frac{1}{2}}}\right) + \frac{t}{3\sqrt{3}}\sec h^2\left(\frac{x}{(3)^{\frac{1}{2}}}\right) + \frac{t}{\sqrt{3}}\tanh^2\left(\frac{x}{(3)^{\frac{1}{2}}}\right)\sec h^2\left(\frac{x}{(3)^{\frac{1}{2}}}\right) \\ & -\frac{t}{3\sqrt{3}}\tanh^2\left(\frac{x}{(3)^{\frac{1}{2}}}\right)\sec h^2\left(\frac{x}{(3)^{\frac{1}{2}}}\right) + \frac{2t}{3\sqrt{3}}\sec h^4\left(\frac{x}{(3)^{\frac{1}{2}}}\right) + \ldots \end{aligned}\right\} \tag{47}$$

$$\beta_{HPM} = \left\{-\frac{1}{6} - \frac{1}{2}\tanh^2\left(\frac{x}{(3)^{\frac{1}{2}}}\right) + \frac{t}{\sqrt{3}}\tanh\left(\frac{x}{(3)^{\frac{1}{2}}}\right)\sec h^2\left(\frac{x}{(3)^{\frac{1}{2}}}\right) + \ldots\right\} \tag{48}$$

Solution by Homotopy Analysis Method

Now we want to find out the analytic approximated solution of the system-II, by Homotopy Analysis Method (HAM), and then to compare it with the the result obtained by Homotopy Perturbation Method (HPM), and also it tendency to the closed form solution, thus we have,

Since we need to define a linear operator , which is already define in Homotopy Perturbation Method (HPM), that is $"L = \frac{\partial}{\partial t}"$ with $"L^{-1} = \int_0^t (.)\,dt"$. Now according to Homotopy Analysis Method (HAM), the zero-order deformation equations becomes, such that,

$$(1-\sigma)L\left[\Psi_1(x,t;\sigma) - \alpha_0\right] = \sigma\eta \mathrm{H}(x,t)\mathrm{N}\Psi_1(x,t;\sigma), \tag{49}$$

$$(1-\sigma)L\left[\Psi_2(x,t;\sigma) - \beta_0\right] = \sigma\eta \mathrm{H}(x,t)\mathrm{N}\Psi_2(x,t;\sigma). \tag{50}$$

In equation (40)-(50), $"\alpha_0"$ and $"\beta_0"$ denote the initial approximation and define as

$$\alpha_0 = \alpha(x,0) = -\tanh\left(\frac{x}{(3)^{\frac{1}{2}}}\right), \tag{51}$$

$$\beta_0 = \beta(x,0) = -\frac{1}{6} - \frac{1}{2}\tanh^2\left(\frac{x}{(3)^{\frac{1}{2}}}\right). \tag{52}$$

Now using the deformation process, the zero order deformation equation becomes,

$$\Psi_1(x,t;1)=\alpha, \tag{53}$$

$$\Psi_2(x,t;1)=\beta, \tag{54}$$

$$\Psi_1(x,t;0)=\alpha_0, \tag{55}$$

$$\Psi_2(x,t;0)=\beta_0. \tag{56}$$

In equation (53)-(56), "α,β" and "α_0,β_0" are the exact solution and initial approximation of the system-ii respectively. Now assume that the solution can be expressed in a series of "σ", such that,

$$\Psi_1(x,t;\sigma)=\alpha_0+\sum_{n=1}^{\infty}\alpha_n\sigma^n, \tag{57}$$

$$\Psi_2(x,t;\sigma)=\beta_0+\sum_{n=1}^{\infty}\beta_n\sigma^n. \tag{58}$$

In equation (57)-(58),

"$\alpha_n=\dfrac{1}{n!}\dfrac{\partial^n\Psi_1(x,t;\sigma)}{\partial\sigma^n},\beta_n=\dfrac{1}{n!}\dfrac{\partial^n\Psi_2(x,t;\sigma)}{\partial\sigma^n}$" at "$\sigma=0$" and exist for "$n\geq 1$" also converges at "$\sigma=1$". Then the solution of the original problem becomes,

$$\alpha=\alpha_0+\sum_{n=1}^{\infty}\alpha_n, \tag{59}$$

$$\beta=\beta_0+\sum_{n=1}^{\infty}\beta_n. \tag{60}$$

Now to find the component of equation (59)-(60), that is $\alpha_1,\alpha_2,\ldots$ and $\beta_1,\beta_2,\ldots$, according to the fundamental theorem of Homotopy Analysis Method (HAM), the nth-order deformation equation for the given system of non-linear PDE,s becomes,

$$L[\alpha_n-\lambda_n\alpha_{n-1}]=\eta\mathrm{H}(x,t)\mathrm{E}_n^{\alpha}\left(\overrightarrow{\alpha_{n-1}}\right),n\geq 1, \tag{67}$$

$$L[\beta_n-\lambda_n\beta_{n-1}]=\eta\mathrm{H}(x,t)\mathrm{E}_n^{\beta}\left(\overrightarrow{\beta_{n-1}}\right),n\geq 1. \tag{68}$$

Now we starting with the initial approximation

$$\alpha_0=-\tanh\left(\frac{x}{(3)^{\frac{1}{2}}}\right), \tag{69}$$

$$\beta_0=-\frac{1}{6}-\frac{1}{2}\tanh^2\left(\frac{x}{(3)^{\frac{1}{2}}}\right). \tag{70}$$

Applying the inverse operator that is $"L^{-1}=\int_0^t(.)\,dt"$ on equation (67)-(68), we obtain,

$$\alpha_n=\lambda_n\alpha_{n-1}+\int_0^t\eta.\mathrm{H}(x,t).\mathrm{E}_n^{\alpha}\left(\overrightarrow{\alpha_{n-1}}\right)dt, \tag{71}$$

$$\beta_n=\lambda_n\beta_{n-1}+\int_0^t\eta.\mathrm{H}(x,t).\mathrm{E}_n^{\beta}\left(\overrightarrow{\beta_{n-1}}\right)dt, \tag{72}$$

