Article

Global behavior of an anti-competitive system of fourth-order rational difference equations

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Abstract

In the present work, we study the qualitative behavior of an anti-competitive system of fourth-order rational difference equations. More precisely, we study the local asymptotic stability, global character of the unique equilibrium point, and the rate of convergence of the positive solutions of the given system. Some numerical examples are given to verify our theoretical results.

Keywords system of rational difference equations; stability; global character; rate of convergence.

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1 Introduction

Difference equations or discrete dynamical systems are diverse field which impact almost every branch of pure and applied mathematics. Every dynamical system $a_{n+1} = f(a_n)$ determines a difference equation and vice versa. Recently, there has been great interest in studying difference equations systems. One of the reasons for this is a necessity for some techniques which can be used in investigating equations arising in mathematical models describing real life situations in many applied sciences. The theory of discrete dynamical systems and difference equations developed greatly during the last twenty-five years of the twentieth century. Applications of discrete dynamical systems and difference equations have appeared recently in many areas. The theory of difference equations occupies a central position in applicable analysis. There is no doubt that the theory of difference equations will continue to play an important role in mathematics as a whole. Nonlinear difference equations of order greater than one are of paramount importance in applications. Such equations also appear naturally as discrete analogues and as numerical solutions of differential and delay differential equations which model various diverse phenomena in biology, ecology, physiology, physics, engineering, economics, probability theory, genetics, psychology and resource management. It is very interesting to investigate the behavior of solutions of a system of higher-order rational difference equations and to discuss the local asymptotic stability of their equilibrium points. Systems of rational difference equations have been studied by several authors. Especially there has been a great interest in the study of the attractivity of the solutions of such systems. For more results for the systems of difference equations, we refer the interested reader to Cinar (2004), Stevic (2012a, b), Bajo and Liz (2011), Kalabusic et al. (2009, 2011), Kurbanli (2011), Kurbanli et al. (2011), Touafek and Elsayed (2012a, b), Elsayed and Ibrahim (in press), Din (a, b; in press).

Zhang et al. (2012) studied the dynamics of a system of rational third-order difference equation:

$$x_{n+1}=\frac{x_{n-2}}{B+y_ny_{n-1}y_{n-2}}\ ,y_{n+1}=\frac{y_{n-2}}{B+x_nx_{n-1}x_{n-2}},n=0,1,\cdots.$$

Din et al. (2012) investigated the dynamics of a system of fourth-order rational difference equations

$$x_{n+1}=\frac{\alpha x_{n-3}}{\beta+\gamma\prod_{i=0}^{3}y_{n-i}}\ ,y_{n+1}=\frac{\alpha_1 y_{n-3}}{\beta_1+\gamma_1\prod_{i=0}^{3}x_{n-i}},n=0,1,\cdots,$$

Our aim in this paper is to investigate the qualitative behavior of an anti-competitive system of fourth order rational difference equations

$$x_{n+1}=\frac{\alpha y_{n-3}}{\beta+\gamma\prod_{i=0}^{3}x_{n-i}}\ ,y_{n+1}=\frac{\alpha_1 x_{n-3}}{\beta_1+\gamma_1\prod_{i=0}^{3}y_{n-i}},n=0,1,\cdots, \quad (1)$$

where the parameters $\alpha,\beta,\gamma,\alpha_1,\beta_1,\gamma_1$ and initial conditions $x_0,x_{-1},x_{-2},x_{-3},y_0,y_{-1},y_{-2},y_{-3}$ are positive real numbers. This paper is natural extension of (Shojaei et al., 2009; Din et al., 2012; Zhang et al., 2012).

Let us consider eight-dimensional discrete dynamical system of the form:

$$x_{n+1}=f(x_n,x_{n-1},x_{n-2},x_{n-3},y_n,y_{n-1},y_{n-2},y_{n-3}),$$
$$y_{n+1}=g(x_n,x_{n-1},x_{n-2},x_{n-3},y_n,y_{n-1},y_{n-2},y_{n-3}),n=0,1,\cdots, \quad (2)$$

where $f:I^4\times J^4\to I$ and $g:I^4\times J^4\to J$ are continuously differentiable functions and I,J are some intervals of real numbers. Furthermore, a solution $\{(x_n,y_n)\}_{n=-3}^{\infty}$ of system (2) is uniquely determined by initial conditions $(x_i,y_i)\in I\times J$ for $i\in\{-3,-2,-1,0\}$. Along with the system (2) we consider the corresponding vector map

$$F=(f,x_n,x_{n-1},x_{n-2},x_{n-3},g,y_n,y_{n-1},y_{n-2},x_{n-3}).$$

An equilibrium point of system (2) is a point $(\bar{x},\bar{y})$ that satisfies

$$\bar{x}=f(\bar{x},\bar{x},\bar{x},\bar{x},\bar{y},\bar{y},\bar{y},\bar{y})$$
$$\bar{y}=g(\bar{x},\bar{x},\bar{x},\bar{x},\bar{y},\bar{y},\bar{y},\bar{y})$$

The point $(\bar{x},\bar{y})$ is also called a fixed point of the vector map F.

Definition 1. Let $(\bar{x}\ ,\bar{y})$ be an equilibrium point of the system (2).

(i) An equilibrium point $(\bar{x}\ ,\bar{y})$ is said to be stable if for every $\varepsilon>0$ there exists $\delta>0$ such thatfor every initial conditions $(x_i,y_i),i\in\{-3,-2,-1,0\}$ if $\left\|\sum_{i=-3}^{0}(x_i,y_i)-(\bar{x},\bar{y})\right\|<\delta$ implies $\|(x_n,y_n)-(\bar{x},\bar{y})\|<\varepsilon$ for all $n>0$, where $\|\cdot\|$ is usual Euclidian norm in $\mathbb{R}^2$.

