Article

Qualitative behavior of an anti-competitive system of third-order rational difference equations

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Abstract
In this paper, our aim is to study the equilibrium points, local asymptotic stability, global behavior of an equilibrium points and rate of convergence of an anti-competitive system of third-order rational difference equations of the form:

$$x_{n+1}=\frac{\alpha y_{n-2}}{\beta+\gamma x_n x_{n-1} x_{n-2}},\, y_{n+1}=\frac{\alpha_1 x_{n-2}}{\beta_1+\gamma_1 y_n y_{n-1} y_{n-2}},\ n=0,1,\cdots,$$

where the parameters $\alpha,\beta,\gamma,\alpha_1,\beta_1,\gamma_1$ and initial conditions $x_0,x_{-1},x_{-2},y_0,y_{-1},y_{-2}$ are positive real numbers. Some numerical examples are given to verify our theoretical results.

Keywords rational difference equations; stability; global character; rate of convergence.

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1 Introduction

Systems of rational difference equations have been studied by several authors. Especially there has been a great interest in the study of the attractivity of the solutions of such systems.Cinar (2004) investigated the periodicity of the positive solutions of the system of rational difference equations:

$$x_{n+1}=\frac{1}{y_n},\, y_{n+1}=\frac{y_n}{x_{n-1}y_{n-1}}.$$

Stevic (2012) studied the system of two nonlinear difference equations:

$$x_{n+1}=\frac{u_n}{1+v_n},\, y_{n+1}=\frac{w_n}{1+s_n},$$

where $u_n,v_n,w_n,s_n$are some sequences$x_n$ory_n. Stevic(2012) studied the system of three nonlinear difference equations:

$$x_{n+1}=\frac{a_1 x_{n-2}}{b_1 y_n z_{n-1} x_{n-2}+c_1},\, y_{n+1}=\frac{a_2 y_{n-2}}{b_2 z_n x_{n-1} y_{n-2}+c_2},\, z_{n+1}=\frac{a_3 z_{n-2}}{b_3 x_n y_{n-1} z_{n-2}+c_3},$$

where the parameters $a_i, b_i, c_i, i \in \{1,2,3\}$ are real numbers.Bajo and Liz (2011) investigated the global behavior of difference equation:

$x_{n+1} = \frac{x_{n-1}}{a+bx_{n-1}x_n}$,

for all values of real parametersa, b.Touafek and Essayed(2012) studied the periodic nature and got the form of the solutions of the following systems of rational difference equations:

$x_{n+1} = \frac{x_{n-3}}{\pm 1 \pm x_{n-3}y_{n-1}}$, $y_{n+1} = \frac{y_{n-3}}{\pm 1 \pm y_{n-3}x_{n-1}}$.

Recently, Zhang et al. (2012) studied the dynamics of a system of rational third-order difference equation:

$$x_{n+1} = \frac{x_{n-2}}{B + y_n y_{n-1} y_{n-2}}, y_{n+1} = \frac{y_{n-2}}{B + x_n x_{n-1} x_{n-2}}, n = 0,1,\cdots.$$

Our aim in this paper is to investigate the qualitative behavior of positive solution for an anti-competitive system of third-order rational difference equations:

$$x_{n+1} = \frac{\alpha y_{n-2}}{\beta+\gamma x_n x_{n-1} x_{n-2}}, y_{n+1} = \frac{\alpha_1 x_{n-2}}{\beta_1+\gamma_1 y_n y_{n-1} y_{n-2}}, n = 0,1,\cdots \quad (1)$$

where $\alpha, \beta, \gamma, \alpha_1, \beta_1, \gamma_1$ and initial conditions $x_0, x_{-1}, x_{-2}, y_0, y_{-1}, y_{-2}$ are positive real numbers.

Let us consider six-dimensional discrete dynamical system of the form:

$$\left.\begin{aligned} x_{n+1} &= f(x_n, x_{n-1}, x_{n-2}, y_n, y_{n-1}, y_{n-2}) \\ y_{n+1} &= g(x_n, x_{n-1}, x_{n-2}, y_n, y_{n-1}, y_{n-2}) \end{aligned}\right\} \quad (2)$$

$n = 0,1,\cdots$, where$f: I^3 \times J^3 \to I$ and $g: I^3 \times J^3 \to J$are continuously differentiable functions and I, J are some intervals of real numbers. Furthermore, a solution $\{(x_n, y_n)\}_{n=-2}^{\infty}$ of system (2) is uniquely determined by initial conditions $(x_i, y_i) \in I \times J$ for $i \in \{-2,-1,0\}$. Along with the system (2), we consider the corresponding vector map $F = (f, x_n, x_{n-1}, x_{n-2}, g, y_n, y_{n-1}, y_{n-2})$. An equilibrium point of system (2) is a point $(\bar{x}, \bar{y})$ that satisfies

$$\bar{x} = f(\bar{x}, \bar{x}, \bar{x}, \bar{y}, \bar{y}, \bar{y})$$
$$\bar{y} = g(\bar{x}, \bar{x}, \bar{x}, \bar{y}, \bar{y}, \bar{y}).$$

The point $(\bar{x}, \bar{y})$ is also called a fixed point of the vector map F.

Definition 1. Let $(\bar{x}, \bar{y})$ be an equilibrium point of the system (2).

(i) An equilibrium point $(\bar{x}, \bar{y})$ is said to be stable if for every $\varepsilon > 0$ there exists $\delta > 0$ such thatfor every initial conditions $(x_i, y_i), i \in \{-2,-1,0\}$ if $\left\|\sum_{i=-2}^{0}(x_i, y_i) - (\bar{x}, \bar{y})\right\| < \delta$ implies $\|(x_n, y_n) - (\bar{x}, \bar{y})\| < \varepsilon$ for all $n > 0$, where$\|\cdot\|$ is usual Euclidian norm in $\mathbb{R}^2$.

(ii) An equilibrium point $(\bar{x}, \bar{y})$ is said to be unstable if it is not stable.

(iii) An equilibrium point $(\bar{x}, \bar{y})$ is said to be asymptotically stable if there exists $\eta > 0$ such that $\left\|\sum_{i=-2}^{0}(x_i, y_i) - (\bar{x}, \bar{y})\right\| < \eta$ and$(x_n, y_n) \to (\bar{x}, \bar{y})$ as $n \to \infty$.