Using $"n=1"$ and $"\mathrm{H}(x,t)=1"$ in equation (71)-(72), we have,

$$\alpha_1=\left\{\begin{array}{l}-\frac{2t}{3\sqrt{3}}\eta\sec h^4\left(\frac{x}{(3)^{\frac{1}{2}}}\right)+\frac{t}{3\sqrt{3}}\eta\tanh^2\left(\frac{x}{(3)^{\frac{1}{2}}}\right)\sec h^2\left(\frac{x}{(3)^{\frac{1}{2}}}\right)\\ -\frac{t}{3\sqrt{3}}\eta\sec h^2\left(\frac{x}{(3)^{\frac{1}{2}}}\right)-\frac{t}{\sqrt{3}}\eta\tan h^2\left(\frac{x}{(3)^{\frac{1}{2}}}\right)\sec h^2\left(\frac{x}{(3)^{\frac{1}{2}}}\right)\end{array}\right\}, \tag{73}$$

$$\beta_1=\left\{-\frac{t}{\sqrt{3}}\eta\tan h\left(\frac{x}{(3)^{\frac{1}{2}}}\right)\sec h^2\left(\frac{x}{(3)^{\frac{1}{2}}}\right)\right\}. \tag{74}$$

Similarly putting $n=2,3,4,\ldots$, in equation (71)-(72), and by using Maple Package we get $\alpha_2,\beta_2,\alpha_3,\beta_{3,}$ and so on. Now since

$$\alpha(x,t)=\alpha_0(x,t)+\alpha_1(x,t)+\alpha_2(x,t)+\ldots \tag{75}$$

$$\beta(x,t)=\beta_0(x,t)+\beta_1(x,t)+\beta_2(x,t)+\ldots \tag{76}$$

Hence using the initial gauss $"\alpha_1,\beta_1"$ and $"\alpha_2,\beta_2"$ obtained through Maple package, included in Appendix in equation (75)-(76), we will get the solution by Homotopy Analysis Method (HAM). Then by using the auxiliary parameter that is $"\eta=-1"$ in the obtained solution by Homotopy Analysis Method (HAM), then it become the solution derived by Homotopy Perturbation Method (HPM).

System-III

In system-iii, we consider a system of three nonlinear wave equation, which is also called the generalized KdV system of coupled Hirota Satsuma, such that,

$$\frac{\partial\alpha}{\partial t}-\frac{1}{2}\frac{\partial^3\alpha}{\partial x^3}+3\alpha\frac{\partial\alpha}{\partial x}-3\frac{\partial}{\partial x}(\beta\gamma)=0, \tag{77}$$

$$\frac{\partial\beta}{\partial t}+\frac{\partial^3\beta}{\partial x^3}-3\alpha\frac{\partial\beta}{\partial x}=0, \tag{78}$$

$$\frac{\partial\gamma}{\partial t}+\frac{\partial^3\gamma}{\partial x^3}-3\alpha\frac{\partial\gamma}{\partial x}=0, \tag{79}$$

subject to initial condition,

$$\alpha_0 = -\frac{1}{3} + 2\tanh^2(x), \tag{80}$$

$$\beta_0 = \tanh(x), \tag{81}$$

$$\gamma_0 = \frac{8}{3}\tanh(x). \tag{82}$$

The closed form solution to the system-III, were seen in Alquran and Muhammad (2011), which is given as

$$\alpha = -\frac{1}{3} + \sqrt{2}\tanh^2(x+t),$$

$$\beta = \tanh(x+t),$$

$$\gamma = \frac{8}{3}\tanh(x+t).$$

and obtained the solution of the above system by Homotopy Perturbation Method (HPM), given as,

Solution by Homotopy Perturbation Method (HPM)

$$\alpha_{HPM} = \left\{\begin{matrix} \frac{1}{3} + 2\tanh^2(x) + 20t\sec h^2(x)\tanh(x) - \\ 16t\sec h^2(x)\tanh^3(x) - 16t\sec h^2(x)\tanh(x) + ... \end{matrix}\right\}, \tag{83}$$

$$\beta_{HPM} = \left\{\begin{matrix} \tanh(x) + 2t\sec h^4(x) + 2t\sec h^4(x) \\ \tanh^4(x) - t\sec h^2(x) + ... \end{matrix}\right\}, \tag{84}$$

$$\gamma_{HPM} = \left\{\begin{matrix} \frac{8}{3}t\tanh(x) + \frac{16}{3}t\sec h^4(x) - \frac{8}{3}t\sec h^2(x) \\ -\frac{16}{3}t\sec h^2(x)\tanh(x) + ... \end{matrix}\right\}. \tag{85}$$

Solution by Homotopy Analysis Method (HAM)

To find the solution of the generalized KdV system of coupled Hirota Satsuma by Homotopy Analysis Method (HAM), first we need to define a linear operator, which is already define in Homotopy Perturbation Method (HPM), that is, $"L = \partial/\partial t"$ with the inverse define as, $"L^{-1} = \int_0^t (.)\,dt"$. Now using the definition of Homotopy Analysis Method (HAM), the zero-order deformation equation for the generalized KdV system of coupled Hirota Satsuma becomes,

$$(1-\sigma)L\left[\Psi_1(x,t;\sigma) - \alpha_0\right] = \sigma\eta \mathrm{H}(x,t)\mathrm{N}\Psi_1(x,t;\sigma), \tag{86}$$

$$(1-\sigma)L\left[\Psi_2(x,t;\sigma) - \beta_0\right] = \sigma\eta \mathrm{H}(x,t)\mathrm{N}\Psi_2(x,t;\sigma), \tag{87}$$

$$(1-\sigma)L\left[\Psi_3(x,t;\sigma) - \gamma_0\right] = \sigma\eta \mathrm{H}(x,t)\mathrm{N}\Psi_3(x,t;\sigma). \tag{88}$$

In equation (86)-(88), $"\alpha_0,"\beta_0"$ and $"\gamma_0"$ denote the initial approximation, define as,