(ii) An equilibrium point $(\bar{x}\ ,\bar{y})$ is said to be unstable if it is not stable.

(iii) An equilibrium point $(\bar{x}\ ,\bar{y})$ is said to be asymptotically stable if there exists $\eta>0$ such that $\left\|\sum_{i=-3}^{0}(x_i,y_i)-(\bar{x},\bar{y})\right\|<\eta$ and$(x_n,y_n)\to(\bar{x}\ ,\bar{y})$as $n\to\infty$.

(iv) An equilibrium point $(\bar{x}\ ,\bar{y})$ is called global attractor if $(x_n,y_n)\to(\bar{x}\ ,\bar{y})$ as $n\to\infty$.

(v) An equilibrium point $(\bar{x}\ ,\bar{y})$ is called asymptotic global attractor if it is a global attractor andstable.

Definition 2. Let$(\bar{x}\ ,\bar{y})$ be an equilibrium point of a map

$$F=(f,x_n,x_{n-1},x_{n-2},x_{n-3},g,y_n,y_{n-1},y_{n-2},x_{n-3})$$

where f and $g$are continuously differentiable functions at$(\bar{x}\ ,\bar{y})$. The linearized system of (2) about the equilibrium point $(\bar{x}, \bar{y})$ is given by

$$X_{n+1} = F(X_n) = F_J X_n,$$

where $X_n = \begin{pmatrix} x_n \\ x_{n-1} \\ x_{n-2} \\ x_{n-3} \\ y_n \\ y_{n-1} \\ y_{n-2} \\ y_{n-3} \end{pmatrix}$ and F_J is Jacobean matrix of the system (2) about the equilibrium point $(\bar{x}, \bar{y})$.

To construct corresponding linearized form of the system (1) we consider the following transformation:

$$(x_n, x_{n-1}, x_{n-2}, x_{n-3}, y_n, y_{n-1}, y_{n-2}, y_{n-3}) \to (f, f_1, f_2, f_3, g, g_1, g_2, g_3), \quad (3)$$

where $f = \frac{\alpha y_{n-3}}{\beta+\gamma \prod_{i=0}^{3} x_{n-i}}$, $f_1 = x_n$, $f_2 = x_{n-1}$, $f_3 = x_{n-2}$, $g = \frac{\alpha_1 x_{n-3}}{\beta_1+\gamma_1 \prod_{i=0}^{3} y_{n-i}}$, $g_1 = y_n$, $g_2 = y_{n-1}$, $g_3 = y_{n-2}$The Jacobian matrix about the fixed point $(\bar{x}, \bar{y})$ under the transformation (3) is given by

$$F_J(\bar{x}, \bar{y}) = \begin{pmatrix} A & A & A & A & 0 & 0 & 0 & B \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & C & D & D & D & D \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

where $A = -\frac{\alpha\gamma\bar{y}\bar{x}^3}{(\beta+\gamma\bar{x}^4)^2}$, $B = \frac{\alpha}{\beta+\gamma\bar{x}^4}$, $C = \frac{\alpha_1}{\beta_1+\gamma_1\bar{y}^4}$ and $D = -\frac{\alpha_1\gamma_1\bar{x}\bar{y}^3}{(\beta_1+\gamma_1\bar{y}^4)^2}$.

Theorem 1. (Sedaghat, 2003) For the system $X_{n+1} = F(X_n)$, $n = 0,1,\cdots$, of difference equations such that $\bar{X}$ be a fixed point of F. If all eigenvalues of the Jacobian matrix J_F about $\bar{X}$ lie inside the open unitdisk $|\lambda| < 1$, then$\bar{X}$ is locally asymptotically stable. If one of them has a modulus greater than one, then $\bar{X}$ is unstable.

2 Main Results

Let $(\bar{x}, \bar{y})$be an equilibrium point of the system (1), then system (1) has only one equilibrium point namely $(0,0)$.

Theorem 2. Let $\{x_n, y_n\}$ be a positive solution of the system (1), then for every $m \geq 0$ the following result hold:

$$(i)\ 0 \le x_n \le \begin{cases} \left(\frac{\alpha}{\beta}\right)^{m+1} \left(\frac{\alpha_1}{\beta_1}\right)^{m} y_{-3}, & n = 8m+1, \\ \left(\frac{\alpha}{\beta}\right)^{m+1} \left(\frac{\alpha_1}{\beta_1}\right)^{m} y_{-2}, & n = 8m+2, \\ \left(\frac{\alpha}{\beta}\right)^{m+1} \left(\frac{\alpha_1}{\beta_1}\right)^{m} y_{-1}, & n = 8m+3, \\ \left(\frac{\alpha}{\beta}\right)^{m+1} \left(\frac{\alpha_1}{\beta_1}\right)^{m} y_{0}, & n = 8m+4, \\ \left(\frac{\alpha\alpha_1}{\beta\beta_1}\right)^{m+1} x_{-3}, & n = 8m+5, \\ \left(\frac{\alpha\alpha_1}{\beta\beta_1}\right)^{m+1} x_{-2}, & n = 8m+6, \\ \left(\frac{\alpha\alpha_1}{\beta\beta_1}\right)^{m+1} x_{-1}, & n = 8m+7, \\ \left(\frac{\alpha\alpha_1}{\beta\beta_1}\right)^{m+1} x_{0}, & n = 8m+8. \end{cases}$$

