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## Turing and Non-Turing patterns in diffusive plankton model

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### Abstract

In this paper, we investigate a Rosenzweig-McAurthur model and its variant for phytoplankton, zooplankton and fish population dynamics with Holling type II and III functional responses. We present the theoretical analysis of processes of pattern formation that involves organism distribution and their interaction of spatially distributed population with local diffusion. The choice of parameter values is important to study the effect of diffusion, also it depends more on the nonlinearity of the system. With the help of numerical simulations, we observe the formation of spatiotemporal patterns both inside and outside the Turing space.

**Keywords** Rosenzweig-McAurthur model; plankton system; functional response; spatiotemporal pattern; Turing domain.

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### 1 Introduction

Conceptual predator-prey models have successfully been used to model phytoplankton-zooplankton interactions and to elucidate mechanisms of spatiotemporal pattern formation like patchiness and blooming (Upadhyay et al., 2010, 2012; Segel and Jackson, 1972; Steele and Henderson, 1981, 1992). The density of plankton population changes not only in time but also in space. The highly inhomogeneous spatial distribution of plankton population in the natural aquatic system called “*Plankton patchiness*” has been observed in many field observations (Greene et al., 1992; Abbott, 1993). A wide variety of spatial and temporal patterns inhibiting dispersal are present in many real ecological systems (Zhang, 2012). The spatiotemporal self organization in plankton communities modeled by reaction-diffusion equations have always been an area of interest for many researchers. The reaction diffusion systems often exhibit diffusion driven instability or Turing instability. It occurs when a homogeneous steady state which is stable for small perturbations in the absence of diffusion becomes unstable in the presence of diffusion. Turing patterns are stationary structures that appear spontaneously upon breaking the symmetry of the medium, which results only from the coupling between the reaction and the diffusion processes. In the absence of diffusion, these systems tend to a linearly

stable uniform steady state. Once Turing patterns arise, they may remain stable until some external perturbation destroys them, but after that perturbation stops, Turing patterns reappear and reorganize themselves. A steady state that is stable in a non-spatial model may become unstable in the corresponding spatial system. Then, after the homogeneity is broken due to Turing instability, the nonlinear interactions between the components drive the system into the formation of standing spatial patterns (Nicolis and Prigogine, 1977) which is an irreversible process. However, a kind of inverse process (anti-Turing phenomena) may take place for some parameter values. That is, a locally unstable equilibrium of the non-spatial system may become dynamically stable in the spatial system. In this case, for certain time and length scale, and equal diffusion coefficients  $d_1 = d_2 = 1$ , formation of spatial patterns is suppressed and homogeneity is restored (Malchow et al., 2008). Malchow et al. (2008) have also observed the following: (i) Turing patterns are sensitive to the boundary conditions. (ii) Boundary conditions may affect both the transient stage and large-time asymptotic. (iii) Generally, initial conditions affect only the transient stage of pattern formation. In a spatially bounded system, a stationary pattern forms only when the boundary conditions are consistent with the intrinsic properties of the pattern, size and shape of the domain. Pattern formation results when a parameter changing with time enters into the domain of parameter space where the system becomes unstable due to perturbation.

Chen and Shi (2012) studied a spatial Holling-Tanner model and proved the global stability of a unique constant equilibrium under a simple parameter condition. Zhang et al. (2008) demonstrated that two species spatial systems could be useful to explain spatio-temporal behavior of populations whose dynamics is strongly affected by noise and the environmental physical variables. These local movements were modeled by Fickian diffusion terms. Applicability of diffusion terms to describe re-distribution of species in space due to random motion of the individuals for any value of population density was shown by Okubo (1980). Since the dispersal rates are assumed to be the same for the prey and predator, the patterns cannot appear due to the Turing instability (Petrovskii and Malchow, 2001; Malchow et al., 2002). Whether the spatial gradient affecting the growth rate of the prey species can provide the coupling mechanism for the oscillatory predator-prey dynamics was examined by Pascual (2002). Two coupled non-chaotic oscillators can admit chaotic dynamics. It was found that the diffusive movements of the species create a system of coupled oscillators that mutually force each other at incommensurate frequencies. This results in chaotic dynamics. Diffusion and spatial heterogeneity introduce qualitatively new types of behavior in predator-prey interaction. Diffusion on a spatial gradient may drive a cyclic predator-prey system into chaotic behavior. Ecological models of diffusion driven instability with spatial heterogeneities have been studied for a variety of reasons (Medvinsky et al., 2002). Malchow et al. (2000) studied the spatiotemporal dynamics of the plankton model under the influence of environmental noise and diffusion in horizontally two-dimensional spaces. Movement of phytoplankton and zooplankton population with different velocities can give rise to spatial patterns (Alonso et al., 2002; Malchow, 2000). Upadhyay et al. (2010, 2011, 2012) investigated the pattern formation in a spatial plankton-fish system.