$\alpha_0 = \alpha(x,0) = -\frac{1}{3} + 2\tanh^2(x),$

$\beta_0 = \beta(x,0) = \tanh(x),$

$\gamma_0 = \gamma(x,0) = \tanh(x).$

Using the deformation process, then equations (86)-(88), becomes

$$\Psi_1(x,t;0) = \alpha_0, \tag{89}$$

$$\Psi_2(x,t;0) = \beta_0, \tag{90}$$

$$\Psi_3(x,t;0) = \gamma_0, \tag{91}$$

$$\Psi_1(x,t;1) = \alpha, \tag{92}$$

$$\Psi_2(x,t;1) = \beta, \tag{93}$$

$$\Psi_3(x,t;1) = \gamma. \tag{94}$$

In equation (89)-(94), "$\alpha_0, \beta_0, \gamma_0$" and "$\alpha, \beta, \gamma$" are the initial approximation and exact solution of the given system respectively. Now assume the solution is of the form, such that,

$$\Psi_1(x,t;\sigma) = \alpha_0 + \sum_{n=1}^{\infty} \alpha_n \sigma^n, \tag{95}$$

$$\Psi_2(x,t;\sigma) = \beta_0 + \sum_{n=1}^{\infty} \beta_n \sigma^n, \tag{96}$$

$$\Psi_3(x,t;\sigma) = \gamma_0 + \sum_{n=1}^{\infty} \gamma_n \sigma^n. \tag{97}$$

In equation (95)-(97),

"$\alpha_n(x,t) = \frac{1}{n!}\frac{\partial^n \Psi_1(x,t;\sigma)}{\partial \sigma^n}, \beta_n(x,t) = \frac{1}{n!}\frac{\partial^n \Psi_2(x,t;\sigma)}{\partial \sigma^n}, \gamma_n(x,t) = \frac{1}{n!}\frac{\partial^n \Psi_3(x,t;\sigma)}{\partial \sigma^n}$"

at "$\sigma = 0$", and exist for "$n \geq 1$" also converges at "$\sigma = 1$". Then the solution of the original problem takes the form,

$$\alpha = \alpha_0 + \sum_{n=1}^{\infty} \alpha_n, \tag{98}$$

$$\beta = \beta_0 + \sum_{n=1}^{\infty} \beta_n, \tag{99}$$

$$\gamma = \gamma_0 + \sum_{n=1}^{\infty} \gamma_n. \tag{100}$$

Now according to the fundamental theorem of Homotopy Analysis Method (HAM), the higher order deformation equations for the generalized *KdV* system of coupled Hirota Satsuma becomes,

$$L\left[\alpha_n-\lambda_n\alpha_{n-1}\right]=\eta\mathrm{H}\left(x,t\right)\mathrm{E}_n^{\alpha}\left(\overrightarrow{\alpha_{n-1}}\right),n\geq 1,$$

$$L\left[\beta_n-\lambda_n\beta_{n-1}\right]=\eta\mathrm{H}\left(x,t\right)\mathrm{E}_n^{\beta}\left(\overrightarrow{\beta_{n-1}}\right),n\geq 1,$$

$$L\left[\gamma_n-\lambda_n\gamma_{n-1}\right]=\eta\mathrm{H}\left(x,t\right)\mathrm{E}_n^{\gamma}\left(\overrightarrow{\gamma_{n-1}}\right),n\geq 1.$$

Applying "L^{-1}" we get the following equation, such that,

$$\alpha_n=\lambda_n\alpha_{n-1}+\int_0^t\eta\mathrm{H}\left(x,t\right)\mathrm{E}_n^{\alpha}\left(\overrightarrow{\alpha_{n-1}}\right)dt, \tag{101}$$

$$\beta_n=\lambda_n\beta_{n-1}+\int_0^t\eta\mathrm{H}\left(x,t\right)\mathrm{E}_n^{\beta}\left(\overrightarrow{\beta_{n-1}}\right)dt, \tag{102}$$

$$\gamma_n=\lambda_n\gamma_{n-1}+\int_0^t\eta\mathrm{H}\left(x,t\right)\mathrm{E}_n^{\gamma}\left(\overrightarrow{\gamma_{n-1}}\right)dt. \tag{103}$$

Here we starting with the initial gauss,

$$\alpha_0=-1/3+2\tanh^2\left(x\right), \tag{104}$$

$$\beta_0=\tanh\left(x\right), \tag{105}$$

$$\gamma_0=\frac{8}{3}\tanh\left(x\right). \tag{106}$$

By using "$n=1$" and "$\mathrm{H}(x,t)=1$" in equation (101)-(103), we obtain,

$$\alpha_1=\begin{Bmatrix}16\eta t\sec h^4\left(x\right)\tanh\left(x\right)+16\eta t\sec h^2\left(x\right)\tanh^3\left(x\right)-20\eta t\\ \sec h^2\left(x\right)\tanh\left(x\right)\end{Bmatrix}, \tag{107}$$

$$\beta_1=\left\{\eta t\sec h^2\left(x\right)-2\eta t\sec h^2\left(x\right)\tanh^2\left(x\right)-2\eta t\sec h^4\left(x\right)\right\}, \tag{108}$$

$$\gamma_1=\left\{\frac{8}{3}\eta t\sec h^2\left(x\right)-\frac{16}{3}\eta t\sec h^2\left(x\right)\tanh^2\left(x\right)-\frac{16}{3}\eta t\sec h^4\left(x\right)\right\}, \tag{109}$$

Now since solution of the given system is given by,

$$\alpha\left(x,t\right)=\alpha_0(x,t)+\alpha_1(x,t)+\alpha_2(x,t)+...$$

$$\beta(x,t)=\beta_0(x,t)+\beta_1(x,t)+\beta_2(x,t)+...$$

$$\gamma(x,t)=\gamma_0(x,t)+\gamma_1(x,t)+\gamma_2(x,t)+...$$

Using the calculated components, that is initial gauss, "α_1,β_1" and "α_2,β_2", which is included in Appendix, we will get the solution of the system-III by Homotopy Analysis Method (HAM). Which will be the required obtain solution of generalized KdV system of coupled Hirota Satsuma equation by Homotopy Analysis Method (HAM).

5 Conclusion

The Homotopy Perturbation Method (HPM), and Homotopy Analysis Method (HAM) was successfully applied to linear and nonlinear problem and do not require any small or large parameter like perturbation methods and avoid the difficulties arising in the perturbation and non perturbation technique. Also the calculation is very simple and straight forward in these methods, but still Homotopy Perturbation Method (HPM) method is not a perfect tool for the solution of nonlinear problem. Homotopy Perturbation Method (HPM) give us a divergent result even for a linear problem some time. Thus it is clear that this method is also not a perfect tool.

