Article

Stability analysis of a discrete ecological model

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Abstract

In this paper, we study the qualitative behavior of following discrete-time population model:

$x_{n+1} = \alpha + \beta x_n + \gamma x_{n-1} e^{-y_n}, y_{n-1} = \delta + \varepsilon y_n + \zeta y_{n-1} e^{-x_n}$,

where parameters $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta$ and initial conditions x_0, x_{-1}, y_0, y_{-1} are positive real numbers. More precisely, we investigate the existence and uniqueness of positive equilibrium point,boundedness character, persistence, local asymptotic stability, global behavior and rate of convergence of unique positive equilibrium point of this model. Some numerical examples are given to verify our theoretical results.

Keywords population models; difference equations; steady-states; boundedness; local and global character.

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1 Introduction

In population dynamics, difference and differential equations are used in many models (Nedorezov, 2012; Nedorezov and Sadykov, 2012; Elena et al., 2013). Exponential difference equations have many applications in population dynamics. One can see the references (El-Metwally et al., 2001; Papaschinopoulos et al., 2011; Papaschinopoulos et al., 2012) for some interesting results related to qualitative behavior of population models. From Zhou and Zou (2003) and Liu (2010) it is clear that difference equations are much better as compared to differential equations, when the populations are of non-overlapping generations. In literature, there are many papers in which discrete dynamical systems are used to study the qualitative behavior of population models (Ahmad, 1993; Zhou and Zou, 2003; Tang andZou, 2006; Din, 2013; Din and. Donchev, 2013; Din et al., 2013; Din, 2014). In case of discrete–time models one can compute mathematical results more efficiently. There are many research papers which are related to mathematical models in population dynamics. During the last few decades, many researchers are attracted by such computational models. For detail of biological models see (Edelstein-Keshet, 1988; Brauer and Castillo-Chavez, 2000; Allen, 2007). Nonlinear difference equations are very important in application. Naturally, such equations appear as discrete approximation of differential and delay differential equations and these models are related to applied sciences such as ecology,biology, physics, chemistry, physiology, engineering, bio-chemistry and economics. We cannot find the solutions of nonlinear difference equations in all cases. So, one can study the behavior of solutions by local asymptotic stability of equilibrium points. El-Metwally et al. (2001) studied the qualitative behavior of following population model:$x_{n+1} = \alpha + \beta x_{n-1} e^{-x_n}$. Papachinopoulous et al. (2011) investigated the dynamics of following population model:$x_{n+1} = a + b x_{n-1} e^{-y_n}$, $y_{n+1} = c + d y_{n-1} e^{-x_n}$. Recently, Papachinopoulous et al. (2012) study following two exponential nonlinear difference equations: $u_{n+1} = \alpha_1 + \beta_1 u_{n-1} e^{-v_n}$,$v_{n+1} = \gamma_1 + \delta_1 u_{n-1} e^{-u_n}$, and $u_{n+1} = \alpha_1 + \beta_1 u_{n-1} e^{-u_n}$, $u_{n+1} = \gamma_1 + \delta_1 u_{n-1} e^{-v_n}$.

Our aim in this paper is to investigate existence and uniqueness of positive equilibrium point, boundedness character, persistence, local asymptotic stability, global behavior of unique positive equilibrium point of following four-dimensional discrete-time population model:

$$x_{n+1} = \alpha + \beta x_n + \gamma x_{n-1} e^{-y_n}, \qquad y_{n+1} = \delta + \varepsilon y_n + \zeta y_{n-1} e^{-x_n}, \tag{1}$$

where parameters $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta$and initial conditions x_0, x_{-1}, y_0, y_{-1} are positive real numbers. System (1) is an extension of (Papachinopoulouset al., 2011).

2 Boundedness and Persistence

The following Theorem shows that every positive solution of (1) is bounded and persists.

Theorem 2.1 Assume that $e^{\delta}\beta + e^{\delta/2}\sqrt{e^{\delta}\beta^2 + 4\gamma} < 2e^{\delta}, e^{\alpha}\varepsilon + e^{\alpha/2}\sqrt{e^{\alpha}\varepsilon^2 + 4\zeta} < 2e^{\alpha}$, then every positive solution $\{(x_n, y_n)\}$ of system (1) is bounded and persists.

Proof. Consider the following second-order difference equations:

$$\bar{x}_{n+1} = \alpha + \beta \bar{x}_n + \gamma \bar{x}_{n-1} e^{-\delta}, \bar{y}_{n+1} = \delta + \varepsilon \bar{y}_n + \zeta \bar{y}_{n-1} e^{-\alpha}. \tag{2.1}$$

The solutions of (2.1) are given by

$$\bar{x}_n = \frac{\alpha}{1-\beta-e^{-\delta}\gamma} + 2^{-n}\left(e^{-\delta}\left(e^{\delta}\beta - e^{\delta/2}\sqrt{e^{\delta}\beta^2+4\gamma}\right)\right)^n c_1 + 2^{-n}\left(e^{-\delta}\left(e^{\delta}\beta + e^{\delta/2}\sqrt{e^{\delta}\beta^2+4\gamma}\right)\right)^n c_2,$$

$$\bar{y}_n = \frac{\delta}{1-\epsilon-e^{-\alpha}\zeta} + 2^{-n}\left(e^{-\alpha}\left(e^{\alpha}\varepsilon - e^{\alpha/2}\sqrt{e^{\alpha}\varepsilon^2+4\zeta}\right)\right)^n c_3 + 2^{-n}\left(e^{-\alpha}\left(e^{\alpha}\varepsilon + e^{\alpha/2}\sqrt{e^{\alpha}\varepsilon^2+4\zeta}\right)\right)^n c_4,$$

where c_1, c_2, c_3, c_4 depend on initial values $\bar{x}_{-1}, \bar{x}_0, \bar{y}_{-1}, \bar{y}_0$.

Assume that $e^{\delta}\beta + e^{\delta/2}\sqrt{e^{\delta}\beta^2 + 4\gamma} < 2e^{\delta}, e^{\alpha}\varepsilon + e^{\alpha/2}\sqrt{e^{\alpha}\varepsilon^2 + 4\zeta} < 2e^{\alpha}$. Then it follows that:

$\bar{x}_n \le \frac{\alpha}{1-\beta-e^{-\delta}\gamma} + c_1 + c_2$ and $\bar{y}_n \le \frac{\delta}{1-\epsilon-e^{-\alpha}\zeta} + c_3 + c_4$ for all $n = 1,2,\cdots$. Evidently, the sequences $\{\bar{x}_n\}$ and $\{\bar{y}_n\}$ are bounded for all $n = 1,2,\cdots$. Then, by comparison we obtain

$x_n \le \frac{\alpha}{1-\beta-e^{-\delta}\gamma}$ and $y_n \le \frac{\delta}{1-\epsilon-e^{-\alpha}\zeta}$ for all $n = 1,2,\cdots$. Hence we have

$\alpha \le x_n \le \frac{\alpha}{1-\beta-e^{-\delta}\gamma}$ and $\delta \le y_n \le \frac{\delta}{1-\epsilon-e^{-\alpha}\zeta}$ for all $n = 1,2,\cdots$. This completes the proof.

3 Existence of Invariant Set for Solutions

Theorem 3.1 Let $\{(x_n, y_n)\}$ be a positive solution of system of (1). Then ,

$\left[\alpha,\frac{\alpha}{1-\beta-e^{-\delta}\gamma}\right]\times\left[\delta,\frac{\delta}{1-\varepsilon-e^{-\alpha}\zeta}\right]$ is invariant set for system (1).