In this paper, we have considered a spatial Rosenzweig-McAurthur model and its variant for modeling the plankton dynamics in aquatic environment. We assume that the local growth of the prey is logistic and that the predator shows the Holling type II functional response for planktonic systems (Medvinsky et al., 2002). Analytically, we have studied the temporal model and obtained the condition for Turing instability. The main objective of this paper is to see the spatiotemporal pattern formation within and outside the Turing space.

## 2 Model System

We consider a reaction diffusion model for plankton system where at any location  $(X, Y)$  and time  $T$ , the phytoplankton  $N(X, Y, T)$  and zooplankton  $P(X, Y, T)$  populations satisfy the following

$$\begin{aligned}\frac{\partial N}{\partial T} &= rN \left(1 - \frac{N}{K}\right) - \frac{cNP}{a+N} + D_N \nabla^2 N, \\ \frac{\partial P}{\partial T} &= \frac{cNP}{a+N} - mP + D_P \nabla^2 P.\end{aligned}\tag{1}$$

The parameters appearing in the equations are explained below

$r$ : the phytoplankton's intrinsic growth rate in the absence of predation,

$K$ : the carrying capacity of phytoplankton population,

$c$ : the rate at which phytoplankton is grazed and it follows Holling type-II functional response,

$a$ : the half-saturation constant for phytoplankton density,

$m$ : the mortality rate of zooplankton,

$D_N$  and  $D_P$ : the diffusion coefficient of both phytoplankton and zooplankton density respectively.

The units of the parameters are as follows. Time  $T$  and length  $X, Y \in [0, L]$  are measured in days [ $d$ ] and meters [ $m$ ].  $r, N, P, K$  and  $a$  are usually measured in mg of dry weight per liter [ $mg \cdot dw/l$ ], the dimension of  $c$  is in [ $d^{-1}$ ],  $m$  is measured in [ $d^{-1}$ ] respectively. The diffusion coefficients  $D_N$  and  $D_P$  are measured in [ $m^2 d^{-1}$ ].

Here  $\nabla^2 = \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)$  in two dimensions and  $\nabla^2 = \frac{\partial^2}{\partial x^2}$  in one dimension case.

We introduce the following substitutions and notations to bring the system of equations into nondimensional form

$$u = \frac{N}{K}, v = \frac{cP}{rK}, t = rT, x = \frac{X}{L}, y = \frac{Y}{L}, \alpha = \frac{a}{K}, \beta = \frac{c}{r}, \delta = \frac{m}{c}, d_1 = \frac{D_N}{rL^2}, d_2 = \frac{D_P}{rL^2}.$$

The model system in dimensionless form can be written as

$$\begin{aligned}\frac{\partial u}{\partial t} &= u(1-u) - \frac{uv}{u+\alpha} + d_1 \nabla^2 u, \\ \frac{\partial v}{\partial t} &= \beta v \left( \frac{u}{u+\alpha} - \delta \right) + d_2 \nabla^2 v,\end{aligned}\tag{2}$$

with non-zero initial conditions

$$u(x, y, 0) > 0, v(x, y, 0) > 0, (x, y) \in \Omega = [0, R] \times [0, R],\tag{3}$$

and the zero-flux boundary conditions

$$\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0, (x, y) \in \partial\Omega \text{ for all } t \text{ and } n \text{ is the outward normal to } \partial\Omega.\tag{4}$$

The zero flux boundary conditions are used for modeling the dynamics of spatially bounded aquatic ecosystems (Scheffer, 1998). It was shown by Dubey et al., (2009), that the results on global asymptotic stability in one dimension are also valid in the two dimensional case. Further, the solutions of the system converge faster to its equilibrium in the case of two-dimensional diffusion in comparison to the one dimensional case.