(iv) An equilibrium point $(\bar{x}, \bar{y})$ is called global attractor if $(x_n, y_n) \to (\bar{x}, \bar{y})$ as $n \to \infty$.

(v) An equilibrium point $(\bar{x}, \bar{y})$ is called asymptotic global attractor if it is a global attractor andstable.

Definition 2. Let$(\bar{x}, \bar{y})$ be an equilibrium point of a map

$$F = (f, x_n, x_{n-1}, x_{n-2}, g, y_n, y_{n-1}, y_{n-2})$$

where$f$ and $g$are continuously differentiable functions at$(\bar{x}, \bar{y})$. The linearized system of (2) about the equilibrium point $(\bar{x}, \bar{y})$ is given by

$$X_{n+1} = F(X_n) = F_J X_n,$$

where $X_n = \begin{pmatrix} x_n \\ x_{n-1} \\ x_{n-2} \\ y_n \\ y_{n-1} \\ y_{n-2} \end{pmatrix}$ and $F_J$is Jacobean matrix of the system (2) about the equilibrium point $(\bar{x}, \bar{y})$.

To construct corresponding linearized form of the system (1) we consider the following transformation:

$$(x_n, x_{n-1}, x_{n-2}, y_n, y_{n-1}, y_{n-2}) \to (f, f_1, f_2, g, g_1, g_2) \qquad (3)$$

where $f = \frac{\alpha y_{n-2}}{\beta+\gamma x_n x_{n-1} x_{n-2}}, g = \frac{\alpha_1 x_{n-2}}{\beta_1+\gamma_1 y_n y_{n-1} y_{n-2}}, f_1 = x_n, f_2 = x_{n-1}, g_1 = y_n, g_2 = y_{n-1}$. The Jacobian matrix about the fixed point $(\bar{x}, \bar{y})$ under the transformation (3) is given by

$$F_J(\bar{x}, \bar{y}) = \begin{pmatrix} A & A & A & 0 & 0 & B \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & C & D & D & D \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

where$A = -\frac{\alpha\gamma\bar{y}\bar{x}^2}{(\beta+\gamma\bar{x}^3)^2}$, $B = \frac{\alpha}{\beta+\gamma\bar{x}^3}$, $C = \frac{\alpha_1}{\beta_1+\gamma_1\bar{y}^3}$ and $D = -\frac{\alpha_1\gamma_1\bar{x}\bar{y}^2}{(\beta_1+\gamma_1\bar{y}^3)^2}$.

Theorem 1. (Sedaghat, 2003). For the system $X_{n+1} = F(X_n)$, $n = 0,1,\cdots$, of difference equations such that $\overline{X}$ be a fixed point of$F$. If all eigenvalues of the Jacobian matrix F_J about $\overline{X}$ lie inside the open unitdisk $|\lambda| < 1$, then$\bar{X}$ is locally asymptotically stable. If one of them has a modulus greater than one, then $\bar{X}$ is unstable.

2 Main Results

Let $(\bar{x}, \bar{y})$be an equilibrium point of the system (1), then system (1) has only(0,0)equilibrium point.

Theorem 2. Let (x_n, y_n) be positive solution of system (1), then for every $m \geq 0$ the following results hold.

$$(i)\ 0 \leq x_n \leq \begin{cases} \left(\frac{\alpha}{\beta}\frac{\alpha_1}{\beta_1}\right)^{m+1} x_{-p}, & n = 6m + (6-p), p = 0,1,2, \\ \left(\frac{\alpha}{\beta}\right)^{m+1}\left(\frac{\alpha_1}{\beta_1}\right)^{m} y_{3-p}, & n = 6m + (6-p), p = 3,4,5. \end{cases}$$

$$(ii)\ 0 \leq y_n \leq \begin{cases} \left(\frac{\alpha}{\beta}\frac{\alpha_1}{\beta_1}\right)^{m+1} y_{-p}, & n = 6m + (6-p), p = 0,1,2, \\ \left(\frac{\alpha}{\beta}\right)^{m}\left(\frac{\alpha_1}{\beta_1}\right)^{m+1} x_{3-p}, & n = 6m + (6-p), p = 3,4,5. \end{cases}$$

Proof. (i)The result is obviously true for $m = 0$. Suppose that results are true for $m = k \geq 1$, $i.e.$,

$$0 \leq x_n \leq \begin{cases} \left(\frac{\alpha}{\beta}\frac{\alpha_1}{\beta_1}\right)^{k+1} x_{-p}, & n = 6k + (6-p), p = 0,1,2, \\ \left(\frac{\alpha}{\beta}\right)^{k+1}\left(\frac{\alpha_1}{\beta_1}\right)^{k} y_{3-p}, & n = 6k + (6-p), p = 3,4,5. \end{cases}$$

and

$$0 \leq y_n \leq \begin{cases} \left(\frac{\alpha}{\beta}\frac{\alpha_1}{\beta_1}\right)^{k+1} y_{-p}, & n = 6k + (6-p), p = 0,1,2, \\ \left(\frac{\alpha}{\beta}\right)^{k}\left(\frac{\alpha_1}{\beta_1}\right)^{k+1} x_{3-p}, & n = 6k + (6-p), p = 3,4,5. \end{cases}$$

Now, for $m = k + 1$ one has

$0 \leq x_{6k+(12-p)} = \frac{\alpha y_{6k+(9-p)}}{\beta+\gamma \prod_{i=9-p}^{11-p} x_{10r+1}} \leq \frac{\alpha y_{6k+(9-p)}}{\beta} \leq \left(\frac{\alpha}{\beta}\right)^{k+2} \left(\frac{\alpha_1}{\beta_1}\right)^{k+1} y_{3-p}, p = 3,4,5,$

and

$0 \leq x_{6k+(12-p)} \leq \left(\frac{\alpha}{\beta}\frac{\alpha_1}{\beta_1}\right)^{k+2} x_{-p}, p = 0,1,2.$

So, the induction is complete. Similarly, inequalities (ii) hold by induction.■

Theorem 3. For the equilibrium point$(0,0)$ of the system (1) the following results hold:

(i)If $\alpha < \beta$ and $\alpha_1 < \beta_1$, then equilibrium point $(0,0)$ of the system (1) is locally asymptotically stable.