$$(ii)\ 0 \le y_n \le \begin{cases} \left(\frac{\alpha}{\beta}\right)^{m} \left(\frac{\alpha_1}{\beta_1}\right)^{m+1} x_{-3}, & n = 8m+1, \\ \left(\frac{\alpha}{\beta}\right)^{m} \left(\frac{\alpha_1}{\beta_1}\right)^{m+1} x_{-2}, & n = 8m+2, \\ \left(\frac{\alpha}{\beta}\right)^{m} \left(\frac{\alpha_1}{\beta_1}\right)^{m+1} x_{-1}, & n = 8m+3, \\ \left(\frac{\alpha}{\beta}\right)^{m} \left(\frac{\alpha_1}{\beta_1}\right)^{m+1} x_{0}, & n = 8m+4, \\ \left(\frac{\alpha\alpha_1}{\beta\beta_1}\right)^{m+1} y_{-3}, & n = 8m+5, \\ \left(\frac{\alpha\alpha_1}{\beta\beta_1}\right)^{m+1} y_{-2}, & n = 8m+6, \\ \left(\frac{\alpha\alpha_1}{\beta\beta_1}\right)^{m+1} y_{-1}, & n = 8m+7, \\ \left(\frac{\alpha\alpha_1}{\beta\beta_1}\right)^{m+1} y_{0}, & n = 8m+8. \end{cases}$$

Proof. It follows from induction.■

Lemma 1.Let $0 \le \frac{\alpha\alpha_1}{\beta\beta_1} < 1$, then every solution $\{x_n, y_n\}_{n=-3}^{\infty}$ of the system (1) is bounded.

Proof. Assume that

$$\lambda_1 = \max\left\{ \frac{\beta_1}{\alpha_1} y_{-3}, \frac{\beta_1}{\alpha_1} y_{-2}, \frac{\beta_1}{\alpha_1} y_{-1}, \frac{\beta_1}{\alpha_1} y_{0}, x_{-3}, x_{-2}, x_{-1}, x_0 \right\},$$

and

$$\lambda_2 = \max\left\{ \frac{\beta}{\alpha} x_{-3}, \frac{\beta}{\alpha} x_{-2}, \frac{\beta}{\alpha} x_{-1}, \frac{\beta}{\alpha} x_{0}, y_{-3}, y_{-2}, y_{-1}, y_0 \right\}.$$

Then from Theorem 2 one can see that $0 \le x_n < \lambda_1$ and $0 \le y_n < \lambda_2$ for all $n = 0,1,\cdots$.■

Theorem 3. If $0 \le \frac{\alpha\alpha_1}{\beta\beta_1} < 1$ then equilibrium point (0, 0) of the system (1) is locally asymptotically stable.

Proof. The linearized system of (1) about the equilibrium point (0, 0) is given by:

$X_{n+1} = F_J(0,0)X_n$,

where$X_n = \begin{pmatrix} x_n \\ x_{n-1} \\ x_{n-2} \\ x_{n-3} \\ y_n \\ y_{n-1} \\ y_{n-2} \\ y_{n-3} \end{pmatrix}$ and $F_J(0,0) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\alpha}{\beta} \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\alpha_1}{\beta_1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$.

The characteristic polynomial of $F_J(0,0)$ is given by

$$P(\lambda) = \lambda^8 - \frac{\alpha\alpha_1}{\beta\beta_1} \tag{4}$$

The roots of $P(\lambda)$

$$\lambda_k = \left(\frac{\alpha\alpha_1}{\beta\beta_1}\right)^{\frac{1}{8}} \exp\left(\frac{k\pi i}{4}\right)$$

for $k = 0,1,\cdots,7$. Now, it is easy to see that $|\lambda_K| < 1$ for all $k = 0,1,\cdots,7$. Since all eigenvalues of Jacobian matrix $F_J(0,0)$ about (0, 0) lie in open unit disk $|\lambda| < 1$. Hence, the equilibrium point (0, 0) is locally asymptotically stable.■

Theorem 4. If $0 \le \frac{\alpha\alpha_1}{\beta\beta_1} < 1$ then equilibrium point (0, 0) of the system (1) is globally asymptotically stable.

Proof. From theorem 3, (0, 0) is locally asymptotically stable. From Lemma 1, every positive solution $\{x_n, y_n\}_{n=-3}^{\infty}$of the system (1) is bounded. Now, it is sufficient to prove that $\{x_n, y_n\}$is decreasing. From system (1) one has

$$x_{n+1} = \frac{\alpha y_{n-3}}{\beta + \gamma \prod_{i=0}^{3} x_{n-i}}$$

$$x_{n+1} \le \frac{\alpha}{\beta} y_{n-3} < y_{n-3}.$$

This implies that $x_{8n+1} < y_{8n-3}$ and $x_{8n+9} < y_{8n+5}$. Also

$$y_{n+1} = \frac{\alpha_1 x_{n-3}}{\beta_1 + \gamma_1 \prod_{i=0}^{3} y_{n-i}}$$

$$y_{n+1} \le \frac{\alpha_1}{\beta_1} x_{n-3} < x_{n-3}.$$

This implies that $y_{8n+1} < x_{8n-3}$ and $y_{8n+9} < x_{8n+5}$. So $x_{8n+9} < y_{8n+5} < x_{8n+1}$ and $y_{8n+9} < x_{8n+5} < y_{8n+1}$. Hence, the subsequences

$\{x_{8n+1}\}, \{x_{8n+2}\}, \{x_{8n+3}\}, \{x_{8n+4}\}, \{x_{8n+5}\}, \{x_{8n+6}\}, \{x_{8n+7}\}, \{x_{8n+8}\}$

and

$\{y_{8n+1}\}, \{y_{8n+2}\}, \{y_{8n+3}\}, \{y_{8n+4}\}, \{y_{8n+5}\}, \{y_{8n+6}\}, \{y_{8n+7}\}, \{y_{8n+8}\}$

are decreasing. Therefore the sequences $\{x_n\}$ and $\{y_n\}$ are decreasing. Hence

$\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = 0.$■

Theorem 5. Let $\alpha > \beta$ and $\alpha_1 > \beta_1$. Then, for solution (x_n, y_n) of system (1) following statements are true:

(i) If $x_n \longrightarrow 0$, then $y_n \longrightarrow \infty$.