### 3 Stability Analysis of the Model System

First, consider the case without diffusion,  $d_1 = d_2 = 0$ , we restrict ourselves to the stability analysis of the

model system in the absence of diffusion in which only the interaction part of the model system are taken into account. We find the non-negative equilibrium states of the model system and discuss their stability properties with respect to variation of several parameters.

In this case, the model system reduces to the form

$$\begin{aligned} \frac{du}{dt} &= u(1-u) - \frac{uv}{u+\alpha}, \\ \frac{dv}{dt} &= \beta v \left( \frac{u}{u+\alpha} - \delta \right). \end{aligned} \tag{5}$$

The equilibrium points of model system (5) can be obtained by solving  $du/dt = 0, dv/dt = 0$ . It can be seen that model system has three non-negative equilibria, namely,  $E_0(0, 0)$ ,  $E_1(1, 0)$  and  $E^*(u^*, v^*)$ . We study the nature of each equilibrium point of the model system; we compute the variational matrices corresponding to each equilibrium point.

From these matrices, the following results are obtained.

- (i)  $E_0$  is a saddle point with an unstable manifold along  $u$ -direction and stable manifold along  $v$ -direction.
- (ii) If  $1 - \delta(1 + \alpha) < 0$ ,  $E_1$  is locally asymptotically stable in  $u$ - $v$  plane. If  $1 - \delta(1 + \alpha) > 0$ , it is a saddle point with an stable manifold along  $u$ -direction and unstable manifold along  $v$ -direction.
- (iii) The existence of positive equilibrium point  $E^*(u^*, v^*)$  can be shown as follows:  
 $u^* = \alpha\delta / (1 - \delta)$ ,  $v^* = (u^* + \alpha)(1 - u^*) = \alpha \{1 - \delta(1 + \alpha)\} / (1 - \delta)^2$ .

The existence of  $E^*$  depends on the condition

$$F_1(\alpha, \delta) = 1 - \delta(1 + \alpha) > 0. \tag{6}$$

We propose the theorem for the necessary and sufficient conditions of the positive equilibrium point  $E^*(u^*, v^*)$  to be locally asymptotically stable. The stability of the steady states is determined by the Routh-Hurwitz criteria.

The Jacobian matrix associated with the system (5) is given by

$$J_{E^*(u^*, v^*)} = \begin{pmatrix} 1 - 2u^* - \frac{\alpha v^*}{(u^* + \alpha)^2} & -\frac{u^*}{u^* + \alpha} \\ \frac{\alpha \beta v^*}{(u^* + \alpha)^2} & \beta \left( \frac{u^*}{u^* + \alpha} - \delta \right) \end{pmatrix} = \begin{pmatrix} -\delta(\delta(1 + \alpha) + \alpha - 1) / (1 - \delta) & -\delta \\ \beta(1 - \delta(1 + \alpha)) & 0 \end{pmatrix}.$$

The eigenvalues are the roots of the characteristic equation  $\lambda^2 + A_1\lambda + A_2 = 0$ .

where

$$A_1 = \delta(\delta(1 + \alpha) + \alpha - 1) / (1 - \delta),$$

$$A_2 = \delta\beta(1 - \delta(1 + \alpha)).$$

Applying the Routh-Hurwitz criteria, the homogeneous state  $(u^*, v^*)$  is stable iff  $A_1 > 0$  and  $A_2 > 0$ .