But the method which have the ability to cover such types of deficiencies appears in the above mentioned technique is known as Homotopy Analysis Method (HAM), this method has successfully applied to all types of equations and give us an opportunity to apply for every type of problem. Also the effectiveness of this method is that, the series obtained by this method is more accurate than numerical solution in many cases.

This method provides us a convergence control parameter known auxiliary parameter. Also Homotopy Perturbation Method (HPM) is a special case of Homotopy Analysis Method (HAM). Thus from all the above it is clear that Homotopy Analysis Method (HAM) is valid in all cases and have a great potential for nonlinear problem.

Appendix

System-I

$$
\begin{aligned}
\alpha_2(x,t) = {} & 12\,t\,\eta \Big[3 - 6\tanh(\tfrac{1}{2}x)^2\Big]\tanh(\tfrac{1}{2}x)\operatorname{sech}(\tfrac{1}{2}x)^2 - 48\,t\,\eta\operatorname{sech}(\tfrac{1}{2}x)^4\tanh(\tfrac{1}{2}x) \\
& + 48\,t\,\eta\tanh(\tfrac{1}{2}x)^3\operatorname{sech}(\tfrac{1}{2}x)^2 + \frac{1}{3}\Big(-\Big(6\,\eta\Big[3 - 6\tanh(\tfrac{1}{2}x)^2\Big]\tanh(\tfrac{1}{2}x)\operatorname{sech}(\tfrac{1}{2} \\
& - 24\,\eta\operatorname{sech}(\tfrac{1}{2}x)^4\tanh(\tfrac{1}{2}x) + 24\,\eta\tanh(\tfrac{1}{2}x)^3\operatorname{sech}(\tfrac{1}{2}x)^2\Big)\Big)\Big(6\,\eta\Big[-12\tanh(\tfrac{1}{2}x) \\
& - \frac{1}{2}\tanh(\tfrac{1}{2}x)^2\Big)\Big]\tanh(\tfrac{1}{2}x)\operatorname{sech}(\tfrac{1}{2}x)^2 + 6\,\eta\Big[3 - 6\tanh(\tfrac{1}{2}x)^2\Big]\Big(\frac{1}{2} \\
& - \frac{1}{2}\tanh(\tfrac{1}{2}x)^2\Big)\operatorname{sech}(\tfrac{1}{2}x)^2 - 6\,\eta\Big[3 - 6\tanh(\tfrac{1}{2}x)^2\Big]\tanh(\tfrac{1}{2}x)^2\operatorname{sech}(\tfrac{1}{2}x)^2 \\
& + 48\,\eta\operatorname{sech}(\tfrac{1}{2}x)^4\tanh(\tfrac{1}{2}x)^2 - 24\,\eta\operatorname{sech}(\tfrac{1}{2}x)^4\Big(\frac{1}{2} - \frac{1}{2}\tanh(\tfrac{1}{2}x)^2\Big) \\
& + 72\,\eta\tanh(\tfrac{1}{2}x)^2\operatorname{sech}(\tfrac{1}{2}x)^2\Big(\frac{1}{2} - \frac{1}{2}\tanh(\tfrac{1}{2}x)^2\Big) - 24\,\eta\tanh(\tfrac{1}{2}x)^4\operatorname{sech}(\tfrac{1}{2}x) \\
& - \Big(-9\sqrt{2}\,i\,\eta\operatorname{sech}(\tfrac{1}{2}x)^2\tanh(\tfrac{1}{2}x) - 24\sqrt{2}\,i\,\eta\operatorname{sech}(\tfrac{1}{2}x)^4\tanh(\tfrac{1}{2}x) \\
& - 12\sqrt{2}\,i\,\eta\operatorname{sech}(\tfrac{1}{2}x)^2\tanh(\tfrac{1}{2}x)^3\Big)\Big(9\sqrt{2}\,i\,\eta\operatorname{sech}(\tfrac{1}{2}x)^2\tanh(\tfrac{1}{2}x)^2 \\
& - 9\sqrt{2}\,i\,\eta\operatorname{sech}(\tfrac{1}{2}x)^2\Big(\frac{1}{2} - \frac{1}{2}\tanh(\tfrac{1}{2}x)^2\Big) + 48\sqrt{2}\,i\,\eta\operatorname{sech}(\tfrac{1}{2}x)^4\tanh(\tfrac{1}{2}x)^2 \\
& - 24\sqrt{2}\,i\,\eta\operatorname{sech}(\tfrac{1}{2}x)^4\Big(\frac{1}{2} - \frac{1}{2}\tanh(\tfrac{1}{2}x)^2\Big) + 12\sqrt{2}\,i\,\eta\operatorname{sech}(\tfrac{1}{2}x)^2\tanh(\tfrac{1}{2}x)^4 \\
& - 36\sqrt{2}\,i\,\eta\operatorname{sech}(\tfrac{1}{2}x)^2\tanh(\tfrac{1}{2}x)^2\Big(\frac{1}{2} - \frac{1}{2}\tanh(\tfrac{1}{2}x)^2\Big)\Big)\Big)\Big)\,t^3 + \frac{1}{2} \\
& \Big(24\,\eta\tanh(\tfrac{1}{2}x)^6\operatorname{sech}(\tfrac{1}{2}x)^2 - 168\,\eta\operatorname{sech}(\tfrac{1}{2}x)^4\Big(\frac{1}{2} - \frac{1}{2}\tanh(\tfrac{1}{2}x)^2\Big)^2 \\
& - 192\,\eta\operatorname{sech}(\tfrac{1}{2}x)^4\tanh(\tfrac{1}{2}x)^4 - 144\,\eta\Big(\frac{1}{2} - \frac{1}{2}\tanh(\tfrac{1}{2}x)^2\Big)^3\operatorname{sech}(\tfrac{1}{2}x)^2 + 18\,\eta\Big[\\
& -12\Big(\frac{1}{2} - \frac{1}{2}\tanh(\tfrac{1}{2}x)^2\Big)^2 + 12\tanh(\tfrac{1}{2}x)^2\Big(\frac{1}{2} - \frac{1}{2}\tanh(\tfrac{1}{2}x)^2\Big)\Big] \\
& \tanh(\tfrac{1}{2}x)^2\operatorname{sech}(\tfrac{1}{2}x)^2 - 18\,\eta\Big[-12\tanh(\tfrac{1}{2}x)\Big(\frac{1}{2} \\
& - \frac{1}{2}\tanh(\tfrac{1}{2}x)^2\Big)\Big]\tanh(\tfrac{1}{2}x)^3\operatorname{sech}(\tfrac{1}{2}x)^2 + 6\,\eta\Big[3 \\
& - 6\tanh(\tfrac{1}{2}x)^2\Big]\tanh(\tfrac{1}{2}x)^4\operatorname{sech}(\tfrac{1}{2}x)^2 - 18\,\eta\Big[-12\Big(\frac{1}{2} - \frac{1}{2}\tanh(\tfrac{1}{2}x)^2\Big)^2 \\
& + 12\tanh(\tfrac{1}{2}x)^2\Big(\frac{1}{2} - \frac{1}{2}\tanh(\tfrac{1}{2}x)^2\Big)\Big]\Big(\frac{1}{2} - \frac{1}{2}\tanh(\tfrac{1}{2}x)^2\Big)\operatorname{sech}(\tfrac{1}{2}x)^2 \\
& - 600\,\eta\tanh(\tfrac{1}{2}x)^4\operatorname{sech}(\tfrac{1}{2}x)^2\Big(\frac{1}{2} - \frac{1}{2}\tanh(\tfrac{1}{2}x)^2\Big) + 72\,\eta\Big[-12\tanh(\tfrac{1}{2}x)\Big(\frac{1}{2} \\
& - \frac{1}{2}\tanh(\tfrac{1}{2}x)^2\Big)\Big]\tanh(\tfrac{1}{2}x)\Big(\frac{1}{2} - \frac{1}{2}\tanh(\tfrac{1}{2}x)^2\Big)\operatorname{sech}(\tfrac{1}{2}x)^2 - 66\,\eta\Big[3
\end{aligned}
$$