(ii) If $y_n \longrightarrow 0$, then $x_n \longrightarrow \infty$.

3 Rate of Convergence

In this section we will determine the rate of convergence of a solution that converges to the equilibrium point (0, 0) of the system (1). The following results give the rate of convergence of solutions of a system of difference equations

$$X_{n+1} = \big(A + B(n)\big)X_n, \tag{5}$$

where X_n is an $m-$dimensional vector, $A \in C^{m\times m}$ is a constant matrix, and $B: Z^+ \to C^{m\times m}$ is a matrix function satisfying

$$\|B(n)\| \to 0 \tag{6}$$

as $n \to \infty$, where $\|.\|$ denotes any matrix norm which is associated with the vector norm

$$\|B(n)\| = \sqrt{x^2 + y^2}$$

Proposition 1. (Perron's theorem) (Pituk, 2002) Suppose that condition (6) holds. If X_n is a solution of (5), then either $X_n = 0$ for all large n or

$$\rho = \lim_{n\to\infty} (\|X_n\|)^{\frac{1}{n}} \tag{7}$$

exist and is equal to the modulus of one the eigenvalues of matrix A.

Proposition 2. (Pituk, 2002)Suppose that condition (6) holds. If X_n is a solution of (5), then either $X_n = 0$ for all large n or

$$\rho = \lim_{n\to\infty} \frac{\|X_{n+1}\|}{\|X_n\|} \tag{8}$$

exist and is equal to the modulus of one the eigenvalues of matrix A

Assume that $\lim\limits_{n\to\infty} x_n = \bar{x}$, $\lim\limits_{n\to\infty} y_n = \bar{y}$. First we will find a system of limiting equations for the map F. The error term are given by

$$x_{n+1} - \bar{x} = \sum_{i=0}^{3} A_i(x_{n-i} - \bar{x}) + \sum_{i=0}^{3} B_i(y_{n-i} - \bar{y}),$$
$$y_{n+1} - \bar{y} = \sum_{i=0}^{3} C_i(x_{n-i} - \bar{x}) + \sum_{i=0}^{3} D_i(y_{n-i} - \bar{y}).$$

Set $e_n^1 = x_n - \bar{x}$ and$e_n^2 = y_n - \bar{y}$, one has

$$e_{n+1}^1 = \sum_{i=0}^{3} A_i\, e_{n-i}^1 + \sum_{i=0}^{3} B_i\, e_{n-i}^2,$$
$$e_{n+1}^2 = \sum_{i=0}^{3} C_i\, e_{n-i}^1 + \sum_{i=0}^{3} D_i\, e_{n-i}^2,$$

where

$$A_0 = -\frac{\alpha\gamma\, \bar{y} \prod_{i=1}^{3} x_{n-i}}{(\beta+\gamma \prod_{i=0}^{3} x_{n-i})(\beta+\gamma\bar{x}^4)}\ ,\ A_1 = -\frac{\alpha\gamma\bar{x}\bar{y}x_{n-2}x_{n-3}}{(\beta+\gamma \prod_{i=0}^{3} x_{n-i})(\beta+\gamma\bar{x}^4)},$$

$$A_2 = -\frac{\alpha\gamma\bar{x}^2\bar{y}x_{n-3}}{(\beta+\gamma \prod_{i=0}^{3} x_{n-i})(\beta+\gamma\bar{x}^4)}\ ,\ A_3 = -\frac{\alpha\gamma\bar{x}^3\bar{y}}{(\beta+\gamma \prod_{i=0}^{3} x_{n-i})(\beta+\gamma\bar{x}^4)},$$

$B_i = 0$for $i \in \{0,1,2\}$,

$$B_3 = \frac{\alpha}{\beta+\gamma \prod_{i=0}^{3} x_{n-i}},$$

$C_i = 0$ for $i \in \{0,1,2\}$,

$C_3 = \frac{\alpha_1}{\beta_1+\gamma_1 \prod_{i=0}^{3} y_{n-i}}$,

$D_0 = -\frac{\alpha_1\gamma_1\bar{x}\prod_{i=1}^{3} y_{n-i}}{(\beta_1+\gamma_1 \prod_{i=0}^{3} y_{n-i})(\beta_1+\gamma_1\bar{y}^4)}$, $D_1 = -\frac{\alpha_1\gamma_1\bar{x}\,\bar{y}y_{n-2}y_{n-3}}{(\beta_1+\gamma_1 \prod_{i=0}^{3} y_{n-i})(\beta_1+\gamma_1\bar{y}^4)}$,

$D_2 = -\frac{\alpha_1\gamma_1\bar{x}\,\bar{y}^2 y_{n-3}}{(\beta_1+\gamma_1 \prod_{i=0}^{3} y_{n-i})(\beta_1+\gamma_1\bar{y}^4)}$,$D_3 = -\frac{\alpha_1\gamma_1\bar{x}\,\bar{y}^3}{(\beta_1+\gamma_1 \prod_{i=0}^{3} y_{n-i})(\beta_1+\gamma_1\bar{y}^4)}$.