(ii)If $\alpha > \beta$or$\alpha_1 > \beta_1$, then equilibrium point $(0,0)$ is unstable.

Proof. (i).The linearized system of (1) about the equilibrium point $(0,0)$ is given by

$$X_{n+1} = F_J(0,0)X_n,$$

where $X_n = \begin{pmatrix} x_n \\ x_{n-1} \\ x_{n-2} \\ y_n \\ y_{n-1} \\ y_{n-2} \end{pmatrix}$ and $F_J(0,0) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \frac{\alpha}{\beta} \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\alpha_1}{\beta_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$.

The characteristic polynomial of $F_J(P_0)$ is given by

$$P(\lambda) = \lambda^6 - \frac{\alpha\alpha_1}{\beta\beta_1}. \quad (4)$$

The roots of $P(\lambda)$ are

$$\lambda_k = \left(\frac{\alpha\alpha_1}{\beta\beta_1}\right)^{\frac{1}{6}} \exp\left(\frac{k\pi i}{3}\right),$$

for $k = 0,1,\cdots,5$. Now, it is easy to see that $|\lambda_K| < 1$ for all $k = 0,1,\cdots,5$. Since all eigenvalues of Jacobian matrix $F_J(0,0)$ about (0, 0) lie in open unit disk $|\lambda| < 1$. Hence, the equilibrium point (0, 0) is locally asymptotically stable.

(ii). It is easy to see that if $\alpha > \beta$ or $\alpha_1 > \beta_1$, then there exist at least one root λ of equation (4) such that $|\lambda| > 1$. Hence by Theorem 1 if $\alpha > \beta$ or $\alpha_1 > \beta_1$, then (0, 0) is unstable.■

Theorem 4. The system (1) has no prime period-two solutions.

Proof. Assume that $(p_1, q_1), (p_2, q_2), (p_1, q_1), \cdots,$ be prime period two solution of system (1) such that $p_i, q_i \neq 0$ for $i \in \{1,2\}$ and $p_1 \neq p_2$, $q_1 \neq q_2$. Then, from system (1) one has

$$p_1 = \frac{\alpha q_2}{\beta+\gamma p_1 p_2^2}, p_2 = \frac{\alpha q_1}{\beta+\gamma p_1^2 p_2}, \quad (5)$$

and

$$q_1 = \frac{\alpha_1 p_2}{\beta_1+\gamma_1 q_1 q_2^2}, q_2 = \frac{\alpha_1 p_1}{\beta_1+\gamma_1 q_1^2 q_2}. \quad (6)$$

After some tedious calculations from (5) and (6), one has

$$(p_1 + p_2)^2 - 4p_1p_2 = 0,$$

and

$$(q_1 + q_2)^2 - 4q_1q_2 = 0,$$

which is contradiction. Hence, system (1) has no prime period-two solutions.■

Theorem 5. Let $\alpha < \beta$ and $\alpha_1 < \beta_1$, then equilibrium point (0, 0) of the system (1) is globally asymptotically stable.

Proof. For $\alpha < \beta$ and $\alpha_1 < \beta_1$, from Theorem 3 (0, 0) is locally asymptotically stable. From Theorem 2, it is easy to see that every positive solution (x_n, y_n) of the system (1) is bounded, $i.e.$, $0 \leq x_n \leq \mu$ and $0 \leq y_n \leq \delta$ for all $n = 0,1,2,\cdots$, where $\mu = \max\{x_{-2}, x_{-1}, x_0\}$ and $\delta = \max\{y_{-2}, y_{-1}, y_0\}$.Now, it is sufficient to prove that (x_n, y_n) is decreasing. From system (1) one has

$$x_{n+1} = \frac{\alpha y_{n-2}}{\beta+\gamma x_n x_{n-1} x_{n-2}},$$

$$\leq \frac{\alpha}{\beta} y_{n-2} < y_{n-2}.$$

This implies that $x_{6n+1} < y_{6n-2}$ and $x_{6n+7} < y_{6n+4}$. Also

$$y_{n+1} = \frac{\alpha_1 x_{n-2}}{\beta_1 + \gamma_1 y_n y_{n-1} y_{n-2}}$$

$$\leq \frac{\alpha_1}{\beta_1} x_{n-2} < x_{n-2}.$$

This implies that $y_{6n+1} < x_{6n-2}$ and $y_{6n+7} < x_{6n+4}$. So $x_{6n+7} < y_{6n+4} < x_{6n+1}$ and $y_{6n+7} < x_{6n+4} < y_{6n+1}$. Hence, the subsequences $\{x_{6n+1}\}, \{x_{6n+2}\}, \{x_{6n+3}\}, \{x_{6n+4}\}, \{x_{6n+5}\}, \{x_{6n+6}\}$ and $\{y_{6n+1}\}, \{y_{6n+2}\}, \{y_{6n+3}\}, \{y_{6n+4}\}, \{y_{6n+5}\}, \{y_{6n+6}\}$ are decreasing. Therefore the sequences $\{x_n\}$ and $\{y_n\}$ are decreasing. Hence $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = 0$.■

3 Rate of Convergence

In this section we will determine the rate of convergence of a solution that converges to the unique equilibrium point of the system (1). The following results give the rate of convergence of solutions of a system of difference equations

$$X_{n+1} = \big(A + B(n)\big) X_n, \tag{7}$$

where X_n is an $m-$dimensional vector, $A \in C^{m\times m}$ is a constant matrix, and $B: Z^+ \to C^{m\times m}$ is a matrix function satisfying

$$\|B(n)\| \to 0 \tag{8}$$

as $n \to \infty$, where $\|\cdot\|$ denotes any matrix norm which is associated with the vector norm

$$\|B(n)\| = \sqrt{x^2 + y^2}.$$

Proposition 1 (Perron's theorem) (Pituk, 2002) Suppose that condition (8) holds. If X_n is a solution of (7), then either $X_n = 0$ for all large n or

$$\rho = \lim_{n\to\infty} (\|X_n\|)^{\frac{1}{n}} \tag{9}$$

or

$$\rho = \lim_{n\to\infty} \frac{\|X_{n+1}\|}{\|X_n\|} \tag{10}$$

exist and is equal to the modulus of one the eigenvalues of matrix A.