$$
\begin{aligned}
&-6\tanh\Big(\frac{1}{2}x\Big)^2\Big]\tanh\Big(\frac{1}{2}x\Big)^2\Big(\frac{1}{2}-\frac{1}{2}\tanh\Big(\frac{1}{2}x\Big)^2\Big)\operatorname{sech}\Big(\frac{1}{2}x\Big)^2\\
&+792\,\eta\operatorname{sech}\Big(\frac{1}{2}x\Big)^4\tanh\Big(\frac{1}{2}x\Big)^2\Big(\frac{1}{2}-\frac{1}{2}\tanh\Big(\frac{1}{2}x\Big)^2\Big)+24\,\eta\Big[3\\
&-6\tanh\Big(\frac{1}{2}x\Big)^2\Big]\Big(\frac{1}{2}-\frac{1}{2}\tanh\Big(\frac{1}{2}x\Big)^2\Big)^2\operatorname{sech}\Big(\frac{1}{2}x\Big)^2\\
&+1152\,\eta\tanh\Big(\frac{1}{2}x\Big)^2\operatorname{sech}\Big(\frac{1}{2}x\Big)^2\Big(\frac{1}{2}-\frac{1}{2}\tanh\Big(\frac{1}{2}x\Big)^2\Big)^2-6\,\eta\Big[48\Big(\frac{1}{2}\\
&-\frac{1}{2}\tanh\Big(\frac{1}{2}x\Big)^2\Big)^2\tanh\Big(\frac{1}{2}x\Big)-12\tanh\Big(\frac{1}{2}x\Big)^3\Big(\frac{1}{2}-\frac{1}{2}\tanh\Big(\frac{1}{2}x\Big)^2\Big)\Big]\\
&\tanh\Big(\frac{1}{2}x\Big)\operatorname{sech}\Big(\frac{1}{2}x\Big)^2+12\Big(3-6\tanh\Big(\frac{1}{2}x\Big)^2\Big)\tanh\Big(\frac{1}{2}x\Big)\Big(\frac{1}{2}-\frac{1}{2}\tanh\Big(\frac{1}{2}x\Big)^2\Big)t\\
&-36\,i^2\tanh\Big(\frac{1}{2}x\Big)^3\Big(\frac{1}{2}-\frac{1}{2}\tanh\Big(\frac{1}{2}x\Big)^2\Big)t.
\end{aligned}
$$