Taking the limit, we obtain $\lim_{n\to\infty} A_i == -\frac{\alpha\gamma\bar{y}\bar{x}^3}{(\beta+\gamma\bar{x}^4)^2}$ for $i \in \{0,1,2,3\}$, $\lim_{n\to\infty} B_i = 0$ for $i \in \{0,1,2\}$, $\lim_{n\to\infty} B_3 = \frac{\alpha}{\beta+\gamma\bar{x}^4}$, $\lim_{n\to\infty} C_i = 0$ for $i \in \{0,1,2\}$, $\lim_{n\to\infty} C_3 = \frac{\alpha_1}{\beta_1+\gamma_1\bar{y}^4}$ and $\lim_{n\to\infty} D_i = -\frac{\alpha_1\gamma_1\bar{x}\bar{y}^3}{(\beta_1+\gamma_1\bar{y}^4)^2}$for $i \in \{0,1,2,3\}$. So, the limiting system of error terms can be written as

$E_{n+1} = KE_n$, where

$$E_n = \begin{pmatrix} e_n^1 \\ e_{n-1}^1 \\ e_{n-2}^1 \\ e_{n-3}^1 \\ e_n^2 \\ e_{n-1}^2 \\ e_{n-2}^2 \\ e_{n-3}^2 \end{pmatrix} \text{ and K=} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\alpha}{\beta} \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\alpha_1}{\beta_1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

which is similar to linearized system of (1) about the equilibrium point $(\bar{x}, \bar{y}) = (0,0)$. Using the Proposition 1, one has following result.

Theorem 6. Assume that $\{(x_n, y_n)\}$ be a positive solution of the system (1) such that $\lim_{n\to\infty} x_n = \bar{x}$ and $\lim_{n\to\infty} y_n = \bar{y}$ where $(\bar{x}, \bar{y}) = (0,0)$. Then, the error term E_n of every solution of (1) satisfies both of the following asymptotic relations

$\lim_{n\to\infty} (\|e_n\|)^{\frac{1}{n}} = |\lambda F_J(\bar{x}, \bar{y})|$, $\lim_{n\to\infty} \frac{\|e_{n+1}\|}{\|e_n\|} = |\lambda F_J(\bar{x}, \bar{y})|$,

where $\lambda F_J(\bar{x}, \bar{y})$ are the characteristic roots of Jacobian matrix $F_J(\bar{x}, \bar{y})$about$(0,0)$.

4 Examples

In order to verify our theoretical results and to support our theoretical discussions, we consider several interesting numerical examples in this section. These examples represent different types of qualitative behavior of solutions to the system of nonlinear difference equations (1). All plots in this section are drawn with Mathematica.

Example 1

Consider the system (1) with initial conditions

$x_{-3} = 2.2$, $x_{-2} = 1.9$, $x_{-1} = 5.8, x_0, = 2.9$, $y_{-3} = 1.8$, $y_{-2} = 3.9$, $y_{-1} = 2.4, y_0 = 1.8$.

Moreover, choosing the parameters $\alpha = 116, \beta = 117, \gamma = 0.9, \alpha_1 = 111, \beta_1 = 112, \gamma_1 = 0.6$. Then , the system (1) can be written as

$$x_{n+1} = \frac{116 y_{n-3}}{117+0.9 \prod_{i=0}^{3} x_{n-i}}, y_{n+1} = \frac{111 x_{n-3}}{112+0.6 \prod_{i=0}^{3} y_{n-i}}, n = 0,1,\cdots, \quad (9)$$

and with initial condition

$x_{-3} = 2.2, x_{-2} = 1.9, x_{-1} = 5.8, x_0, = 2.9, \ y_{-3} = 1.8, \ y_{-2} = 3.9, y_{-1} = 2.4, y_0 = 1.8$. Moreover, in Fig. 1 the plot of x_n is shown in Fig. 1a, the plot of y_n is shown in Fig. 1b and an attractor of the system (9) is shown in Fig. 1c.

Example 2

Consider the system (1) with initial conditions

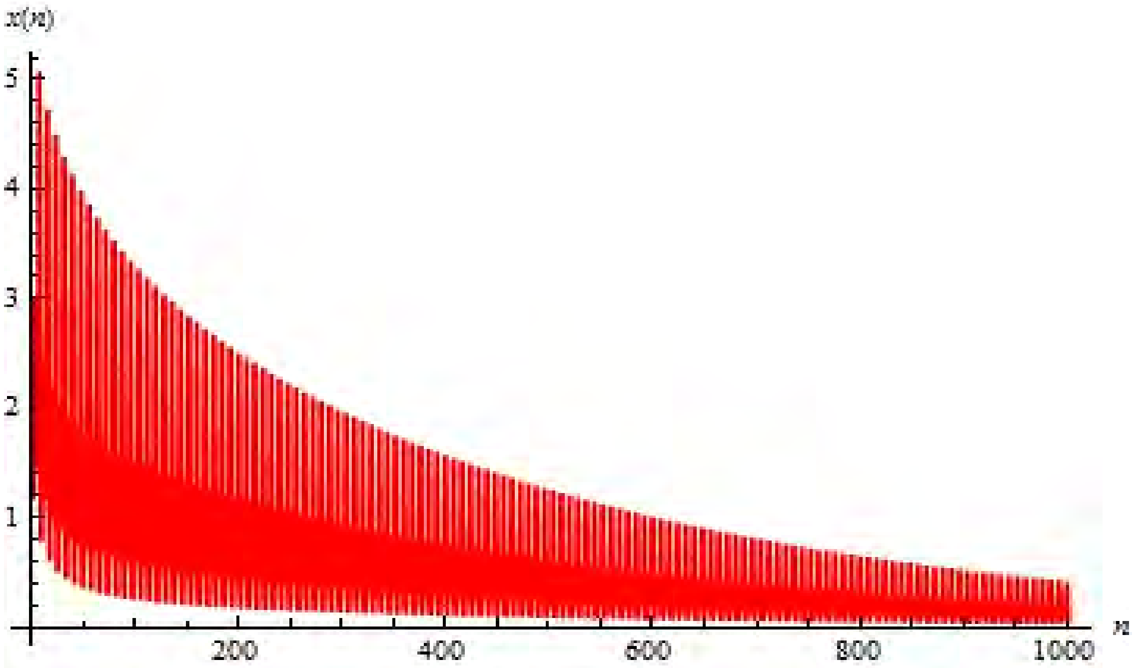
$x_{-3} = 0.3, x_{-2} = 1.8, x_{-1} = 1.9, x_0, = 3.2, \ y_{-3} = 1.1, \ y_{-2} = 1.9, y_{-1} = 0.1, y_0 = 1.8$. Moreover, choosing the parameters