Assume that $\lim\limits_{n\to\infty} x_n = \bar{x}$, $\lim\limits_{n\to\infty} y_n = \bar{y}$.Then error terms for the system (1) are given by

$x_{n+1} - \bar{x} = \sum_{i=0}^{2} A_i (x_{n-i} - \bar{x}) + \sum_{i=0}^{2} B_i (y_{n-i} - \bar{y}),$
$y_{n+1} - \bar{y} = \sum_{i=0}^{2} C_i (x_{n-i} - \bar{x}) + \sum_{i=0}^{2} D_i (y_{n-i} - \bar{y}).$
Set $e_n^1 = x_n - \bar{x}$ and$e_n^2 = y_n - \bar{y}$, one has
$e_{n+1}^1 = \sum_{i=0}^{2} A_i\, e_{n-i}^1 + \sum_{i=0}^{2} B_i\, e_{n-i}^2\,, e_{n+1}^2 = \sum_{i=0}^{2} C_i\, e_{n-i}^1 + \sum_{i=0}^{2} D_i\, e_{n-i}^2.$
where

$$A_0 = -\frac{\alpha\gamma\bar{y}\prod_{i=1}^{2} x_{n-i}}{(\beta+\gamma\prod_{i=0}^{2} x_{n-i})(\beta+\gamma\bar{x}^3)}\ , \ A_1 = -\frac{\alpha\gamma\bar{x}\bar{y}x_{n-2}}{(\beta+\gamma\prod_{i=0}^{2} x_{n-i})(\beta+\gamma\bar{x}^3)},\ A_2 = -\frac{\alpha\gamma\bar{x}^2\bar{y}}{(\beta+\gamma\prod_{i=0}^{2} x_{n-i})(\beta+\gamma\bar{x}^3)},$$

$B_i = 0$ for $i \in \{0,1\}$,$B_2 = \frac{\alpha}{\beta+\gamma\prod_{i=0}^{2} x_{n-i}}$,$C_i = 0$for $i \in \{0,1\}$, $C_2 = \frac{\alpha_1}{\beta_1+\gamma_1\prod_{i=0}^{2} y_{n-i}}$,

$D_0 = -\frac{\alpha_1\gamma_1\bar{x}\prod_{i=1}^{2} y_{n-i}}{(\beta_1+\gamma_1\prod_{i=0}^{2} y_{n-i})(\beta_1+\gamma_1\bar{y}^3)}$,$D_1 = -\frac{\alpha_1\gamma_1\bar{x}\bar{y}y_{n-2}}{(\beta_1+\gamma_1\prod_{i=0}^{2} y_{n-i})(\beta_1+\gamma_1\bar{y}^3)}$,$D_2 = \frac{\alpha_1\gamma_1 x\bar{y}^2}{(\beta_1+\gamma_1\prod_{i=0}^{2} y_{n-i})(\beta_1+\gamma_1\bar{y}^3)}$.

Taking the limit, we obtain $\lim_{n\to\infty} A_i == -\frac{\alpha\gamma\bar{y}\bar{x}^2}{(\beta+\gamma\bar{x}^3)^2}$ for $i \in \{0,1,2\}$, $\lim_{n\to\infty} B_i = 0$ for $i \in \{0,1\}$, $\lim_{n\to\infty} B_2 = \frac{\alpha}{\beta+\gamma\bar{x}^3}$, $\lim_{n\to\infty} C_i = 0$ for $i \in \{0,1\}$, $\lim_{n\to\infty} C_2 = \frac{\alpha_1}{\beta_1+\gamma_1\bar{y}^3}$ and $\lim_{n\to\infty} D_i == -\frac{\alpha_1\gamma_1\bar{x}\bar{y}^3}{(\beta_1+\gamma_1\bar{y}^3)^2}$for $i \in \{0,1,2\}$. So, the limiting system of error terms can be written as $E_{n+1} = F_J(P_0)E_n$,where $E_n = \begin{pmatrix} e_n^1 \\ e_{n-1}^1 \\ e_{n-2}^1 \\ e_n^2 \\ e_{n-1}^2 \\ e_{n-2}^2 \end{pmatrix}$.

Using the Proposition (1), one has following result.

Theorem 6. Assume that $\{(x_n, y_n)\}$ be a positive solution of the system (1) such that $\lim_{n\to\infty} x_n = \bar{x}$ and $\lim_{n\to\infty} y_n = \bar{y}$ where $(\bar{x}, \bar{y}) = (0,0)$. Then, the error term E_n of every solution of (1) satisfies both of the following asymptotic relations

$\lim_{n\to\infty}(\|e_n\|)^{\frac{1}{n}} = |\lambda F_J(\bar{x}, \bar{y})|$, $\lim_{n\to\infty}\frac{\|e_{n+1}\|}{\|e_n\|} = |\lambda F_J(\bar{x}, \bar{y})|$,

where$\lambda F_J(\bar{x}, \bar{y})$ are the characteristic roots of Jacobian matrix $F_J(\bar{x}, \bar{y})$about(0,0) .

4 Examples

In order to verify our theoretical results we consider several interesting numerical examples in this section. These examples represent different types of qualitative behavior of solutions to the system of nonlinear difference equations (1). All plots in this section are drawn with Mathematica.

Example 1. Consider the system (1) with initial conditions $x_{-2} = 1.9, x_{-1} = 0.9, x_0 = 0.8, y_{-2} = 1.9, y_{-1} = 1.5, y_0 = 1.9$. Moreover, choosing the parameters $\alpha = 31, \beta = 32, \gamma = 1.9, \alpha_1 = 521, \beta_1 = 522, \gamma_1 = 1.8$.Then , the system (1) can be written as

$$x_{n+1} = \frac{31y_{n-2}}{32+1.9x_n x_{n-1}x_{n-2}}, y_{n+1} = \frac{521x_{n-2}}{522+1.8y_n y_{n-1}y_{n-2}},\ n = 0,1,\cdots, \tag{11}$$

and with initial condition $x_{-2} = 1.9, x_{-1} = 0.9, x_0 = 0.8, y_{-2} = 1.9, y_{-1} = 1.5, y_0 = 1.9$. . Moreover, in Fig. 1 the plot of x_n is shown in Fig. 1a, the plot of y_n is shown in Fig. 1b and an attractor of the system (9) is shown in Fig. 1c.