Proof. Let $\{(x_n,y_n)\}$ be a solution of system (1) with initial conditions $x_{-1},x_0\in I=\left[\alpha,\frac{\alpha}{1-\beta-e^{-\delta}\gamma}\right]$ and $y_{-1},y_0\in J=\left[\delta,\frac{\delta}{1-\varepsilon-e^{-\alpha}\zeta}\right]$. Then, it follows from system (1) that

$$x_1=\alpha+\beta x_0+\gamma x_{-1}e^{y_0}$$
$$\leq\alpha+\beta\frac{\alpha}{1-\beta-e^{-\delta}\gamma}+\gamma\frac{\alpha}{1-\beta-e^{-\delta}\gamma}e^{-\delta}=\frac{\alpha}{1-\beta-e^{-\delta}\gamma},$$

and

$$y_1=\delta+\varepsilon y_0+\zeta y_{-1}e^{x_0}$$
$$\leq\delta+\varepsilon\frac{\delta}{1-\varepsilon-e^{-\alpha}\zeta}+\zeta\frac{\delta}{1-\varepsilon-e^{-\alpha}\zeta}=\frac{\delta}{1-\varepsilon-e^{-\alpha}\zeta}.$$

Hence, $x_1\in I$ and $y_1\in J$. Suppose that result is true for $n=k>1$, i.e., $x_k\in I$ and $y_k\in J$. Then, for $n=k+1$ from system (1) one has:

$$x_{k+1}=\alpha+\beta x_k+\gamma x_{k-1}e^{-y_k}$$
$$\leq\alpha+\beta\frac{\alpha}{1-\beta-e^{-\delta}\gamma}+\gamma\frac{\alpha}{1-\beta-e^{-\delta}\gamma}e^{-\delta}=\frac{\alpha}{1-\beta-e^{-\delta}\gamma},$$

$$y_{k+1}=\delta+\varepsilon y_k+\zeta y_{k-1}e^{-x_k}\leq\delta+\varepsilon\frac{\delta}{1-\varepsilon-e^{-\alpha}\zeta}+\zeta\frac{\delta}{1-\varepsilon-e^{-\alpha}\zeta}=\frac{\delta}{1-\varepsilon-e^{-\alpha}\zeta}s$$

Hence, the proof follows by induction.

4 Existence and Uniqueness of Positive Equilibrium and Local Stability

Theorem 4.1 Assume that $e^{\delta}\beta+e^{\delta/2}\sqrt{e^{\delta}\beta^2+4\gamma}<2e^{\delta},e^{\alpha}\varepsilon+e^{\alpha/2}\sqrt{e^{\alpha}\varepsilon^2+4\zeta}<2e^{\alpha},0<\beta,\in<1$ and $\zeta\exp\left(\frac{-\varepsilon\delta-\delta\zeta\exp\left(-\frac{\alpha}{1-\beta-e^{-\delta}\gamma}\right)}{1-\varepsilon-\zeta\exp\left(-\frac{\alpha}{1-\beta-e^{-\delta}\gamma}\right)}\right)<\gamma<(1-\beta)\exp\left(\frac{\delta}{1-\varepsilon-e^{-\alpha}\zeta}\right)$.

Then the system (1) has a unique positive equilibrium point $(\bar{x},\bar{y})$ such that $\bar{x}\in\left[\alpha,\frac{\alpha}{1-\beta-\gamma e^{-\delta}}\right]=I$ and $\bar{y}\in\left[\delta,\frac{\delta}{1-\varepsilon-\zeta e^{-\alpha}}\right]=J$.

Proof. Consider the following system of equations:

$$x=\alpha+\beta x+\gamma xe^{-y},y=\delta+\varepsilon y+\gamma ye^{-x}$$

Then, one has $x=\frac{\alpha}{1-\beta-\gamma e^{-y}}$ and $y=\frac{\delta}{1-\varepsilon-\zeta e^{-x}}$. Taking $F(x)=\frac{\alpha}{1-\beta-\gamma e^{f(x)}}-x$, where $f(x)=\frac{\delta}{1-\varepsilon-\zeta e^{-x}}$ and $x\in I$. Then, it follows that

$F(\alpha)=\frac{\alpha(\beta e^{f(\alpha)+\zeta})}{(1-\beta)e^{f(\alpha)}-\zeta}$. Now, $F(\alpha)>0$ if and only if $(1-\beta)e^{f(\alpha)}-\zeta>0, i.e., \zeta<(1-\beta)\exp\left(\frac{\delta}{1-\varepsilon-\zeta e^{-\alpha}}\right)$.

Hence, it follows that $F(\alpha)>0$ if and only if $\zeta<(1-\beta)\exp\left(\frac{\delta}{1-\varepsilon-\zeta e^{-\alpha}}\right)$.

Furthermore, we have

$$\mathrm{F}\left(\frac{\alpha}{1-\beta-\gamma e^{-\delta}}\right)=\alpha\left(\frac{1}{1-\beta-\zeta e^{-f\left(\frac{\alpha}{1-\beta-\gamma e^{-\delta}}\right)}}-\frac{1}{1-\beta-\gamma e^{-\delta}}\right),$$

where $f\left(\frac{\alpha}{1-\beta-e^{-\delta}\gamma}\right)=\frac{\delta}{1-\varepsilon-\zeta e^{-\frac{\alpha}{1-\beta-\gamma e^{-\delta}}}}$. It is easy to see that $F\left(\frac{\alpha}{1-\beta-\gamma e^{-\delta}}\right)<0$ if and only if $\zeta e^{-f\left(\frac{\alpha}{1-\beta-\gamma e^{-\delta}}\right)}<\gamma e^{-\delta}$, i.e., $\zeta\exp\left(\frac{-\varepsilon\delta-\delta\zeta\exp\left(-\frac{\alpha}{1-\beta-\gamma e^{-\delta}}\right)}{1-\varepsilon-\zeta\exp\left(-\frac{\alpha}{1-\beta-\gamma e^{-\delta}}\right)}\right)<\gamma$.

It follows that $F\left(\frac{\alpha}{1-\beta-e^{-\delta}\gamma}\right)<0$, if and only if $\zeta\exp\left(\frac{-\varepsilon\delta-\delta\zeta\exp\left(-\frac{\alpha}{1-\beta-\gamma e^{-\delta}}\right)}{1-\varepsilon-\zeta\exp\left(-\frac{\alpha}{1-\beta-\gamma e^{-\delta}}\right)}\right)<\gamma$.

Hence, $F(x)$ has positive solution in I.

Moreover, we have that

$$\frac{dF}{dx}(x)=-1-\frac{e^{f(x)}\alpha\zeta f'(x)}{(e^{f(x)}(-1+\beta)+\zeta)^2}$$

$$=-1+\frac{e^{x+\frac{e^x\delta}{e^x(-1+\varepsilon)+\zeta}}\alpha\delta\zeta^2}{(e^x(-1+\varepsilon)+\zeta)^2(-1+\beta+e^{\frac{e^x\delta}{e^{x(-1+\varepsilon)+\zeta}}}\zeta)^2}<0.$$

Hence, $F(x)$ has a unique positive solution in I. The proof is therefore completed.

Consider a fourth-dimensional discrete dynamical system of the form:

$$\left.\begin{aligned}x_{n+1}&=f(x_n,y_n,x_{n-1},y_{n-1})\\ y_{n+1}&=g(x_n,y_n,x_{n-1},y_{n-1}),n=0,1,\cdots,\end{aligned}\right\}\qquad(4.1)$$

where $f\colon I^2\times J^2\to I$ and $g\colon I^2\times J^2\to J$ are continuously differentiable functions, and I,J are real intervals. Furthermore, a solution $\{(x_n,y_n)\}$ of system (4.1) is uniquely determined by initial conditions $(x_i,y_i)\in I\times J$ for $i\in\{-1,0\}$. Along with system (4.1) we consider a vector map of the form$F=(f,g,x_n,y_n)$. A constant solution or equilibrium point (fixed-point) of (4.1) is point that satisfies:

$$\bar{x}=f(\bar{x},\bar{y},\bar{x},\bar{y}),$$

$$\bar{y}=g(\bar{x},\bar{y},\bar{x},\bar{y}).$$

The point $(\bar{x},\bar{y})$ is also point called a fixed point of the vector map F.

Definition 4.1 Assume that $(\bar{x},\bar{y})$ be aconstant solution (steady-state) of (4.1).