$\alpha = 1.1, \beta = 1.12, \gamma = 0.001, \alpha_1 = 0.9, \beta_1 = 0.91, \gamma_1 = 0.007$.Then , the system (1) can be written as

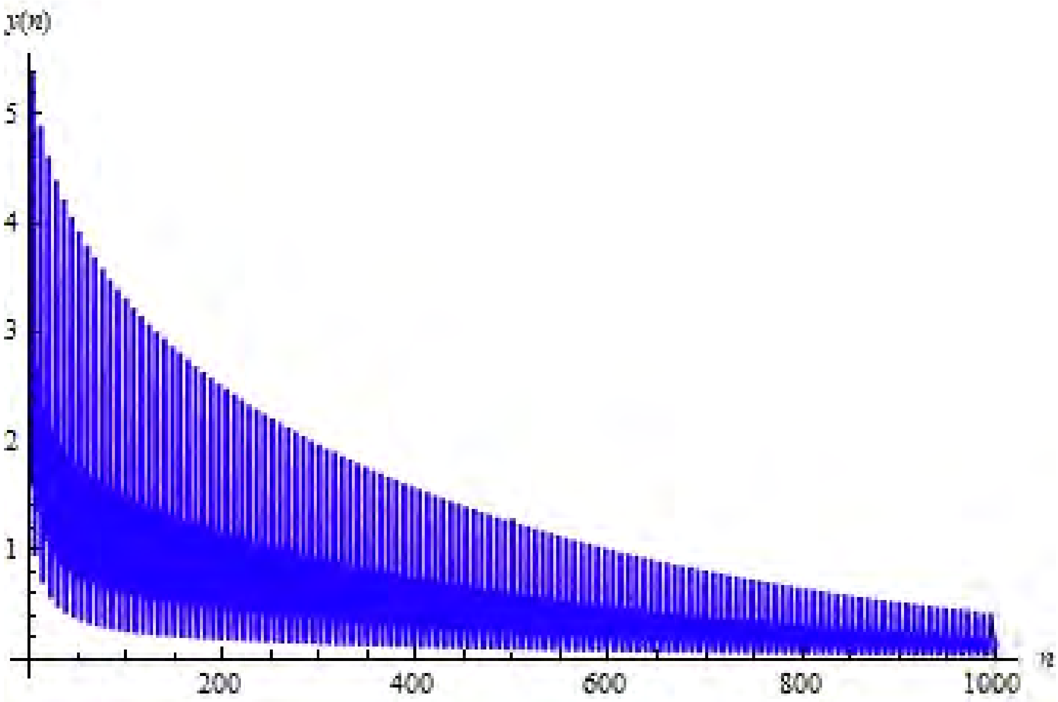
$$x_{n+1} = \frac{1.1 y_{n-3}}{1.12+0.001 \prod_{i=0}^{3} x_{n-i}}, y_{n+1} = \frac{0.9 x_{n-3}}{0.91+0.007 \prod_{i=0}^{3} y_{n-i}}, n = 0,1,\cdots, \qquad (9)$$