$$
\begin{aligned}
&\beta_2(x,t)-18\sqrt{2}\,i\,\eta\,t\operatorname{sech}\Big(\frac{1}{2}x\Big)^2\tanh\Big(\frac{1}{2}x\Big)-48\sqrt{2}\,i\,\eta\,t\operatorname{sech}\Big(\frac{1}{2}x\Big)^4\tanh\Big(\frac{1}{2}x\Big)\\
&\quad-24\sqrt{2}\,i\,\eta\,t\operatorname{sech}\Big(\frac{1}{2}x\Big)^2\tanh\Big(\frac{1}{2}x\Big)^3+\frac{1}{3}\Big(6\,\eta\Big[3\\
&\quad-6\tanh\Big(\frac{1}{2}x\Big)^2\Big]\tanh\Big(\frac{1}{2}x\Big)\operatorname{sech}\Big(\frac{1}{2}x\Big)^2-24\,\eta\operatorname{sech}\Big(\frac{1}{2}x\Big)^4\tanh\Big(\frac{1}{2}x\Big)\\
&\quad+24\,\eta\tanh\Big(\frac{1}{2}x\Big)^3\operatorname{sech}\Big(\frac{1}{2}x\Big)^2\Big)\Big(9\sqrt{2}\,i\,\eta\operatorname{sech}\Big(\frac{1}{2}x\Big)^2\tanh\Big(\frac{1}{2}x\Big)^2\\
&\quad-9\sqrt{2}\,i\,\eta\operatorname{sech}\Big(\frac{1}{2}x\Big)^2\Big(\frac{1}{2}-\frac{1}{2}\tanh\Big(\frac{1}{2}x\Big)^2\Big)+48\sqrt{2}\,i\,\eta\operatorname{sech}\Big(\frac{1}{2}x\Big)^4\tanh\Big(\frac{1}{2}x\Big)^2\\
&\quad-24\sqrt{2}\,i\,\eta\operatorname{sech}\Big(\frac{1}{2}x\Big)^4\Big(\frac{1}{2}-\frac{1}{2}\tanh\Big(\frac{1}{2}x\Big)^2\Big)+12\sqrt{2}\,i\,\eta\operatorname{sech}\Big(\frac{1}{2}x\Big)^2\tanh\Big(\frac{1}{2}x\Big)^4\\
&\quad-36\sqrt{2}\,i\,\eta\operatorname{sech}\Big(\frac{1}{2}x\Big)^2\tanh\Big(\frac{1}{2}x\Big)^2\Big(\frac{1}{2}-\frac{1}{2}\tanh\Big(\frac{1}{2}x\Big)^2\Big)\Big)t^3\\
&\quad+\frac{1}{2}\Big(18\sqrt{2}\,i\,\eta\operatorname{sech}\Big(\frac{1}{2}x\Big)^2\tanh\Big(\frac{1}{2}x\Big)^4-198\sqrt{2}\,i\,\eta\operatorname{sech}\Big(\frac{1}{2}x\Big)^2\tanh\Big(\frac{1}{2}x\Big)^2\Big(\frac{1}{2}\\
&\quad-\frac{1}{2}\tanh\Big(\frac{1}{2}x\Big)^2\Big)+72\sqrt{2}\,i\,\eta\operatorname{sech}\Big(\frac{1}{2}x\Big)^2\Big(\frac{1}{2}-\frac{1}{2}\tanh\Big(\frac{1}{2}x\Big)^2\Big)^2\\
&\quad-1584\sqrt{2}\,i\,\eta\operatorname{sech}\Big(\frac{1}{2}x\Big)^4\tanh\Big(\frac{1}{2}x\Big)^2\Big(\frac{1}{2}-\frac{1}{2}\tanh\Big(\frac{1}{2}x\Big)^2\Big)\\
&\quad+336\sqrt{2}\,i\,\eta\operatorname{sech}\Big(\frac{1}{2}x\Big)^4\Big(\frac{1}{2}-\frac{1}{2}\tanh\Big(\frac{1}{2}x\Big)^2\Big)^2+24\sqrt{2}\,i\,\eta\operatorname{sech}\Big(\frac{1}{2}x\Big)^2\tanh\Big(\frac{1}{2}x\Big)^6\\
&\quad+384\sqrt{2}\,i\,\eta\operatorname{sech}\Big(\frac{1}{2}x\Big)^4\tanh\Big(\frac{1}{2}x\Big)^4+1152\sqrt{2}\,i\,\eta\operatorname{sech}\Big(\frac{1}{2}x\Big)^2\tanh\Big(\frac{1}{2}x\Big)^2\Big(\frac{1}{2}\\
&\quad-\frac{1}{2}\tanh\Big(\frac{1}{2}x\Big)^2\Big)^2-144\sqrt{2}\,i\,\eta\operatorname{sech}\Big(\frac{1}{2}x\Big)^2\Big(\frac{1}{2}-\frac{1}{2}\tanh\Big(\frac{1}{2}x\Big)^2\Big)^3\\
&\quad-600\sqrt{2}\,i\,\eta\operatorname{sech}\Big(\frac{1}{2}x\Big)^2\tanh\Big(\frac{1}{2}x\Big)^4\Big(\frac{1}{2}-\frac{1}{2}\tanh\Big(\frac{1}{2}x\Big)^2\Big)\Big)t^2-6\Big(3\\
&\quad-6\tanh\Big(\frac{1}{2}x\Big)^2\Big)i\sqrt{2}\tanh\Big(\frac{1}{2}x\Big)\Big(\frac{1}{2}-\frac{1}{2}\tanh\Big(\frac{1}{2}x\Big)^2\Big)t.
\end{aligned}
$$

System-II

$$
\begin{aligned}
\alpha_2(x,t) = &-\frac{4}{9}\, t\,\eta\, \operatorname{sech}\left(\frac{1}{3}x\sqrt{3}\right)^4 \sqrt{3} - \frac{4}{9}\sqrt{3}\, t\,\eta\, \operatorname{sech}\left(\frac{1}{3}x\sqrt{3}\right)^2 \tanh\left(\frac{1}{3}x\sqrt{3}\right)^2 \\
&- \frac{2}{9}\sqrt{3}\, t\,\eta\, \operatorname{sech}\left(\frac{1}{3}x\sqrt{3}\right)^2 \\
&+ \frac{1}{3}\Bigg(\frac{2}{3}\sqrt{3}\,\eta\, \operatorname{sech}\left(\frac{1}{3}x\sqrt{3}\right)^2 \tanh\left(\frac{1}{3}x\sqrt{3}\right)\Bigg(\frac{8}{9}\eta\, \operatorname{sech}\left(\frac{1}{3}x\sqrt{3}\right)^4 \tanh\left(\frac{1}{3}x\sqrt{3}\right) \\
&+ \frac{4}{9}\eta\, \operatorname{sech}\left(\frac{1}{3}x\sqrt{3}\right)^2 \tanh\left(\frac{1}{3}x\sqrt{3}\right)^3 - \frac{4}{9}\eta\, \operatorname{sech}\left(\frac{1}{3}x\sqrt{3}\right)^2 \tanh\left(\frac{1}{3}x\sqrt{3}\right)\Bigg(1 \\
&- \tanh\left(\frac{1}{3}x\sqrt{3}\right)^2\Bigg) + \frac{2}{9}\eta\, \operatorname{sech}\left(\frac{1}{3}x\sqrt{3}\right)^2 \tanh\left(\frac{1}{3}x\sqrt{3}\right)\Bigg) - \Bigg(\\
&-\frac{2}{9}\eta\, \operatorname{sech}\left(\frac{1}{3}x\sqrt{3}\right)^4 \sqrt{3} - \frac{2}{9}\sqrt{3}\,\eta\, \operatorname{sech}\left(\frac{1}{3}x\sqrt{3}\right)^2 \tanh\left(\frac{1}{3}x\sqrt{3}\right)^2 \\
&- \frac{1}{9}\sqrt{3}\,\eta\, \operatorname{sech}\left(\frac{1}{3}x\sqrt{3}\right)^2\Bigg)\Bigg(\frac{2}{3}\eta\, \operatorname{sech}\left(\frac{1}{3}x\sqrt{3}\right)^2 \tanh\left(\frac{1}{3}x\sqrt{3}\right)^2 \\
&- \frac{1}{3}\eta\, \operatorname{sech}\left(\frac{1}{3}x\sqrt{3}\right)^2 \left(1 - \tanh\left(\frac{1}{3}x\sqrt{3}\right)^2\right)\Bigg)\Bigg)\, t^3 \\
&+ \frac{1}{2}\Bigg(\frac{112}{27}\eta\, \operatorname{sech}\left(\frac{1}{3}x\sqrt{3}\right)^4 \tanh\left(\frac{1}{3}x\sqrt{3}\right)\left(1 - \tanh\left(\frac{1}{3}x\sqrt{3}\right)^2\right) \\
&+ \frac{16}{3}\eta\, \operatorname{sech}\left(\frac{1}{3}x\sqrt{3}\right)^2 \tanh\left(\frac{1}{3}x\sqrt{3}\right)^3 \left(1 - \tanh\left(\frac{1}{3}x\sqrt{3}\right)^2\right) \\
&- \frac{128}{27}\eta\, \operatorname{sech}\left(\frac{1}{3}x\sqrt{3}\right)^4 \tanh\left(\frac{1}{3}x\sqrt{3}\right)^3 - \frac{8}{27}\eta\, \operatorname{sech}\left(\frac{1}{3}x\sqrt{3}\right)^2 \tanh\left(\frac{1}{3}x\sqrt{3}\right)^3 \\
&- \frac{16}{27}\eta\, \operatorname{sech}\left(\frac{1}{3}x\sqrt{3}\right)^2 \tanh\left(\frac{1}{3}x\sqrt{3}\right)^5 - \frac{80}{27}\eta\, \operatorname{sech}\left(\frac{1}{3}x\sqrt{3}\right)^2 \tanh\left(\frac{1}{3}x\sqrt{3}\right)\Bigg(1 \\
&- \tanh\left(\frac{1}{3}x\sqrt{3}\right)^2\Bigg)^2 + \frac{16}{27}\eta\, \operatorname{sech}\left(\frac{1}{3}x\sqrt{3}\right)^2 \tanh\left(\frac{1}{3}x\sqrt{3}\right)\left(1 - \tanh\left(\frac{1}{3}x\sqrt{3}\right)^2\right)\Bigg)\, t^2 \\
&+ \frac{2}{3}\left(-\frac{1}{6} - \frac{1}{2}\tanh\left(\frac{1}{3}x\sqrt{3}\right)^2\right)\left(1 - \tanh\left(\frac{1}{3}x\sqrt{3}\right)^2\right)\sqrt{3}\, t - \frac{1}{3}\tanh\left(\frac{1}{3}x\sqrt{3}\right)^2 \Bigg(1 \\
&- \tanh\left(\frac{1}{3}x\sqrt{3}\right)^2\Bigg)\sqrt{3}\, t.
\end{aligned}
$$