Example 2. Consider the system (1) with initial conditions $x_{-2} = 3.5, x_{-1} = 2.6, x_0 = 2.1, y_{-2} = 1.2, y_{-1} = 2.8, y_0 = 2.9$. Moreover, choosing the parameters $\alpha = 1.1, \beta = 1.12, \gamma = 0.004, \alpha_1 = 0.8, \beta_1 = 0.81, \gamma_1 = 0.007$. Then , the system (1) can be written as

$$x_{n+1} = \frac{1.1y_{n-2}}{1.12+0.004x_n x_{n-1}x_{n-2}}, y_{n+1} = \frac{0.8x_{n-2}}{0.81+0.007y_n y_{n-1}y_{n-2}},\ n = 0,1,\cdots, \tag{12}$$

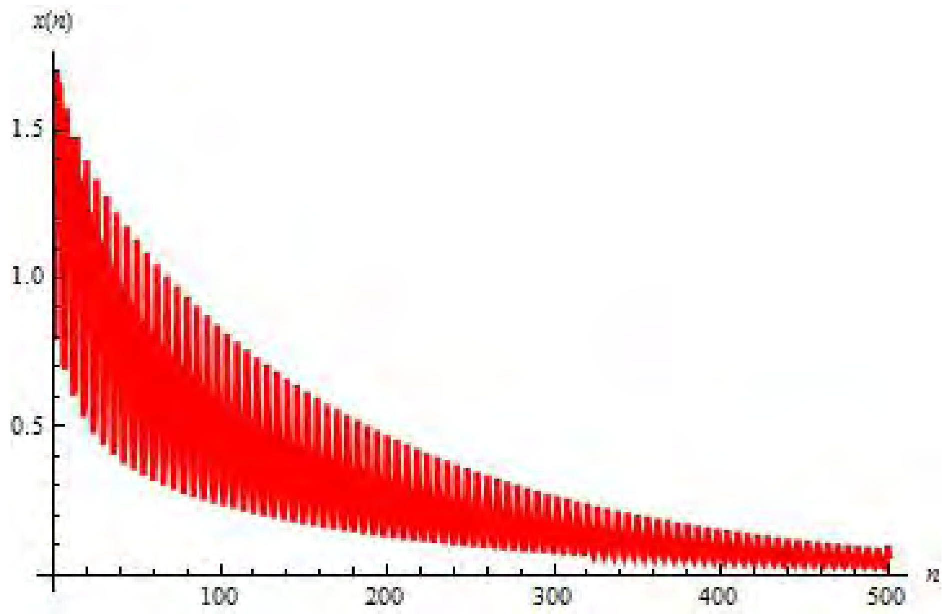
and with initial condition $x_{-2} = 3.5, x_{-1} = 2.6, x_0 = 2.1, y_{-2} = 1.2, y_{-1} = 2.8, y_0 = 2.9$. Moreover, in Fig. 2 the plot of x_n is shown in Fig. 2a, the plot of y_n is shown in Fig. 2b and an attractor of the system (12) is shown in Fig. 2c.

Example 3. Consider the system (1) with initial conditions $x_{-2} = 3.4, x_{-1} = 2.6, x_0 = 2.1, y_{-2} = 1.2, y_{-1} =$

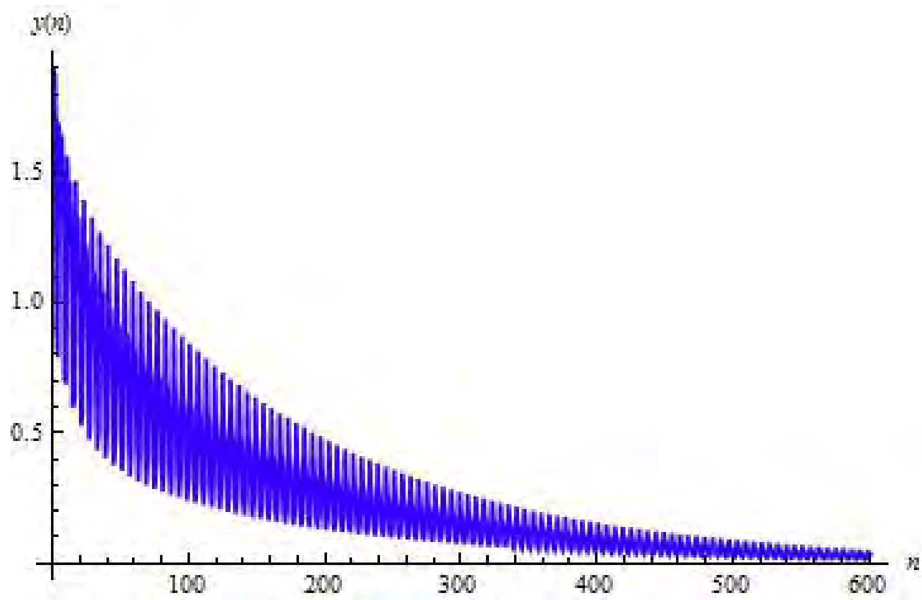
$2.8, y_0 = 2.7$. Moreover, choosing the parameters $\alpha = 1500, \beta = 1535, \gamma = 1.9, \alpha_1 = 1400, \beta_1 = 1450, \gamma_1 = 350$. Then, the system (1) can be written as

$$x_{n+1} = \frac{1500y_{n-2}}{1535+1.9x_nx_{n-1}x_{n-2}}, y_{n+1} = \frac{1400x_{n-2}}{1450+350y_ny_{n-1}y_{n-2}}, \ n = 0,1,\cdots, \quad (13)$$