and with initial condition

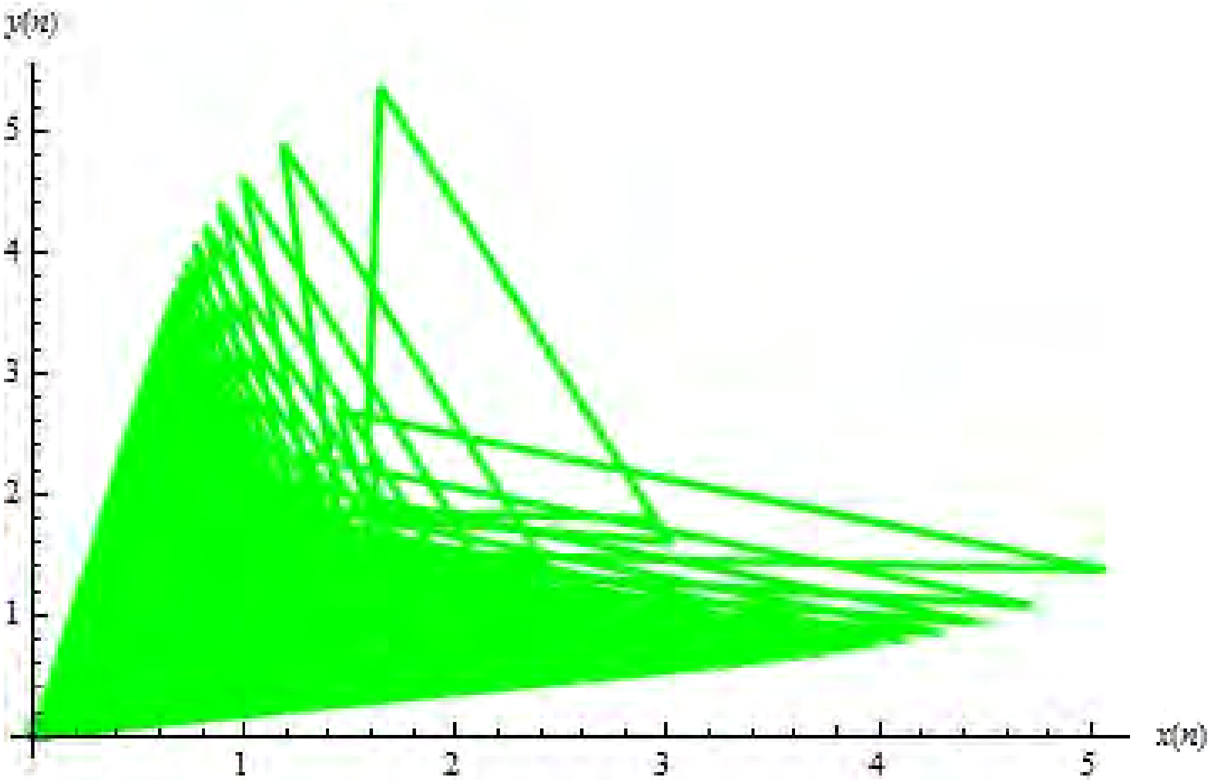
$x_{-3} = 0.3, x_{-2} = 1.8, x_{-1} = 1.9, x_0, = 3.2, \ y_{-3} = 1.1, \ y_{-2} = 1.9, y_{-1} = 0.1, y_0 = 1.8$. Moreover, in Fig. 2 the plot of x_n is shown in Fig. 2a, the plot of y_n is shown in Fig. 2b and an attractor of the system (9) is shown in Fig. 2c.



(a) Plot of x_n for the system (9)

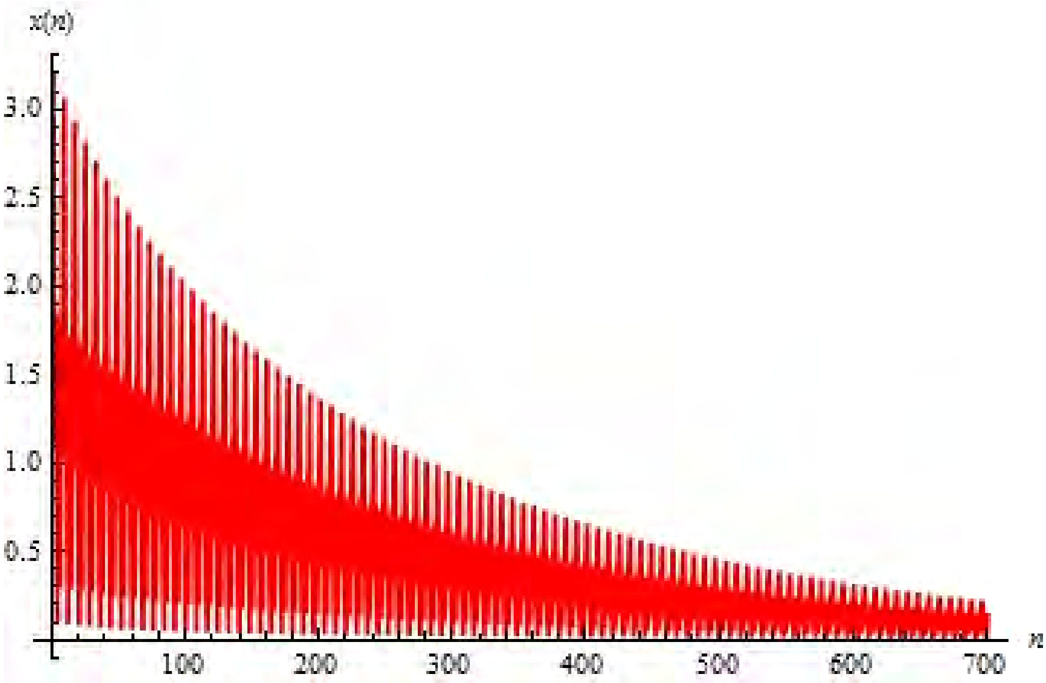


(b) Plot of y_n for the system (9)