$$
\begin{aligned}
\beta_2(x,t) = &-\frac{2}{3}\sqrt{3}\, t\,\eta\, \operatorname{sech}\left(\frac{1}{3}x\sqrt{3}\right)^2 \tanh\left(\frac{1}{3}x\sqrt{3}\right) - \frac{1}{3}\Bigg(-\frac{2}{9}\eta\, \operatorname{sech}\left(\frac{1}{3}x\sqrt{3}\right)^4 \sqrt{3} \\
&- \frac{2}{9}\sqrt{3}\,\eta\, \operatorname{sech}\left(\frac{1}{3}x\sqrt{3}\right)^2 \tanh\left(\frac{1}{3}x\sqrt{3}\right)^2 - \frac{1}{9}\sqrt{3}\,\eta\, \operatorname{sech}\left(\frac{1}{3}x\sqrt{3}\right)^2\Bigg) \\
&\Bigg(\frac{8}{9}\eta\, \operatorname{sech}\left(\frac{1}{3}x\sqrt{3}\right)^4 \tanh\left(\frac{1}{3}x\sqrt{3}\right) + \frac{4}{9}\eta\, \operatorname{sech}\left(\frac{1}{3}x\sqrt{3}\right)^2 \tanh\left(\frac{1}{3}x\sqrt{3}\right)^3 \\
&- \frac{4}{9}\eta\, \operatorname{sech}\left(\frac{1}{3}x\sqrt{3}\right)^2 \tanh\left(\frac{1}{3}x\sqrt{3}\right)\left(1 - \tanh\left(\frac{1}{3}x\sqrt{3}\right)^2\right) \\
&+ \frac{2}{9}\eta\, \operatorname{sech}\left(\frac{1}{3}x\sqrt{3}\right)^2 \tanh\left(\frac{1}{3}x\sqrt{3}\right)\Bigg)\, t^3 - \frac{1}{3}\tanh\left(\frac{1}{3}x\sqrt{3}\right)\Bigg(1 \\
&- \tanh\left(\frac{1}{3}x\sqrt{3}\right)^2\Bigg)\sqrt{3}\, t.
\end{aligned}
$$