1. A constant solution$(\bar{x},\bar{y})$ is called stable, if for an arbitrary $\varepsilon>0$ there exists a positiveδ such that for any initial condition $(\bar{x}_i,\bar{y}_i), i\in\{-1,0\}$ if $\|\sum_{i=-1}^{0}(x_i,y_i)-(\bar{x},\bar{y})\|<\delta$, then one has $\|(x_n,y_n)-(\bar{x},\bar{y})\|<\varepsilon$ for all $n>0$, where $\|\cdot\|$ denotes usual Euclidean norm in $\mathbb{R}^2$.
2. A constant solution$(\bar{x},\bar{y})$ is called unstable if it did not satisfy stability condition.
3. A constant solution$(\bar{x},\bar{y})$ is locally asymptotically stable, if there exists a positive η such that $\sum_{i=-1}^{0}\|(x_i,y_i)-(\bar{x},\bar{y})\|<\eta$ and $(x_n,y_n)\to(\bar{x},\bar{y})$ as $n\to\infty$.
4. A constant solution$(\bar{x},\bar{y})$ is said to be global attracter if $(x_n,y_n)\to(\bar{x},\bar{y})$ as $n\to\infty$.
5. A constant solution $(\bar{x},\bar{y})$ is said to be asymptotically global attractor if it is global attractor and stable.

Assume that$(\bar{x},\bar{y})$ be a fixed-point of map defined by F=(f,x_n,g,y_n), where f and g are continuously differentiable functions about $(\bar{x},\bar{y})$. The linearized system of (4.1)about the fixed-point $(\bar{x},\bar{y})$ is given by:

$$X_{n+1}=F(X_n)=F_JX_n\,,$$

where $X_n=\begin{pmatrix}x_n\\ y_n\\ x_{n-1}\\ y_{n-1}\end{pmatrix}$ and F_J is a Jacobain matrix of (4.1) about the fixed-point $(\bar{x},\bar{y})$.

Lemma 4.1 (Sedaghat, 2003) Consider the discrete dynamical system of the form $X_{n+1}=F(X_n), n=0,1,\cdots$, such that $\bar{X}$ be an equilibrium point of (4.1). Then following statements are true:

(i) If all eigenvalues of Jacobian J_F at $\bar{X}$ lie inside an open unit disc $|\lambda|<1$, then the fixed-point $\bar{X}$ is locally asymptotically stable.

(ii) If one of the eigenvalue of Jacobian matrix J_F has an absolute value greater than one, then $\bar{X}$ is unstable.

To construct corresponding linearized form of system (1) we consider the following transformation:

$$(x_n, y_n, x_{n-1}, y_{n-1}) \mapsto (f, g, f_1, g_1), \quad (4.2)$$

where $f = x_{n+1}, g = y_{n+1}, f_1 = x_n$ and $g_1 = y_n$. The Jacobian matrix about the fixed point $(\bar{x}, \bar{y})$ of (1) under the transformation (4.2) is given by

$$F_J(\bar{x}, \bar{y}) = \begin{pmatrix} \beta & -\gamma\bar{x}e^{-\bar{y}} & \gamma e^{-\bar{y}} & 0 \\ -\zeta\bar{y}e^{-\bar{x}} & \varepsilon & 0 & \zeta e^{-\bar{x}} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Theorem 4.2 The unique positive equilibrium point $(\bar{x}, \bar{y})$ of system (1) is locally asymptotically stable if the following condition is satisfied:

$$\beta + \epsilon + \beta\epsilon + \zeta(1+\beta)e^{-\alpha} + \gamma(1+\epsilon)e^{-\delta} + \gamma\zeta e^{-\alpha-\delta}\left(1 + \frac{\alpha\delta}{(1-\beta-e^{-\delta}\gamma)(1-\varepsilon-e^{-\alpha}\zeta)}\right) < 1. \quad (4.3)$$

Proof. The characteristic polynomial of Jacobain matrix $F_J(\bar{x}, \bar{y})$ about $(\bar{x}, \bar{y})$ is given by

$$P(\lambda) = \lambda^4 - (\beta+\epsilon)\lambda^3 - A\lambda^2 + B\lambda + \gamma\zeta e^{-\bar{x}-\bar{y}}, \quad (4.4)$$

where $A = \zeta e^{-\bar{x}} + \gamma e^{-\bar{y}} + \gamma\zeta\bar{x}\bar{y}e^{-\bar{x}-\bar{y}} - \beta\epsilon$ and $B = \beta\zeta e^{-\bar{x}} + \gamma\epsilon e^{-\bar{y}}$.

Let $\phi(\lambda) = \lambda^4$ and $\psi(\lambda) = (\beta+\epsilon)\lambda^3 + A\lambda^2 - B\lambda - \gamma\zeta e^{-\bar{x}-\bar{y}}$.

Assume that condition (4.3) is satisfied and $|\lambda| = 1$, then one has

$$\begin{aligned} |\psi(\lambda)| &< \beta + \epsilon + |A| + B + \gamma\zeta e^{-\bar{x}-\bar{y}} \\ &= \beta + \epsilon + |\zeta e^{-\bar{x}} + \gamma e^{-\bar{y}} + \gamma\zeta\bar{x}\bar{y}e^{-\bar{x}-\bar{y}} - \beta\epsilon| \\ &\quad + \beta\zeta e^{-\bar{x}} + \gamma\epsilon e^{-\bar{y}} + \gamma\zeta e^{-\bar{x}-\bar{y}} \\ &< \beta + \epsilon + \zeta e^{-\bar{x}} + \gamma e^{-\bar{y}} + \gamma\zeta\bar{x}\bar{y}e^{-\bar{x}-\bar{y}} + \beta\epsilon \\ &\quad + \beta\zeta e^{-\bar{x}} + \gamma\epsilon e^{-\bar{y}} + \gamma\zeta e^{-\bar{x}-\bar{y}} \\ &< \beta + \epsilon + \beta\epsilon + \zeta(1+\beta)e^{-\alpha} + \gamma(1+\epsilon)e^{-\delta} \\ &\quad + \gamma\zeta e^{-\alpha-\delta}\left(1 + \frac{\alpha\delta}{(1-\beta-e^{-\delta}\gamma)(1-\varepsilon-e^{-\alpha}\zeta)}\right) < 1. \end{aligned}$$

Then, by Rouche's theorem $\phi(\lambda)$ and $\phi(\lambda) - \psi(\lambda)$ have same number of zeroes in an open unit disc $|\lambda| < 1$. Hence, all the roots of (4.3) satisfies $|\lambda| < 1$, and it follows from Lemma 4.1 that the unique positive equilibrium point $(\bar{x}, \bar{y})$ of system (1) is locally asymptotically stable.

5 Global Stability

Theorem 5.1 The unique positive equilibrium point $(\bar{x}, \bar{y})$ of system (1) is globally asymptotically stable.