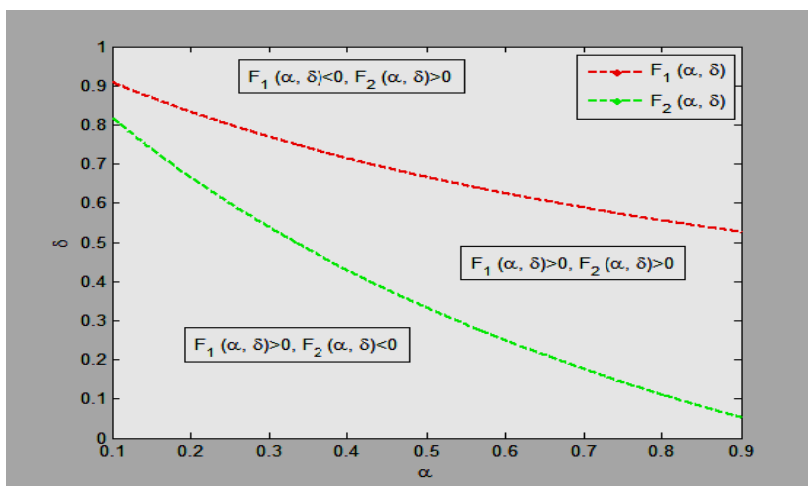
Now  $A_2 > 0$ , if eq. (6) is satisfied. The condition  $A_1 > 0$  is satisfied if

$$F_2(\alpha, \delta) = \delta(1 + \alpha) + \alpha - 1 > 0, \text{ provided that } \delta < 1. \tag{7}$$

**Theorem 1** The unique nontrivial positive equilibrium point  $E^*(u^*, v^*)$  is locally asymptotically stable provided the following inequality hold:

$$(1 - \alpha) < \delta(1 + \alpha) < 1. \quad (8)$$

The equilibrium point  $E^*(u^*, v^*)$  is locally asymptotically stable if it satisfies the eq. (6) and (7). From Fig. 1 we observe that if parameters value lies above the green line and below the red line, the equilibrium point  $E^*(u^*, v^*)$  corresponding to the model system (5) is locally asymptotically stable.



**Fig.1** The parametric domain for  $F_1$  and  $F_2$  provided  $\delta < 1$ .

To study the effect of diffusion on the model system, consider the linearized form of the system about the positive equilibrium point  $E^*(u^*, v^*)$ . Instability will occur due to diffusion when a parameter varies slowly in such a way that a stability condition is suddenly violated and it can bring about a situation wherein perturbation of a non-zero (finite) wavelength starts growing (perturbations of zero wave number are stable when diffusive instability sets in) (Dubey et al., 2001; Serizawa et al., 2008). Turing instability can occur for the model system because the equation for predator is nonlinear with respect to predator population,  $P$  (Serizawa et al., 2008).

To study the effect of diffusion on the model system, we perturb the steady state  $(u^*, v^*)$  by setting  $u = u^* + U$ ,  $v = v^* + V$ . The linearized form of the equations are obtained as

$$\begin{aligned} \frac{\partial U}{\partial t} &= b_{11}U + b_{12}V + d_1 \nabla^2 u, \\ \frac{\partial V}{\partial t} &= b_{21}U + b_{22}V + d_2 \nabla^2 v. \end{aligned} \quad (9)$$

where

$$\begin{aligned} b_{11} &= 1 - 2u^* - \frac{\alpha v^*}{(u^* + \alpha)^2} = -\delta(\delta(1 + \alpha) + \alpha - 1)/(1 - \delta), & b_{12} &= -\frac{u^*}{u^* + \alpha} = -\delta, \\ b_{21} &= \frac{\alpha \beta v^*}{(u^* + \alpha)^2} = \beta(1 - \delta(1 + \alpha)), & b_{22} &= \beta \left( \frac{u^*}{u^* + \alpha} - \delta \right) = 0. \end{aligned}$$

The solution to the above equations can be assumed in the following form:

$$\begin{aligned} U(t, x, y) &= se^{\lambda t} \cos(k_x x) \cos(k_y y), \\ V(t, x, y) &= we^{\lambda t} \cos(k_x x) \cos(k_y y). \end{aligned} \quad (10)$$

where  $\lambda$  and  $k$  (where  $k^2 = k_x^2 + k_y^2$ ) are the frequency and wave number respectively. Substituting the expressions for  $U$  and  $V$  in eq. (9). The homogeneous equation in  $s$  and  $w$  have solution if determinant of the coefficient matrix is zero. We get