System-III

$$\alpha_2(x,t) = 32\,\eta\,t\,\mathrm{sech}(x)^4\tanh(x) + 32\,\eta\,t\,\mathrm{sech}(x)^2\tanh(x)^3 - 40\,\eta\,t\,\mathrm{sech}(x)^2\tanh(x) + \frac{1}{3}\Big(-3\,\big(-2\,\eta\,\mathrm{sech}(x)^2\tanh(x) + 4\,\eta\,\mathrm{sech}(x)^2\tanh(x)^3 - 4\,\eta\,\mathrm{sech}(x)^2\tanh(x)\,(1-\tanh(x)^2) + 8\,\eta\,\mathrm{sech}(x)^4\tanh(x)\big)\left(\frac{8}{3}\,\eta\,\mathrm{sech}(x)^2 - \frac{16}{3}\,\eta\,\mathrm{sech}(x)^2\tanh(x)^2 - \frac{16}{3}\,\eta\,\mathrm{sech}(x)^4\right) - 3\,\big(\eta\,\mathrm{sech}(x)^2 - 2\,\eta\,\mathrm{sech}(x)^2\tanh(x)^2 - 2\,\eta\,\mathrm{sech}(x)^4\big)\left(-\frac{16}{3}\,\eta\,\mathrm{sech}(x)^2\tanh(x) + \frac{32}{3}\,\eta\,\mathrm{sech}(x)^2\tanh(x)^3 - \frac{32}{3}\,\eta\,\mathrm{sech}(x)^2\tanh(x)\,(1-\tanh(x)^2) + \frac{64}{3}\,\eta\,\mathrm{sech}(x)^4\tanh(x)\right) + 3\,\big(16\,\eta\,\mathrm{sech}(x)^4\tanh(x) + 16\,\eta\,\mathrm{sech}(x)^2\tanh(x)^3 - 20\,\eta\,\mathrm{sech}(x)^2\tanh(x)\big)\,\big(-64\,\eta\,\mathrm{sech}(x)^4\tanh(x)^2 + 16\,\eta\,\mathrm{sech}(x)^4\,(1-\tanh(x)^2) - 32\,\eta\,\mathrm{sech}(x)^2\tanh(x)^4 + 48\,\eta\,\mathrm{sech}(x)^2\tanh(x)^2\,(1-\tanh(x)^2) + 40\,\eta\,\mathrm{sech}(x)^2\tanh(x)^2 - 20\,\eta\,\mathrm{sech}(x)^2\,(1-\tanh(x)^2)\big)\Big)\,t^3 + \frac{1}{2}\Big(512\,\eta\,\mathrm{sech}(x)^4\tanh(x)^4 + 112\,\eta\,\mathrm{sech}(x)^4\,(1-\tanh(x)^2)^2 - 800\,\eta\,\mathrm{sech}(x)^2\tanh(x)^4\,(1-\tanh(x)^2) + 768\,\eta\,\mathrm{sech}(x)^2\tanh(x)^2\,(1-\tanh(x)^2)^2 - 48\,\eta\,\mathrm{sech}(x)^2\,(1-\tanh(x)^2)^3 + 64\,\eta\,\mathrm{sech}(x)^2\tanh(x)^6 + 440\,\eta\,\mathrm{sech}(x)^2\tanh(x)^2\,(1-\tanh(x)^2) - 80\,\eta\,\mathrm{sech}(x)^2\,(1-\tanh(x)^2)^2 - 1056\,\eta\,\mathrm{sech}(x)^4\tanh(x)^2\,(1-\tanh(x)^2) - 80\,\eta\,\mathrm{sech}(x)^2\tanh(x)^4\Big)\,t^2 - 16\tanh(x)\,(1-\tanh(x)^2)\,t + 12\left(-\frac{1}{3} + 2\tanh(x)^2\right)\tanh(x)\,(1-\tanh(x)^2)\,t.$$

$$\beta_2(x,t) = 2\,\eta\,t\,\mathrm{sech}(x)^2 - 4\,\eta\,t\,\mathrm{sech}(x)^2\tanh(x)^2 - 4\,\eta\,t\,\mathrm{sech}(x)^4 - \big(16\,\eta\,\mathrm{sech}(x)^4\tanh(x) + 16\,\eta\,\mathrm{sech}(x)^2\tanh(x)^3 - 20\,\eta\,\mathrm{sech}(x)^2\tanh(x)\big)\,\big(-2\,\eta\,\mathrm{sech}(x)^2\tanh(x) + 4\,\eta\,\mathrm{sech}(x)^2\tanh(x)^3 - 4\,\eta\,\mathrm{sech}(x)^2\tanh(x)\,(1-\tanh(x)^2) + 8\,\eta\,\mathrm{sech}(x)^4\tanh(x)\big)\,t^3 + \frac{1}{2}\Big(16\,\eta\,\mathrm{sech}(x)^2\tanh(x)\,(1-\tanh(x)^2) + 16\,\eta\,\mathrm{sech}(x)^2\tanh(x)^5 - 144\,\eta\,\mathrm{sech}(x)^2\tanh(x)^3\,(1-\tanh(x)^2) - 8\,\eta\,\mathrm{sech}(x)^2\tanh(x)^3 + 128\,\eta\,\mathrm{sech}(x)^4\tanh(x)^3 - 112\,\eta\,\mathrm{sech}(x)^4\tanh(x)\,(1-\tanh(x)^2) + 80\,\eta\,\mathrm{sech}(x)^2\tanh(x)\,(1-\tanh(x)^2)^2\Big)\,t^2 - 3\left(-\frac{1}{3} + 2\tanh(x)^2\right)(1-\tanh(x)^2)\,t.$$

$$
\begin{aligned}
\gamma_2(x,t) = {} & \frac{16}{3}\,\eta\, t\operatorname{sech}(x)^2 - \frac{32}{3}\,\eta\, t\operatorname{sech}(x)^2\tanh(x)^2 - \frac{32}{3}\,\eta\, t\operatorname{sech}(x)^4 - \Big(16\,\eta\operatorname{sech}(x)^4\tanh(x) \\
& + 16\,\eta\operatorname{sech}(x)^2\tanh(x)^3 - 20\,\eta\operatorname{sech}(x)^2\tanh(x)\Big)\Big(-\frac{16}{3}\,\eta\operatorname{sech}(x)^2\tanh(x) \\
& + \frac{32}{3}\,\eta\operatorname{sech}(x)^2\tanh(x)^3 - \frac{32}{3}\,\eta\operatorname{sech}(x)^2\tanh(x)\left(1-\tanh(x)^2\right) + \frac{64}{3}\,\eta\operatorname{sech}(x)^4\tanh(x)\Big) \\
& t^3 + \frac{1}{2}\Big(\frac{128}{3}\,\eta\operatorname{sech}(x)^2\tanh(x)\left(1-\tanh(x)^2\right) + \frac{128}{3}\,\eta\operatorname{sech}(x)^2\tanh(x)^5 \\
& - 384\,\eta\operatorname{sech}(x)^2\tanh(x)^3\left(1-\tanh(x)^2\right) - \frac{64}{3}\,\eta\operatorname{sech}(x)^2\tanh(x)^3 \\
& + \frac{1024}{3}\,\eta\operatorname{sech}(x)^4\tanh(x)^3 - \frac{896}{3}\,\eta\operatorname{sech}(x)^4\tanh(x)\left(1-\tanh(x)^2\right) \\
& + \frac{640}{3}\,\eta\operatorname{sech}(x)^2\tanh(x)\left(1-\tanh(x)^2\right)^2\Big)\, t^2 - 3\left(-\frac{1}{3} + 2\tanh(x)^2\right)\left(\frac{8}{3} - \frac{8}{3}\tanh(x)^2\right) t.
\end{aligned}
$$

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