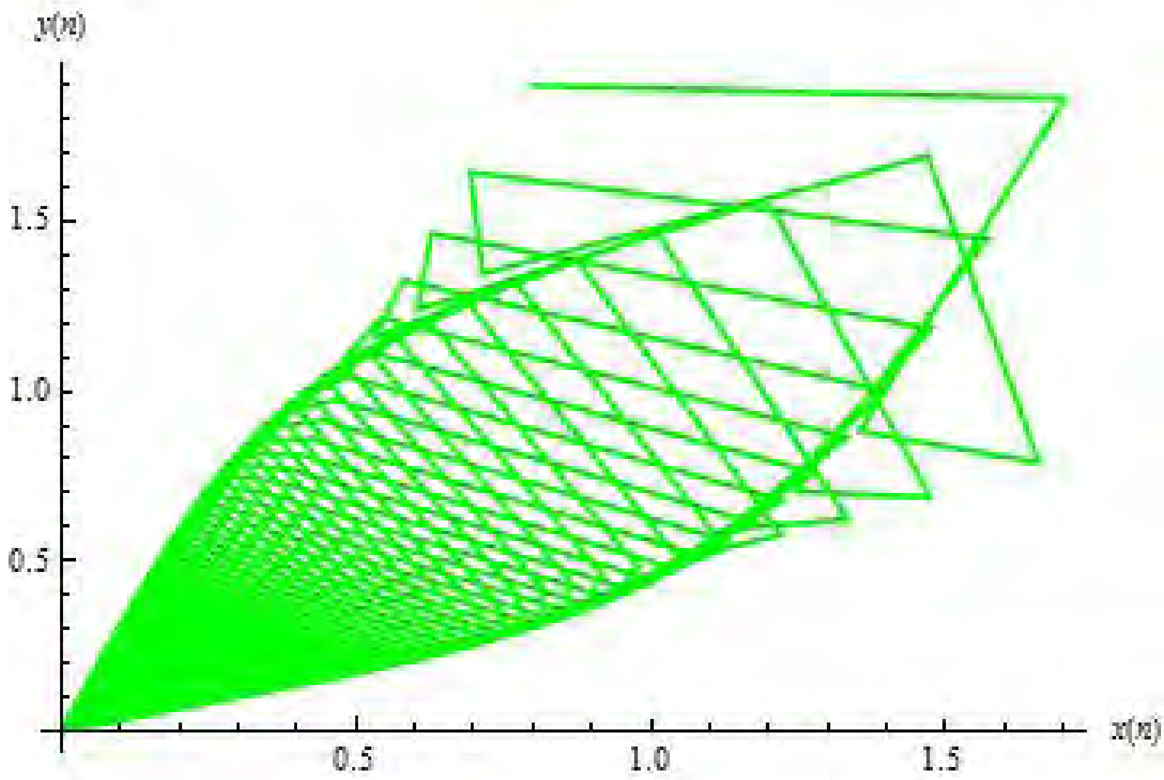
and with initial condition $x_{-2} = 3.4, x_{-1} = 2.6, x_0 = 2.1, y_{-2} = 1.2, y_{-1} = 2.8, y_0 = 2.7$. Moreover, in Fig. 3 the plot of x_n is shown in Fig. 3a, the plot of y_n is shown in Fig. 3b and an attractor of the system (13) is shown in Fig. 3c.



(a) Plot of x_n for the system (11)

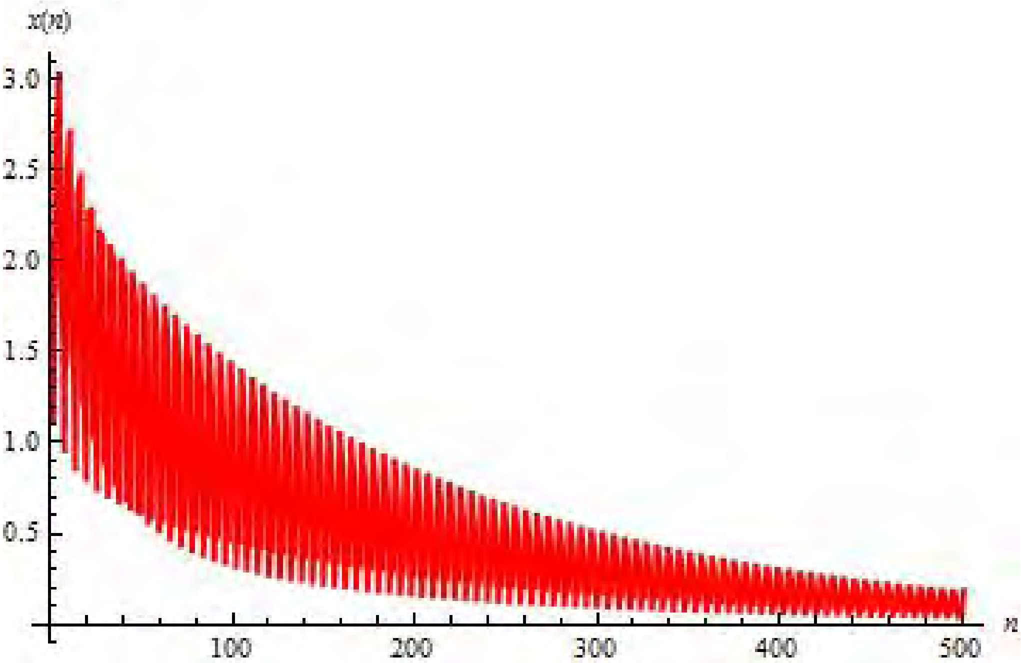


(b) Plot of y_n for the system (11)