$$\begin{vmatrix} b_{11} - d_1 k^2 - \lambda & b_{12} \\ b_{21} & b_{22} - d_2 k^2 - \lambda \end{vmatrix} = 0, \quad (11)$$

$$\begin{aligned} (\lambda - b_{11} + d_1 k^2)(\lambda - b_{22} + d_2 k^2) - b_{12} b_{21} &= 0, \\ \text{or, } \lambda^2 + \lambda(k^2(d_1 + d_2) - b_{11}) + (d_1 d_2 k^4 - b_{11} d_2 k^2 - b_{12} b_{21}) &= 0, \\ \text{or, } \lambda^2 + \rho_1 \lambda + \rho_2 &= 0, \end{aligned}$$

where  $\rho_1 = A_1 + (d_1 + d_2)k^2$ ,  $\rho_2 = d_1 d_2 k^4 - b_{11} d_2 k^2 + A_2$ .

The positive equilibrium  $E^*$  is locally asymptotically stable in the presence of diffusion if and only if  $\rho_1 > 0$  and  $\rho_2 > 0$ .

#### 4 Turing Instability

Now, in order to have Turing instability we require that at least one of the roots of the above eq. (11) should have a positive roots or positive real part or in other words,  $\text{Re}(\lambda) > 0$  for some  $k \neq 0$ . Irrespective of the sign of  $\rho_1$ , the equation has a positive root if  $\rho_2 < 0$ . Therefore, diffusion driven instability occurs when  $\rho_2(k^2) < 0$ . Hence the condition for diffusive instability is given by

$$H(k^2) = d_1 d_2 k^4 - b_{11} d_2 k^2 + A_2 < 0. \quad (12)$$

$H$  is a quadratic in  $k^2$  and the graph of  $y = H(k^2)$  is a parabola. The minimum of  $y = H(k^2)$  occurs at  $k^2 = k_{cr}^2$ , where

$$k_{cr}^2 = \frac{\delta(1 - \alpha - \delta(1 + \alpha))}{2d_1(1 - \delta)} > 0. \quad (13)$$

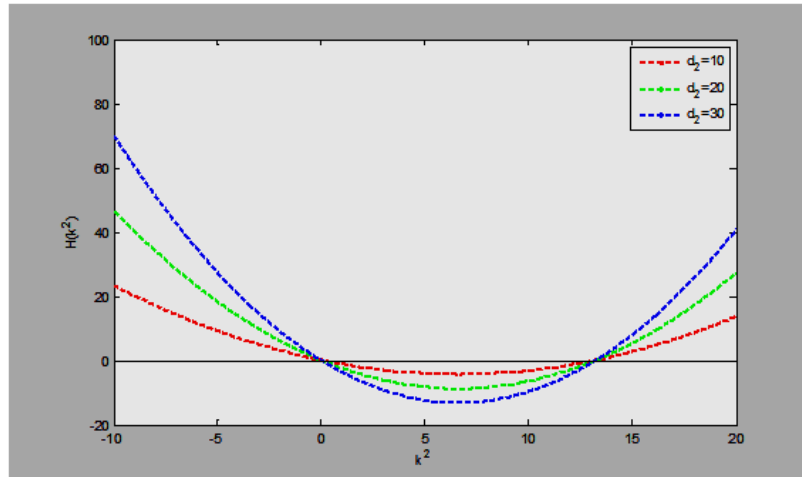
Then,  $H(k^2) < 0$ , when  $k_1^2 < k^2 < k_2^2$ , where

$$k_1^2, k_2^2 = [b_{11} d_2 \mp \sqrt{(b_{11} d_2)^2 - 4d_1 d_2 A_2}] / (2d_1 d_2).$$

Consequently, the condition for diffusive instability is  $H(k_{cr}^2) < 0$ . Therefore