Proof. Let $\{(x_n, y_n)\}$ be an arbitrary positive solution of system (1) and let $0 < \mathrm{I}_1 = \lim_{n\to\infty} inf\, x_n$, $0 < \mathrm{I}_2 = \lim_{n\to\infty} inf\, y_n$, $\mathrm{S}_1 = \lim_{n\to\infty} sup\, x_n < \infty$, $\mathrm{S}_2 = \lim_{n\to\infty} sup\, y_n < \infty$. Then, from (1) one has:

$$\begin{aligned} \mathrm{I}_1 &\geq \alpha + \beta\mathrm{I}_1 + \gamma\mathrm{I}_1 e^{-S_2} \\ \mathrm{I}_2 &\geq \alpha + \beta\mathrm{I}_2 + \gamma\mathrm{I}_2 e^{-S_1} \\ \mathrm{S}_1 &\leq \alpha + \beta\mathrm{S}_1 + \gamma\mathrm{S}_1 e^{-\mathrm{I}_2} \\ \mathrm{S}_2 &\leq \alpha + \beta\mathrm{S}_2 + \gamma\mathrm{S}_2 e^{-\mathrm{I}_1} \end{aligned}$$

Furthermore,

$$\mathrm{I}_1 \geq \frac{\alpha}{1-\beta-\gamma e^{-S_2}},$$

$$\mathrm{I}_2 \geq \frac{\delta}{1-\epsilon-\zeta e^{-S_1}},$$

$$S_1 \le \frac{\alpha}{1-\beta-\gamma e^{-I_2}},$$

$$S_2 \le \frac{\delta}{1-\epsilon-\zeta e^{-I_1}}.$$

Then, it follows that:

$$I_1 S_2 \ge \frac{\alpha S_2}{1-\beta-\gamma e^{-S_2}},$$

$$I_2 S_1 \ge \frac{\delta S_1}{1-\epsilon-\zeta e^{-S_1}},$$

$$I_2 S_1 \le \frac{\alpha I_2}{1-\beta-\gamma e^{-I_2}},$$

$$I_1 S_2 \le \frac{\delta I_1}{1-\epsilon-\zeta e^{-I_1}}.$$

Hence one has $\frac{\alpha S_2}{1-\beta-\gamma e^{-S_2}} \le \frac{\delta I_1}{1-\epsilon-\zeta e^{-I_1}}and\frac{\delta S_1}{1-\epsilon-\zeta e^{-S_1}} \le \frac{\alpha I_2}{1-\beta-\gamma e^{-I_2}}$. Now consider the following functions:

$$f(x) = \frac{\delta x}{1-\epsilon-\zeta e^{-x}}\, and g(y) = \frac{\alpha y}{1-\beta-\gamma e^{-y}}, x\epsilon I, y \in J.$$

Moreover, for every $(x, y)\epsilon I \times J$:

$$\frac{df}{dx}(x) = -\frac{e^{-x} x\delta\zeta}{(1-\epsilon-e^{-x}\zeta)^2} + \frac{\delta}{(1-\epsilon-e^{-x}\zeta)} > 0,$$

$$\frac{dg}{dy}(y) = -\frac{e^{-y} y\alpha\gamma}{(1-\beta-e^{-y}\gamma)^2} + \frac{\delta}{(1-\beta-e^{-y}\gamma)} > 0.$$

Hence, $I_1 = S_1$ and $I_2 = S_2$.

6 Existence of Unbounded Solutions

Theorem 6.1 Let $\beta, \epsilon \in (0,1)$ and $\{(x_n, y_n)\}$ be a positive solution of system (3.1) Then following are true:

(i) If $\gamma > e^{\frac{\delta}{1-\epsilon}}, then\ x_n \to \infty, y_n \to \delta + \epsilon In(\gamma) as\ n \to \infty$.

(ii) If $\zeta > e^{\frac{\alpha}{1-\gamma}}, then\ x_n \to \alpha + \beta In(\zeta), y_n \to \infty\ as\ n \to \infty$.

Proof. (i) Let $\gamma > e^{\frac{\delta}{1-\epsilon}}$. Choosing the initial conditions x_0, x_{-1}, y_0, y_{-1} for the system (1) such that $x_0, x_{-1} > a$ and $y_0, y_{-1} < b$ with $= In\frac{b\zeta}{b-\delta-\epsilon b}, b = In\gamma$. Then from system (1) one has:

$y_1 = \delta + y_1\epsilon y_0 + \zeta y_{-1} e^{-x_0} < \delta + \epsilon b + \zeta b e^{-a} = b,$
and
$x_1 = \alpha + \beta x_0 + \gamma x_{-1} e^{y_0} > \alpha + \beta a + \gamma a e^{-b} = \alpha + a\beta + a > a.$

Suppose that result is true for $n = k > 1$, i.e., $x_k > a$and $y_k < b$. Then for $n = k + 1$ from system (1) one has:

$$x_{k+1} = \alpha + \beta x_k + \gamma x_{k-1} e^{-y_k} > \alpha + \beta a + \gamma a e^{-b} = \alpha + a\beta + a > a,$$

and

$$y_{k+1} = \delta + \epsilon y_k + \zeta y_{k-1} e^{-x_k} < \delta + \epsilon b + \zeta b e^{-a} = b.$$

Hence, $x_n > a, y_n < b$ for all $n = 1,2,\cdots$. Furthermore, $x_{n+1} = \alpha + \beta x_n + \gamma x_{n-1} e^{-y_n} > \alpha + \beta x_n + \gamma x_{n-1} e^{-b} = \alpha + \beta x_n + x_{n-1}$. Let $z_{n+1} = \alpha + \beta z_n + z_{n-1}$ with initial conditions z_{-1}, z_0. Then,$z_n = -\frac{\alpha}{\beta} +$

$2^{-n}(\beta - \sqrt{4+\beta^2})^n z_0 + 2^{-n}(\beta + \sqrt{4+\beta^2})^n z_{-1}$. Hence by comparison $x_n > z_n$. But $z_n \to \infty asn \to \infty$, so $\lim_{n\to\infty} x_n = \infty$. Moreover, $\delta \le y_{n+1} = \delta + \epsilon y_n + \zeta y_{n-1} e^{-x_n} < \delta + \epsilon b + \zeta b e^{-x_n}$. Hence, $\lim_{n\to\infty} y_n = \delta + \epsilon In(\gamma)$.

(ii) The proof of (ii) is similar, therefore it is omitted.

7 Rate of Convergence

To discuss the rate of convergence of discrete dynamical system, we consider the following system:

$$Z_{n+1} = \big(K + L(n)\big) Z_n, \tag{7.1}$$

where Z_n is a k-dimensional vector, $K \in C^{k\times k}$ is a constant matrix, and $L: \mathbb{Z}^+ \to C^{k\times k}$ is a matrix function such that:

$$\|L(n)\| \to 0, \tag{7.2}$$

as$n \to \infty$, where $\|.\|$ denotes a matrix norm which associated with the vector norm of the form:

$$\|(u,v)\| = \sqrt{u^2+v^2}.$$

Lemma 7.1 (Perron's theorem) (Pituk, 2002) Assume that the condition (7.2) holds true. If Z_n be a solution of (7.1) then either $Z_n = 0$ for all$n \to \infty$, or

$$\tau = \lim_{n\to\infty} (\|Z_n\|)^{1/n}$$

is defined and it is equal to the absolute value of one of the eigenvalues of matrix K.

Lemma 7.2 (Pituk, 2002) Assume that condition (7.2) holds. If Z_n is a solution of (7.1), then either $Z_n = 0$ for all$n \to \infty$, or

$$\tau = \lim_{n\to\infty} \frac{\|Z_{n+1}\|}{\|Z_n\|},$$

is defined and is equal to the absolute value of one the eigenvalues of matrix K.

Let $\{(x_n, y_n)\}$ be any solution of the system (1) such that $\lim_{n\to\infty} x_n = \bar{x}$ and $\lim_{n\to\infty} y_n = \bar{y}$, where $\bar{x} \in \left[\alpha, \frac{\alpha}{1-\beta-\gamma e^{-\delta}}\right]$ and $\bar{y} \in \left[\delta, \frac{\delta}{1-\beta-\gamma e^{-\delta}}\right]$. To find the error terms, one has from the system (1)

$$x_{n+1} - \bar{x} = \beta x_n + \gamma x_{n-1} e^{-y_n} - \beta\bar{x} - \gamma\bar{x} e^{-\bar{y}}$$

$$= \beta(x_n - \bar{x}) + \frac{\gamma\bar{x}(e^{-y_n} - e^{-\bar{y}})}{y_n - \bar{y}}(y_n - \bar{y}) + \gamma e^{-y_n}(x_{n-1} - \bar{x}),$$

and

$$y_{n+1} - \bar{y} = \epsilon y_n + \zeta y_{n-1} e^{-x_n} - \epsilon\bar{y} - \zeta\bar{y} e^{-\bar{x}}$$

$$= \frac{\zeta\bar{y}(e^{x_n} - e^{\bar{x}})}{x_n - \bar{x}}(x_n - \bar{x}) + \epsilon(y_n - \bar{y}) + \zeta e^{-x_n}(y_{n-1} - \bar{y}).$$