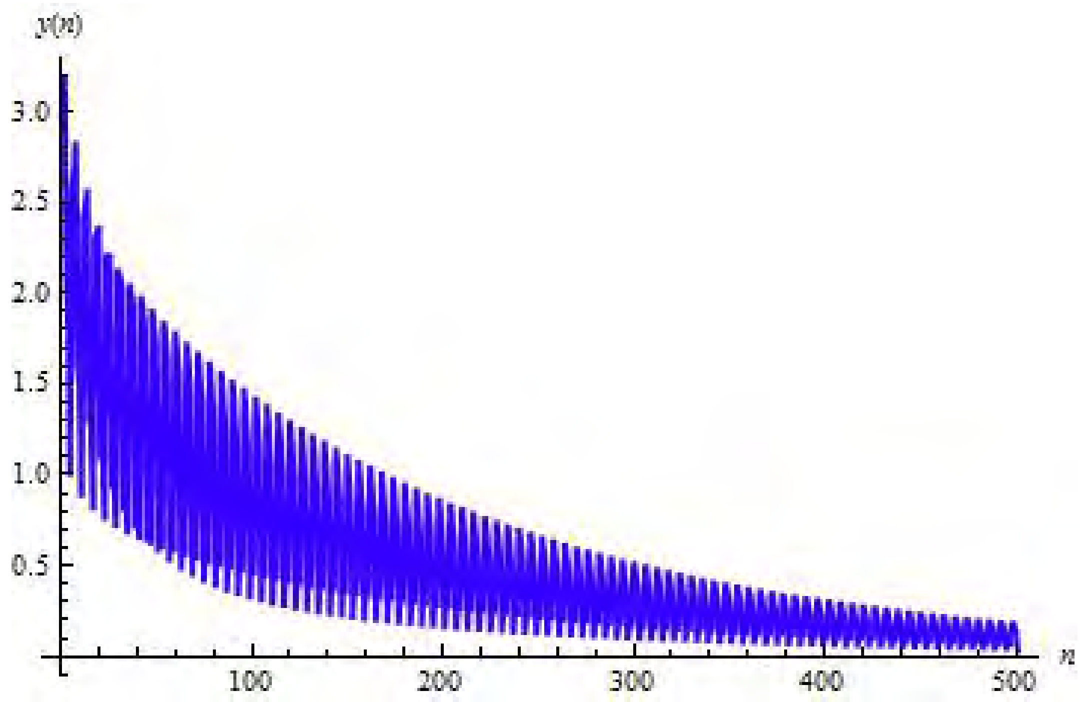


(c) An attractor of the system (11)

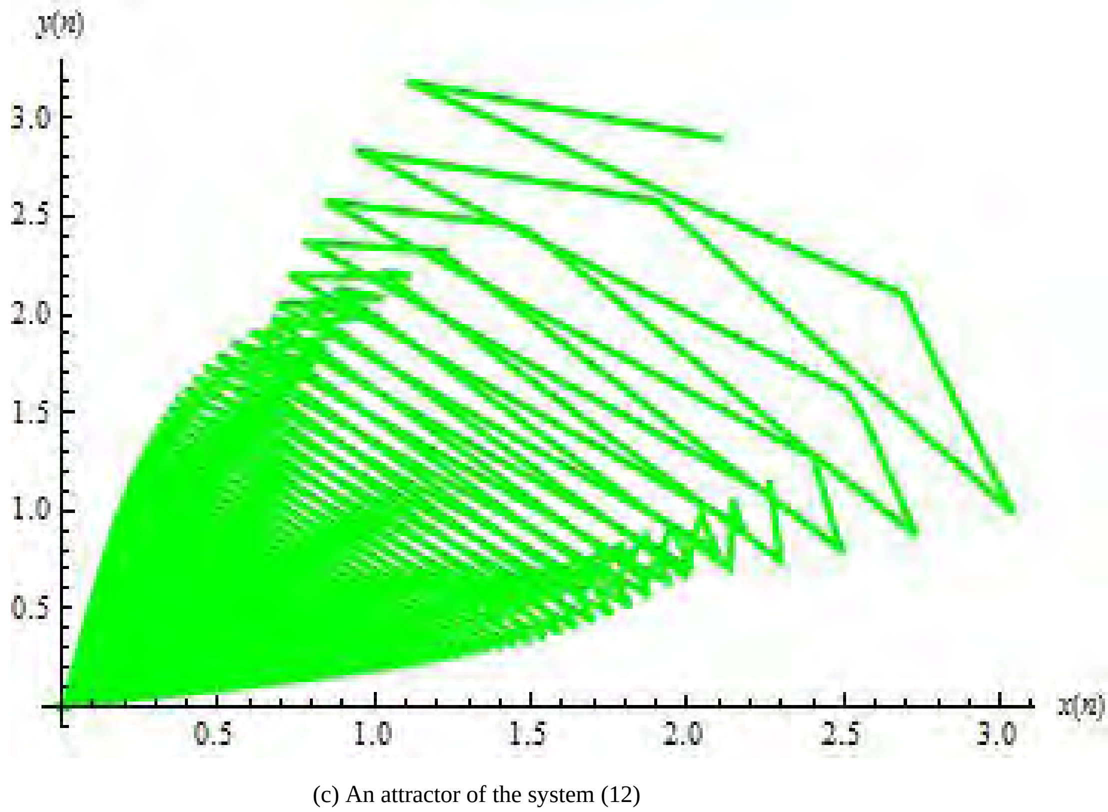
Fig. 1 Plots for the system (11).



(a) Plot of x_n for the system (12)

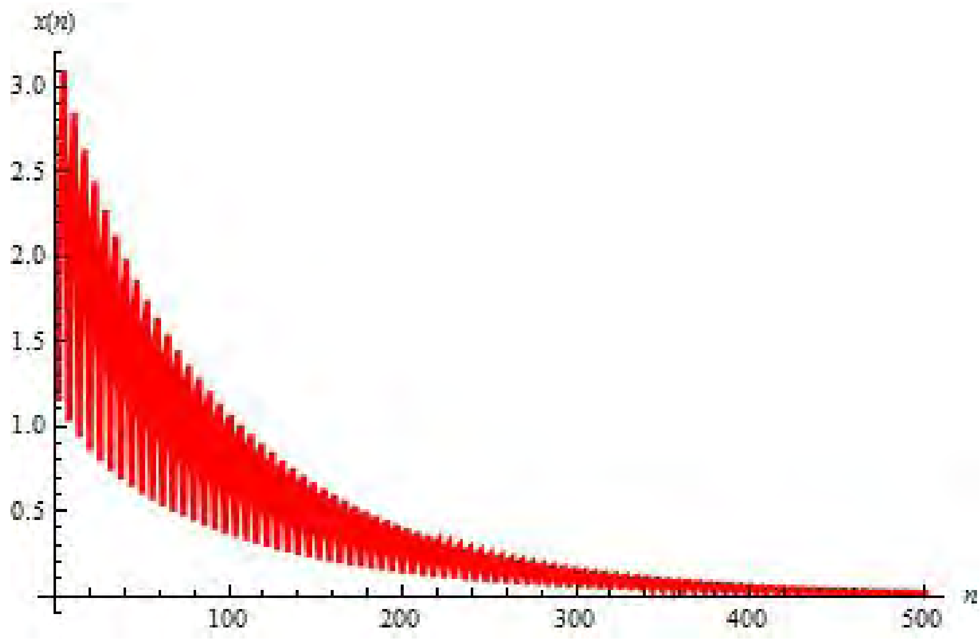


(b) Plot of y_n for the system (12)