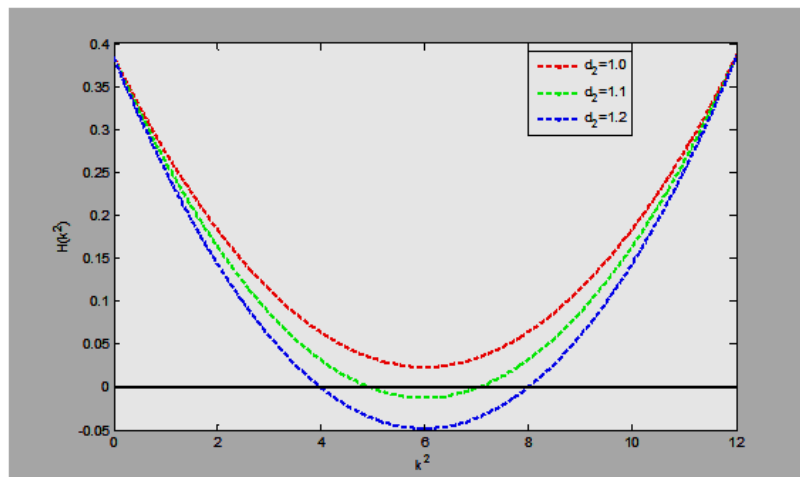
$$\left\{ \delta d_2 (\delta(1 + \alpha) + \alpha - 1) / (1 - \delta) \right\}^2 > 4d_1 d_2 \delta \beta (\delta(1 + \alpha) - 1). \quad (14)$$

Now, consider the set of parameter values  $\alpha = 0.3$ ,  $\beta = 2$ ,  $\delta = 0.3$ . We obtain  $u^* = 0.1286$  and  $v^* = 0.3735$ . For the set of parameter values  $\alpha = 0.3$ ,  $\beta = 2$ ,  $\delta = 0.3$ ,  $d_1 = 0.01$ ,  $d_2 = 10$ , we obtain  $k_1^2 = 0.2814$  and  $k_2^2 = 13.0043$  and corresponding critical value is  $(k_{cr}^2, H(k_{cr}^2)) = (6.6429, -4.0468)$ . The graph of  $H(k^2)$  vs  $k^2$  has been plotted for different values of  $d_2$  in Fig. 2. For all values of  $k^2$  lying in the range (0.2814, 13.0043), the plankton system (2) is unstable.



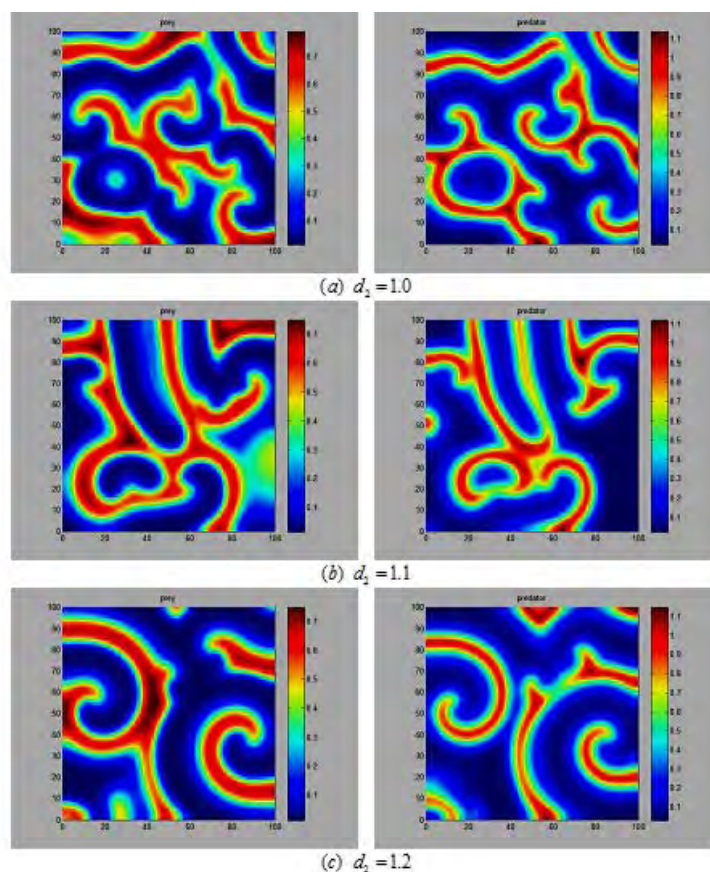
**Fig. 2** The graph of the function  $H(k^2)$  vs  $k^2$  for  $\alpha = 0.3$ ,  $\beta = 2$ ,  $\delta = 0.3$ ,  $d_1 = 0.01$  and for different values of  $d_2 = 10, 20, 30$ .