Let $e_n^1 = x_n - \bar{x}$, and $e_n^2 = y_n - \bar{y}$, then one has

$$e_{n+1}^1 = a_n e_n^1 + b_n e_n^2 + c_n e_{n-1}^1 + d_n e_{n-1}^2,$$

and

$$e_{n+1}^2 = f_n e_n^1 + g_n e_n^2 + h_n e_{n-1}^1 + k_n e_{n-1}^2,$$

where

$$a_n = \beta, b_n = \frac{\gamma\bar{x}(e^{-y_n} - e^{-\bar{y}})}{y_n - \bar{y}}, c_n = \gamma e^{-y_n}, d_n = 0, f_n = \frac{\zeta\bar{y}(e^{-x_n} - e^{-\bar{x}})}{x_n - \bar{x}}, g_n = \zeta e^{-x_n},$$
$$h_n = 0, \quad k_n = \epsilon.$$

Moreover,

$$\lim_{n\to\infty} a_n = \beta, \lim_{n\to\infty} b_n = -\gamma\bar{x}e^{-\bar{y}}, \lim_{n\to\infty} c_n = \gamma e^{-\bar{y}}, \lim_{n\to\infty} d_n = 0, \lim_{n\to\infty} f_n = -\zeta\bar{y}e^{-\bar{x}},$$

$lim_{n\to\infty} g_n = \epsilon, lim_{n\to\infty} h_n = 0, lim_{n\to\infty} k_n = \zeta e^{-\bar{x}}.$

Now the limiting system of error terms can be written as

$$\begin{bmatrix} e^1_{n+1} \\ e^2_{n+1} \\ e^1_n \\ e^2_n \end{bmatrix} = \begin{pmatrix} \beta & -\gamma\bar{x}e^{-\bar{y}} & \gamma e^{-\bar{y}} & 0 \\ -\zeta\bar{y}e^{-\bar{x}} & \varepsilon & 0 & \zeta e^{-\bar{x}} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{bmatrix} e^1_n \\ e^2_n \\ e^1_{n-1} \\ e^2_{n-1} \end{bmatrix},$$

which is similar to linearized system of (1) about the equilibrium point $(\bar{x}, \bar{y})$.

Using Lemma 7.1 one has the following result.

Theorem 7.1 Assume that $\{(x_n, y_n)\}$ be a solution of (1) such that $\lim_{n\to\infty} x_n = \bar{x}$, $\lim_{n\to\infty} y_n = \bar{y}$, where $(\bar{x}, \bar{y})$ be unique equilibrium point of (1), then the error vector $E_n = \begin{bmatrix} e^1_{n+1} \\ e^1_n \\ e^2_{n+1} \\ e^2_n \end{bmatrix}$ of every solution of system (1) satisfies both of the following asymptotic relations:

$$\lim_{n\to\infty}(\|E_n\|)^{1/n} = |\lambda_{1,2,3,4}F_J(\bar{x}, \bar{y})|,$$

$$\lim_{n\to\infty} \frac{\|E_{n+1}\|}{\|E_n\|} = |\lambda_{1,2,3,4}F_J(\bar{x}, \bar{y})|,$$

where $\lambda_{1,2,3,4}F_J(\bar{x}, \bar{y})$ are the characteristic roots of the Jacobian matrix$F_J(\bar{x}, \bar{y})$.

8 Examples

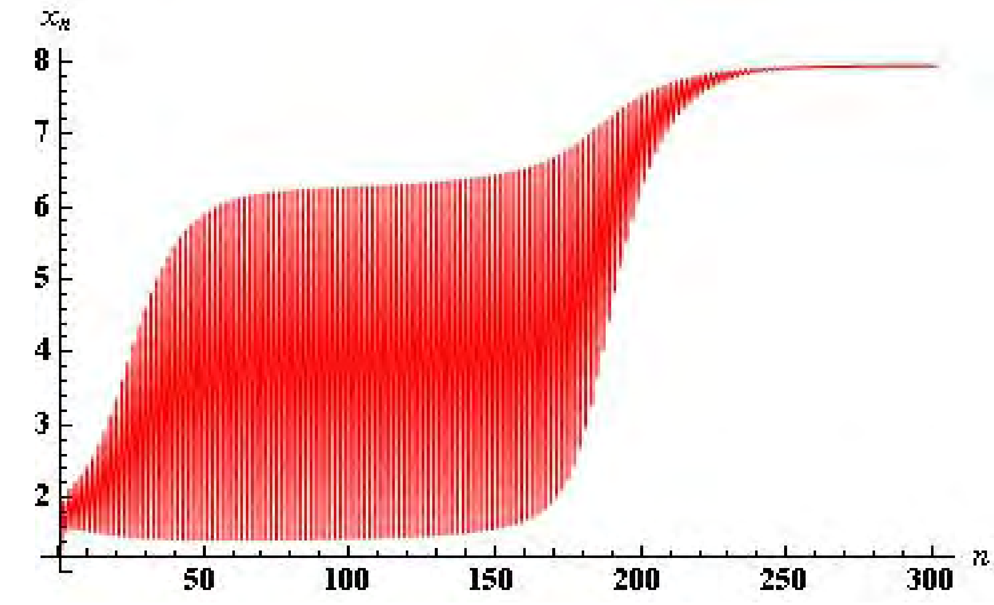
In this section, we consider the following numerical examples.

Example 8.1 Let $\alpha = 1.1$, $\beta = 0.01, \gamma = 2.9, \delta = 1.2, \epsilon = 0.02, \zeta = 2.304$.Then, system (1) can be written as

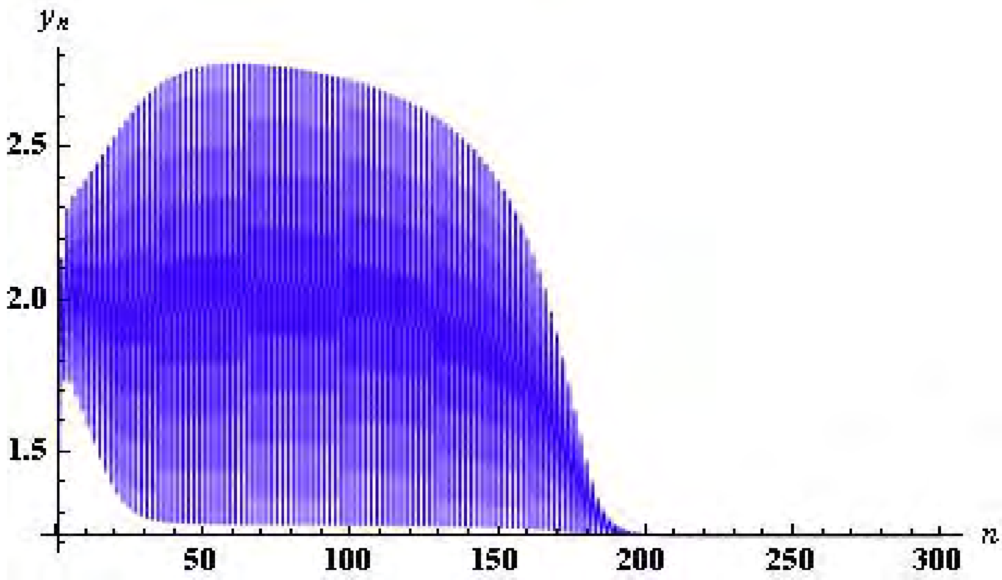
$x_{n+1} = 1.1 + 0.01x_n + 2.9x_{n-1}e^{-y_n}, y_{n+1} = 1.2 + 0.02y_n + 2.9y_{n-1}e^{-x_n}$(8.1)

with initial conditions $x_{-1} = 1.3, x_0 = 1.2, y_{-1} = 1.3, y_0 = 1.5$.