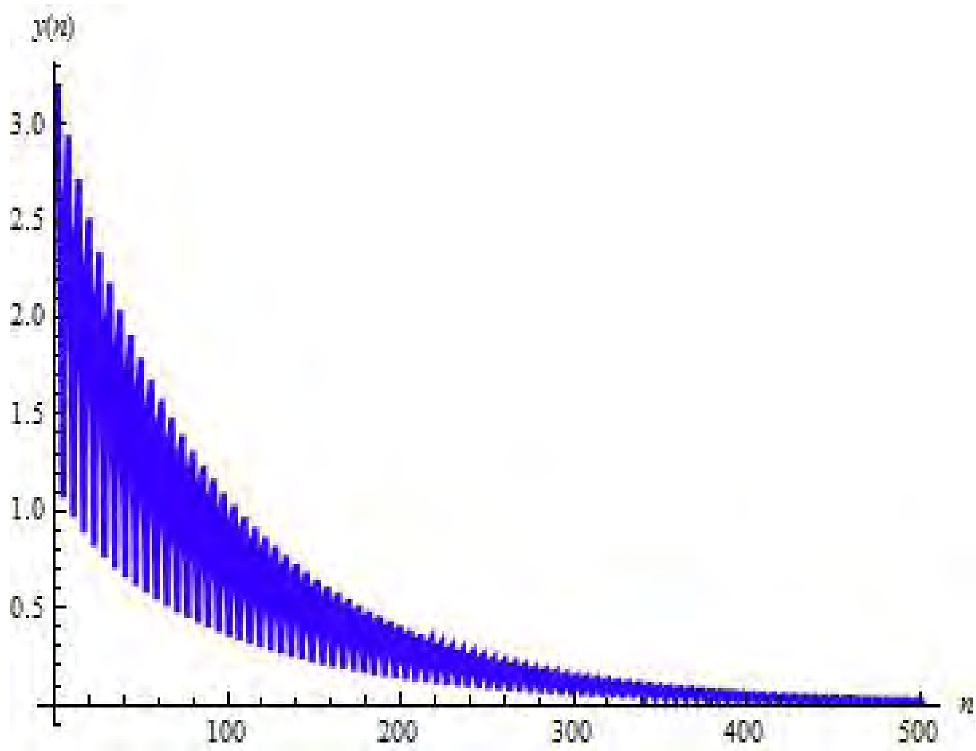


(c) An attractor of the system (12)

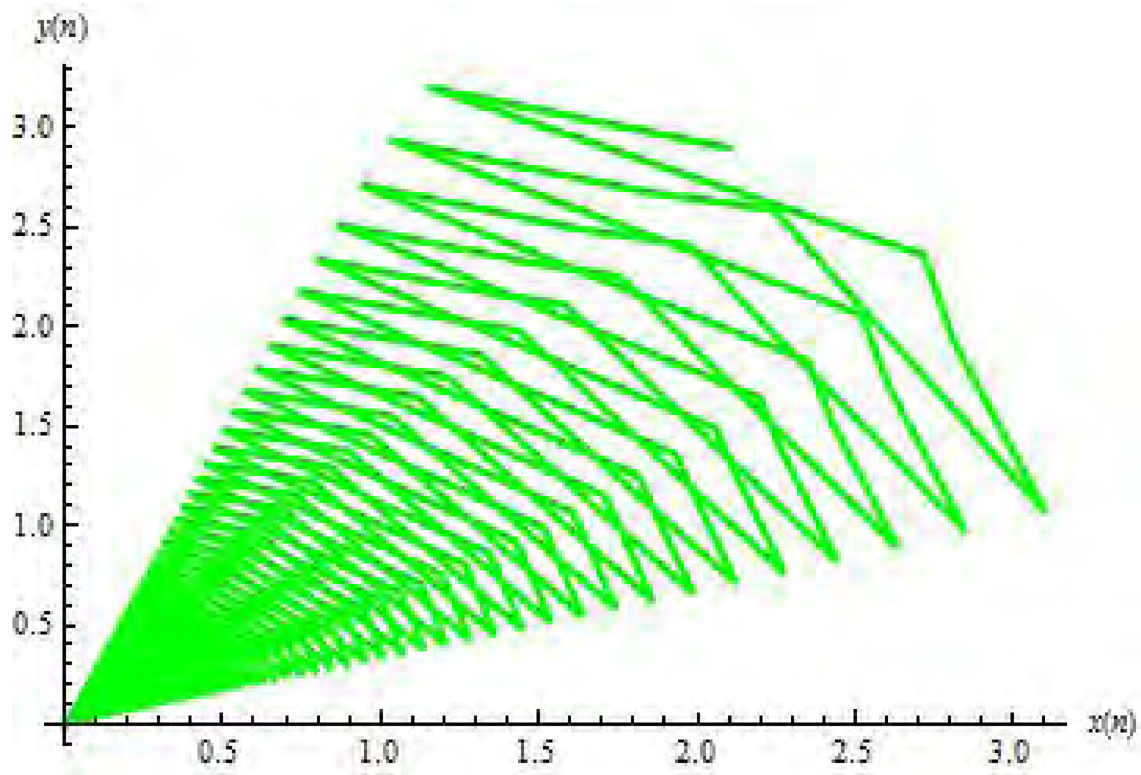
Fig. 2 Plots for the system (12).



(a) Plot of x_n for the system (13)



(b) Plot of y_n for the system (13)



(c) An attractor of the system (13)

Fig.3 Plots for the system (13).

5 Conclusions

In the paper, we investigate some dynamics of a six-dimensional discrete system. The system has only one equilibrium point $(0,0)$. The linearization method is used to show that equilibrium point $(0,0)$is locally asymptotically stable. Also system has no prime period-two solutions. The main objective of dynamical systems theory is to predict the global behavior and rate of convergence of the system. In the paper, we use simple technique to prove the global asymptotic stability of equilibrium point $(0,0)$. Some numerical examples are provided to support our theoretical results.

Acknowledgements

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