Again, we consider the another set of parameter values  $\alpha = 0.3$ ,  $\beta = 2$ ,  $\delta = 0.4$ . We obtain  $u^* = 0.2$  and  $v^* = 0.4$ . For the set of parameter values  $\alpha = 0.3$ ,  $\beta = 2$ ,  $\delta = 0.4$ ,  $d_1 = 0.01$ , and for different values of  $d_2 = 1.0, 1.1, 1.2$ , we observe that for  $d_2 = 1.0$ , the minimum value of  $H(k^2) = 0.024$  is positive at  $k^2 = 6.0$  (c.f. Fig. 3). As we increase the value of  $d_2 = 1.1, 1.2$ , the graph of the function  $H(k^2)$  crosses the line  $k^2$  (for  $d_2 = 1.1$ ,  $k^2 = 6.979$ ,  $H(k^2) = -0.012$ ).



**Fig. 3** The graph of the function  $H(k^2)$  vs  $k^2$  for  $\alpha = 0.3$ ,  $\beta = 2$ ,  $\delta = 0.4$ ,  $d_1 = 0.01$  and for different values of  $d_2 = 1.0, 1.1, 1.2$ .

The spatiotemporal patterns are plotted for the same set of parameter values as taken for plotting the Fig. 3 at fixed time  $t = 1000$ , and for different values of predator's diffusivity constant  $d_2 = 1.0, 1.1$  and  $1.2$ . The initial condition ( $u_0 = 0.2(X/100) + 0.19$ ,  $v_0 = 0.4((100-Y)/100) + 0.31$ ), considered for generating the spatiotemporal patterns and is presented in Fig. 4.



**Fig. 4** Snapshots of phytoplankton [first column figures] and zooplankton population [second column figures] is plotted at fixed time  $t = 1000$  at the different values of diffusivity constant (a)  $d_2 = 1.0$ , (b)  $d_2 = 1.1$ , and (c)  $d_2 = 1.2$ .

### 5 Variant of the Rosenzweig-MacArthur Model with Diffusion

The choice of parameter values is important to study the effect of diffusion and also whether the system without diffusion is stable or not. Also, it would depend more on the nonlinearity of the system. To see this we include the fish predation in the model system (2) and consider a diffusive plankton-fish model as

$$\begin{aligned} \frac{\partial u}{\partial t} &= u(1-u) - \frac{uv}{u+\alpha} + d_1 \nabla^2 u, \\ \frac{\partial v}{\partial t} &= \beta v \left( \frac{u}{u+\alpha} - \delta \right) - f \frac{v^2}{v^2 + \eta^2} + d_2 \nabla^2 v. \end{aligned} \quad (15)$$

where  $f$  is the predation rate of zooplankton by fish population which follows Holling type III functional response,  $\eta$  half-saturation constant for zooplankton density. The initial condition and no-flux boundary conditions are

$$u(x, y, 0) > 0, \quad v(x, y, 0) > 0, \quad \text{for } x, y \in [0, R] \times [0, R],$$

$$\text{and } (\partial u / \partial x) = 0, \quad (\partial v / \partial x) = 0, \quad \text{at } (x, y) \in \partial \Omega.$$

Stability analysis to study the effect of nonlinearity introduced by discrete value of fish predation on the model system, we consider the linearized form of the system about the positive equilibrium point  $E^*(u^*, v^*)$ . Set  $u = u^* + U$ ,  $v = v^* + V$ . We obtain the linearized form as

$$\frac{\partial U}{\partial t} = M_{11}U + M_{12}V + d_1 \nabla^2 u, \quad \frac{\partial V}{\partial t} = M_{21}U + M_{22}V + d_2 \nabla^2 v, \quad (16)$$



$$\text{where } M_{11} = -u^* \left( 1 - \frac{v^*}{(u^* + \alpha)^2} \right), \quad M_{12} = -\frac{u^*}{u^* + \alpha}, \quad M_{21} = \frac{\alpha \beta v^*}{(u^* + \alpha)^2}, \quad M_{22} = \frac{f v^* (v^{*2} - \eta^2)}{(v^{*2} + \eta^2)^2}.$$