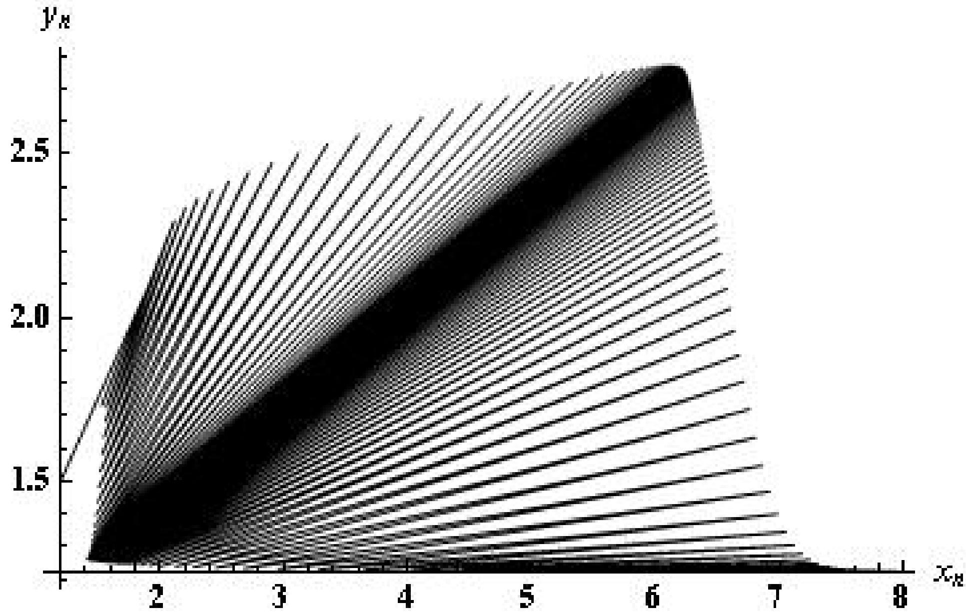
In this case the unique positive equilibrium point of the system (8.1)is given by $(\bar{x}, \bar{y}) = (7.93981, 1.22552)$. Moreover, in Fig. 8.1 the plot of x_n is shown in Fig. 8.1a, the plot of y_n is shown in Fig. 8.1b, and an attractor of the system is shown in Fig. 8.1c.



(8.1a) Plot of x_n for the system (8.1)



(8.1b) Plot of y_n for the system (8.1)



(8.1c) An attractor of the system (8.1)

Fig. 8.1 Plots for the system (8.1).

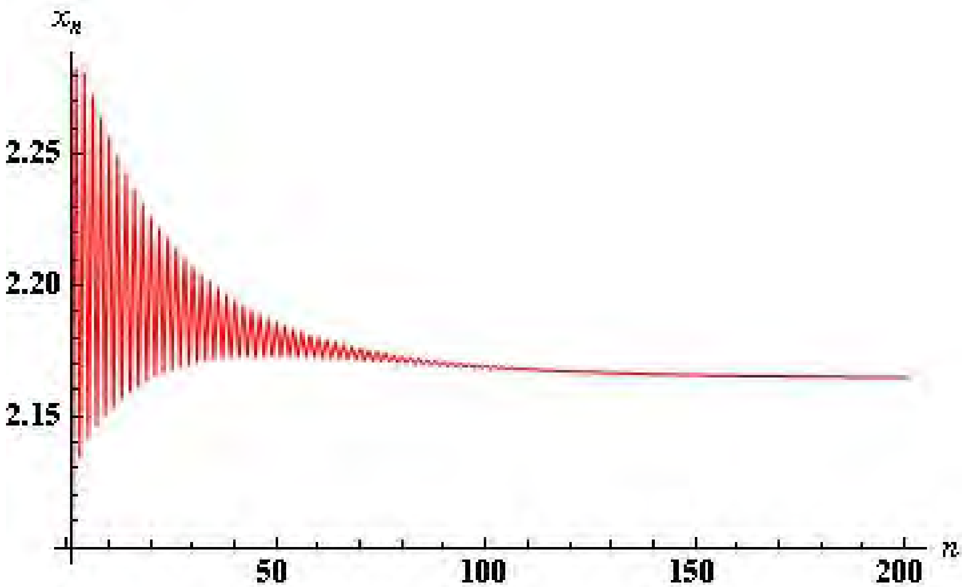
Example 8.2 Let $\alpha = 1.5$, $\beta = 0.026, \gamma = 3.14, \delta = 1.67, \epsilon = 0.0096, \zeta = 2.6$. Then, system (1) can be written as

$x_{n+1} = 1.5 + 0.026x_n + 3.14x_{n-1}e^{-y_n}$,

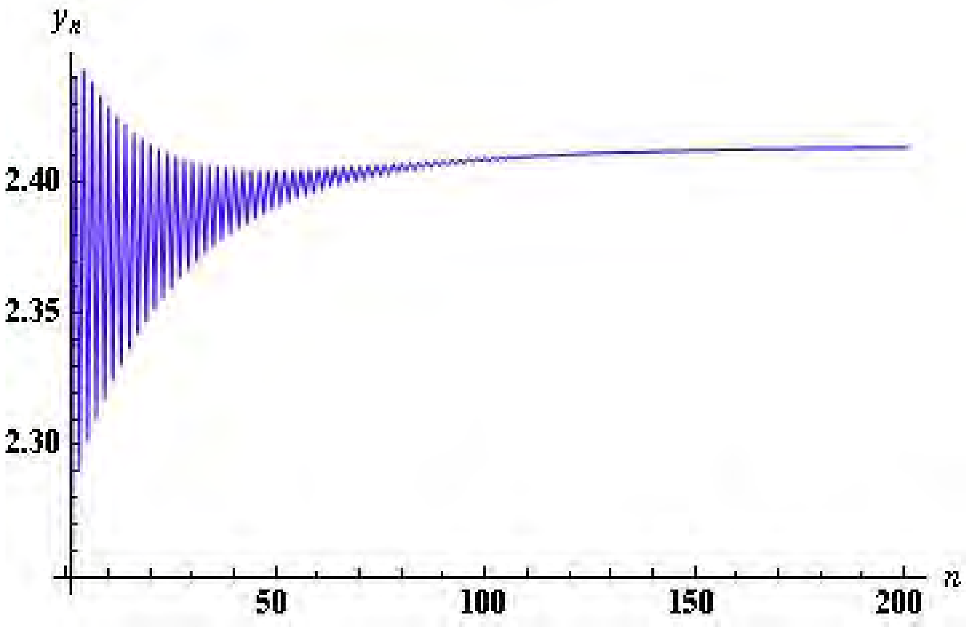
$y_{n+1} = 1.67 + 0.0096y_n + 2.6y_{n-1}e^{-x_n}$ (8.2)

with initial conditions $x_{-1} = 2.2, x_0 = 2.1, y_{-1} = 2.35, y_0 = 2.25$.

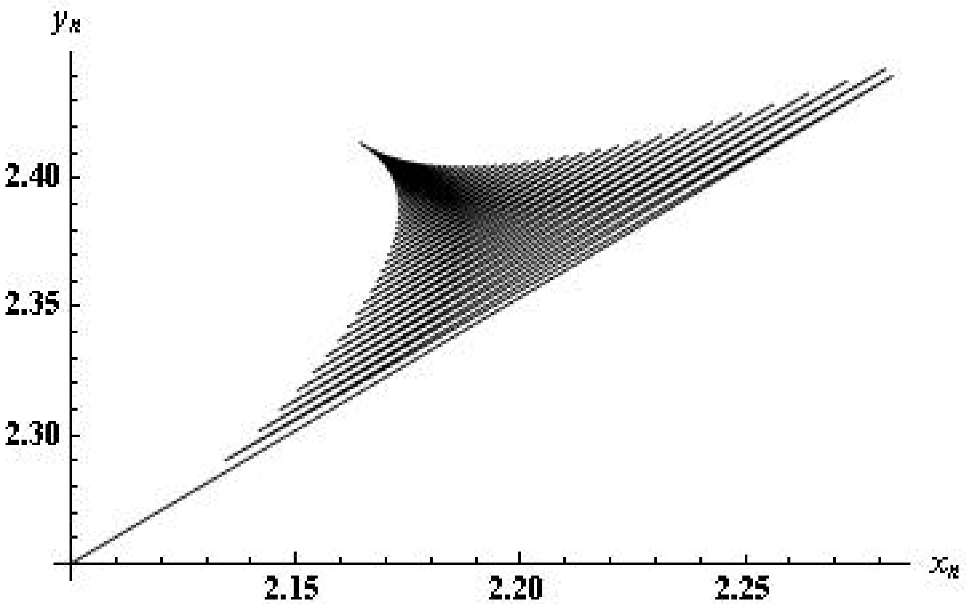
In this case the unique positive equilibrium point of the system (8.2) is given by $(\bar{x}, \bar{y}) = (2.16435, 2.41379)$. Moreover, in Fig. 8.2 the plot of x_n is shown in Fig. 8.2a, the plot of y_n is shown in Fig. 8.2b, and an attractor of the system is shown in Fig. 8.2c.