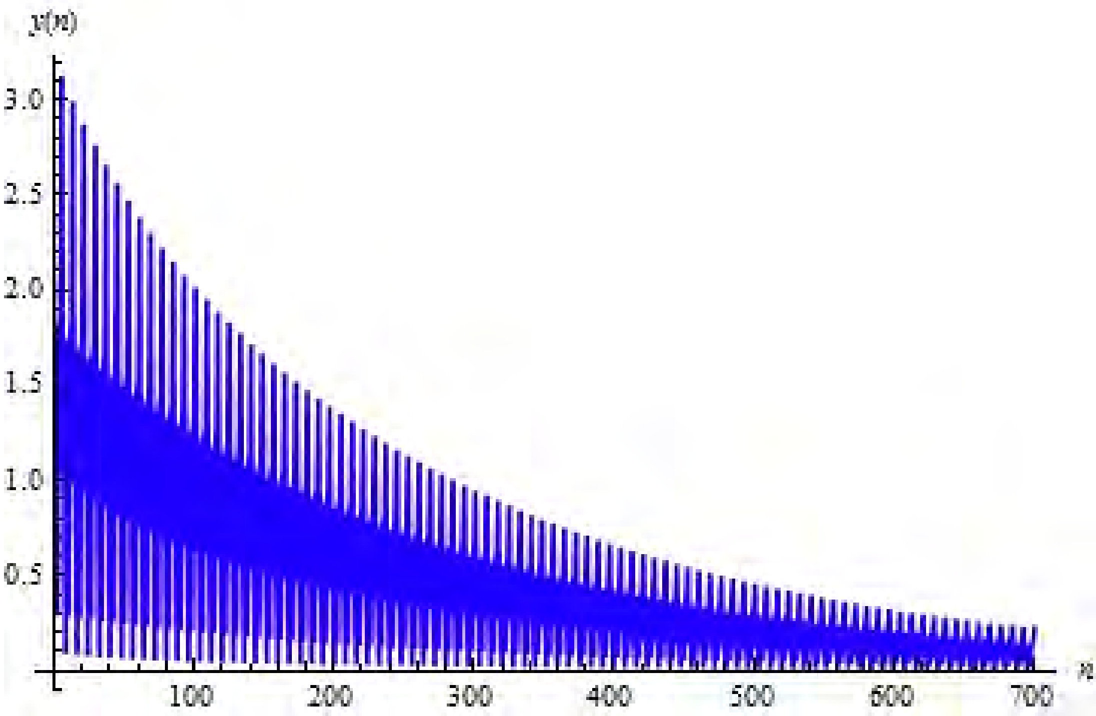


(c) An attractor for the system (9)

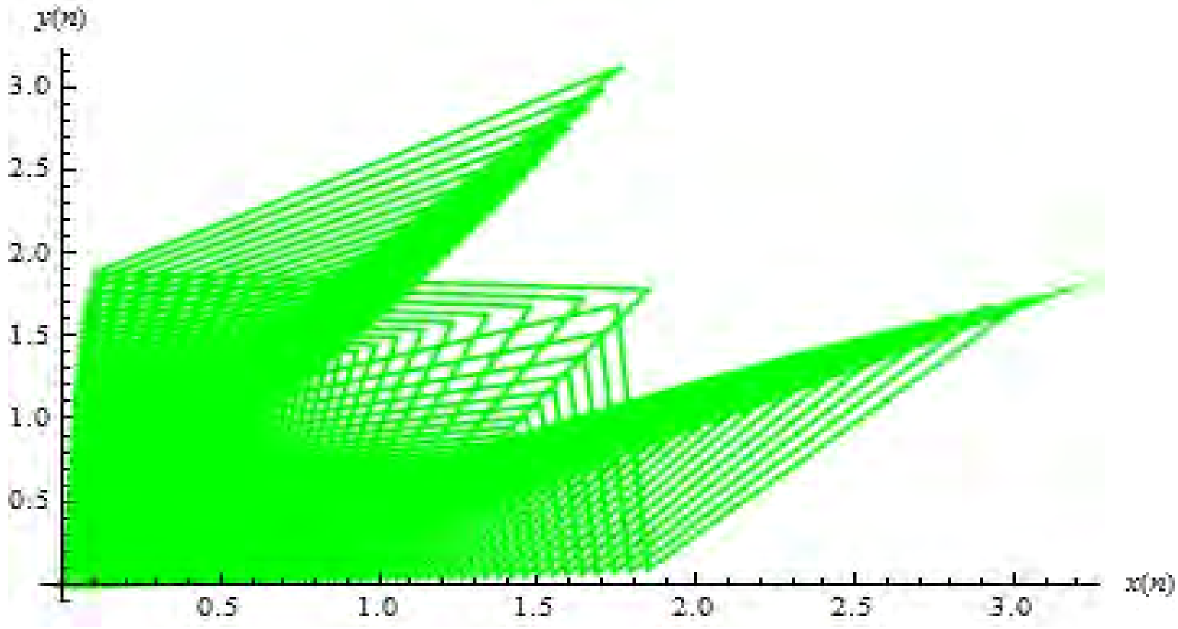
Fig. 1 Plots for the system (9).



(a) Plot of x_n for the system (10)



(b) Plot of y_n for the system (10)



(c) An attractor of the system (10)

Fig. 2 Plots for the system (10)

5 Conclusions

This work is natural extension of (Shojaei et al., 2009; Din et al., 2012; Zhang et al., 2012). In the paper, we have investigated the qualitative behavior of an eight-dimensional discrete dynamical system. The system has only one equilibrium point which is stable under some restriction to parameters. The linearization method is used to show that equilibrium point (0, 0) is locally asymptotically stable. The main objectiveof dynamical systems theory is to predict the global behavior of a system based on the knowledge of its present state. An approach to this problem consists of determining the possible global behaviors of the system and determining which initial conditions lead to these long-term behaviors. In case of higher-order dynamical systems, it is very crucial to discuss global behavior of the system. Some powerful tools such as semiconjugacy and weak contraction cannot be used to analyze global behavior of the system (1). In the paper, we prove the global asymptotic stability of equilibrium point (0, 0) by using simple techniques. Due to the simplicity of our model, we have carried out a systematical local and global stability analysis of it. The most important finding here is that the unique equilibrium point (0, 0) can be a global asymptotic attractor for the system (1). Moreover, we have determined the rate of convergence of a solution that converges to the equilibrium point (0, 0) of the system (1). Some numerical examples are provided to support our theoretical results. These examples are experimental verifications of theoretical discussions.

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