(8.2a) Plot of x_n for the system (8.2)



(8.2b) Plot of y_n for the system (8.2)



(8.2c) An attractor of the system (8.2)

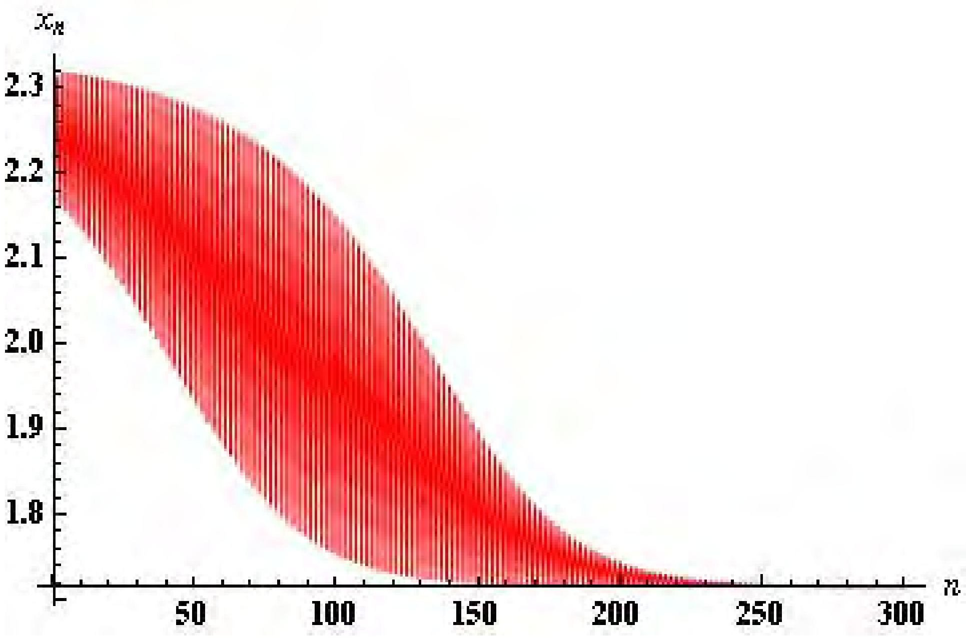
Fig.8.2 Plots for the system (8.2)

Example 8.3 Let $\alpha = 1.5,\ \beta = 0.0001, \gamma = 3.5, \delta = 1.67, \epsilon = 0.00002, \zeta = 2.77$. Then, system (1) can be written as

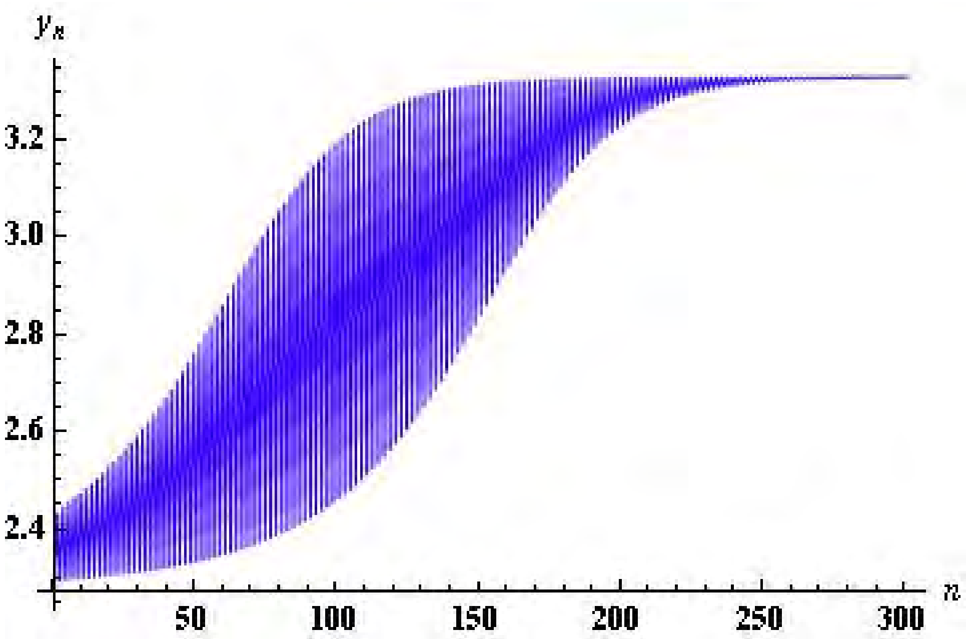
$$x_{n+1} = 2.3 + 0.0001x_n + 3.5x_{n-1}e^{-y_n}$$
$$y_{n+1} = 1.67 + 0.00002y_n + 2.77y_{n-1}e^{-x_n} \quad (8.3)$$

with initial conditions $x_{-1} = 2.25, x_0 = 2.15, y_{-1} = 2.36, y_0 = 2.27$. In this case the unique positive equilibrium point of the system (8.3) is given by $(\bar{x}, \bar{y}) = (1.71494, 3.33035)$.

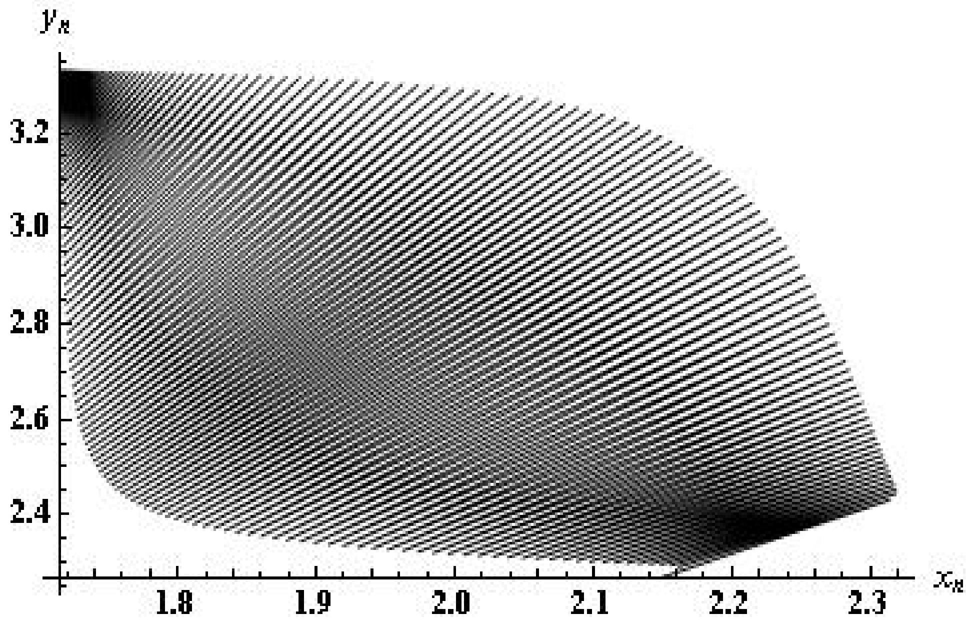
Moreover, in Fig. 8.3 the plot of x_n is shown in Fig. 8.3a, the plot of y_n is shown in Fig. 8.3b, and an attractor of the system is shown in Fig. 8.3c.



(8.3a) Plot of x_n for the system (8.3)



(8.3b) Plot of y_n for the system (8.3)



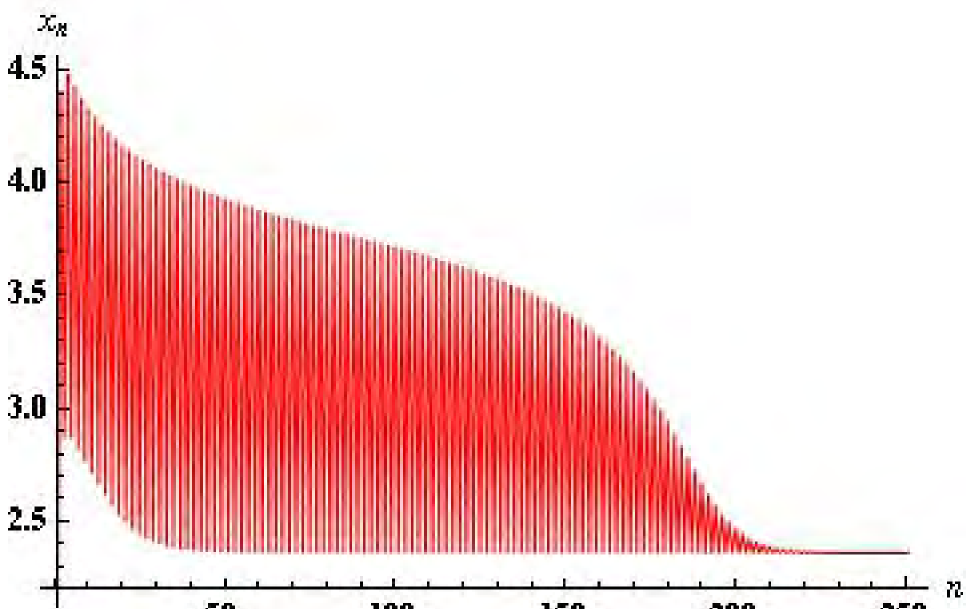
(8.3c) An attractor of the system (8.3)

Fig. 8.3 Plots for the system (8.3).

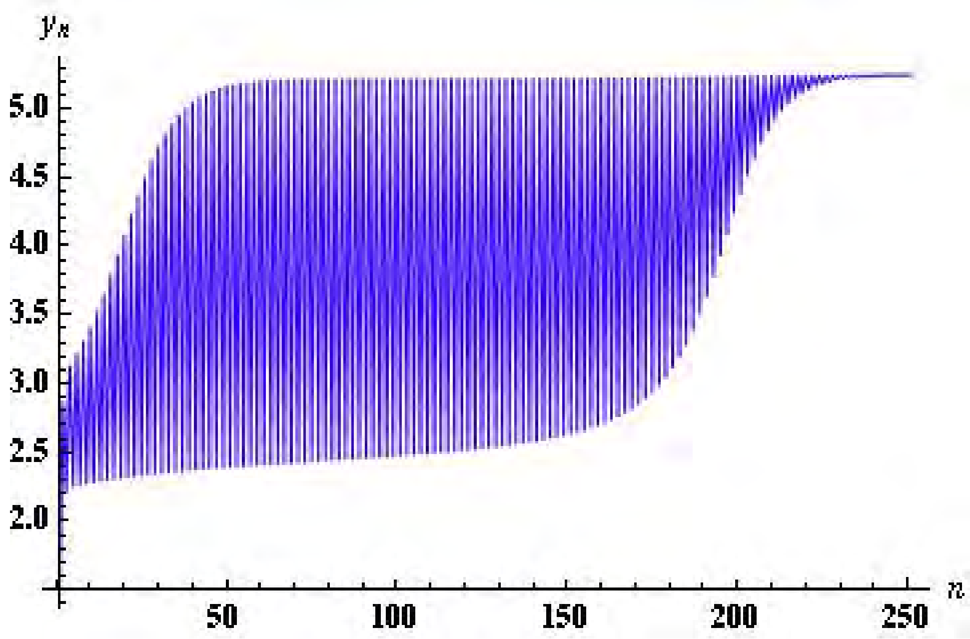
Example 8.4 Let $\alpha = 2.3, \beta = 0.0009, \gamma = 4.5, \delta = 2.09, \epsilon = 0.0008, \zeta = 6.35$. Then system (1) can be written as:

$$x_{n+1} = 2.3 + 0.0009x_n + 4.5x_{n-1}e^{-y_n},$$
$$y_{n+1} = 2.09 + 0.0008y_n + 6.35y_{n-1}e^{-x_n} \quad (8.4)$$

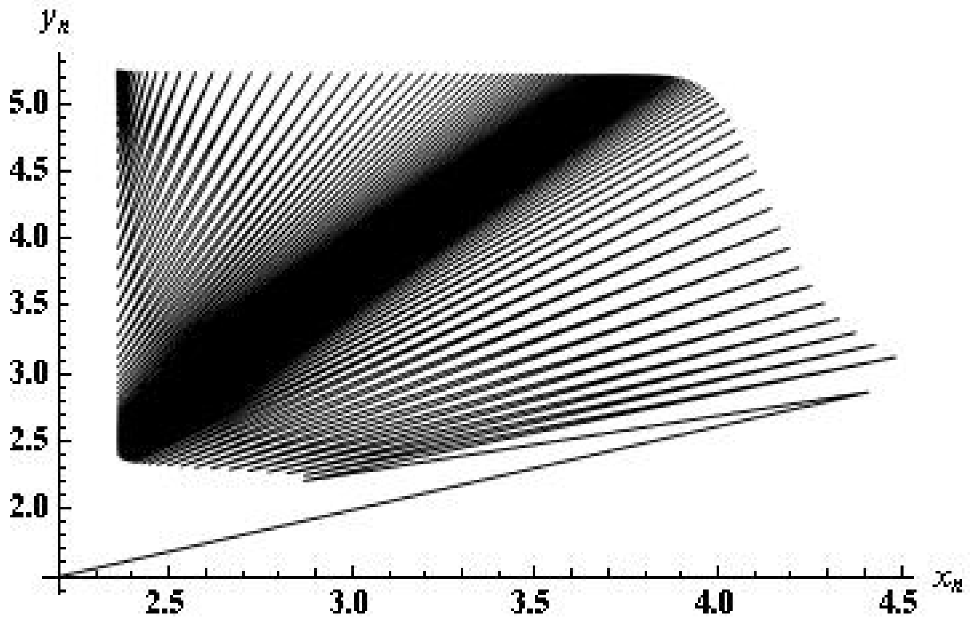
with initial conditions $x_{-1} = 2.1, x_0 = 2.2, y_{-1} = 1.1, y_0 = 1.5$. In this case the unique positive equilibrium point of the system (8.4) is given by $(\bar{x}, \bar{y}) = (2.3581, 5.24482)$. Moreover, in Fig. 8.4 the plot of x_n is shown in Fig. 8.4a,the plot of y_n is shown in Fig. 8.4b,and an attractor of the system (8.4) is shown in Fig. 8.4c.



(8.4a) Plot of x_n for the system (8.4)



(8.4b) Plot of y_n for the system (8.4)



(8.4c) An attractor of the system (8.4)

Fig. 8.4 Plots for the system (8.4).

9 Conclusion

This work is related to qualitative behavior of a four-dimensional exponential discrete population model. We prove that system (1) has a unique positive equilibrium point which is globally asymptotically stable. Usually, biologists believe that a globally asymptotically stable equilibrium point is very important in ecological point of view. Method of linearization is used for local asymptotic stability of steady-state of (1). Furthermore, we prove the boundedness and persistence of positive solutions of (1), and an invariant set for its solutions is investigated. We prove the rate of convergence of positive solutions of system (1) which converge to its unique positive equilibrium point.

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