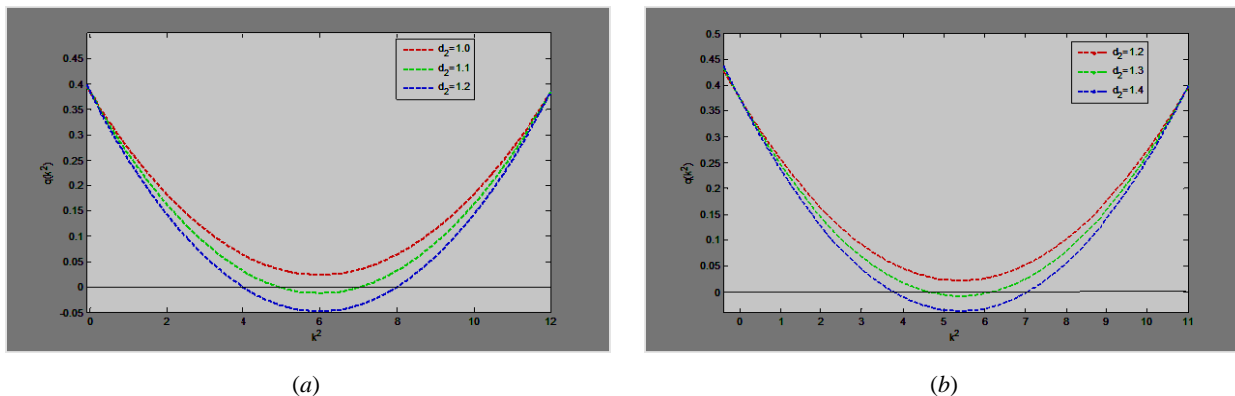
Write the solution in the form  $U = se^{\lambda t + ikx}$ ,  $V = we^{\lambda t + ikx}$ , where  $\lambda$  and  $k$  are the frequency and wave number respectively. Substitute the expressions for  $U, V$  in eq. (16). The homogeneous equations in  $s$  and  $w$  have solution if determinant of the coefficient matrix is zero. We get

$$(\lambda - M_{11} + d_1 k^2)(\lambda - M_{22} + d_2 k^2) - M_{12} M_{21} = 0, \text{ or } \lambda^2 + p\lambda + q = 0, \quad (17)$$

where  $p = (d_1 + d_2)k^2 - (M_{11} + M_{22})$ ,  $q = d_1 d_2 k^4 - k^2(d_2 M_{11} + d_1 M_{22}) + (M_{11} M_{22} - M_{12} M_{21})$ . By Routh-Hurwitz criterion, the roots of eq. (17) are negative or have negative real parts if  $p > 0$  and  $q > 0$ . A sufficient condition for  $p > 0$  is  $(M_{11} + M_{22}) < 0$ . The sufficient conditions are  $v^* < (u^* + \alpha)^2$ , and  $v^{*2} < \eta^2$ . For these values,  $M_{11} < 0$ ,  $M_{22} < 0$ . Also,  $M_{12} < 0$  and  $M_{21} > 0$ . Hence,  $q > 0$ . Therefore, the positive equilibrium  $E^*$  is locally asymptotically stable in the presence of diffusion if the above two conditions are satisfied. The positive equilibrium  $E^*$  may or may not be locally asymptotically stable in the absence of diffusion. Suppose that the sufficient conditions are not met. Now, irrespective of the sign of  $p$ , diffusive instability can arise if  $q < 0$ , (one root of the eq. (17) is positive), that is if  $q(k^2) = Dk^4 - Ck^2 + B < 0$ , where  $D = d_1 d_2$ ,  $C = (d_2 M_{11} + d_1 M_{22})$ , and  $B = M_{11} M_{22} - M_{12} M_{21}$ . The roots of this equation in  $k^2$  are real and positive when (i)  $B > 0$ , (ii)  $C > 0$  and (iii)  $C^2 - 4BD > 0$ . Then,  $q < 0$  when  $k_1^2 < k^2 < k_2^2$ , where  $k_1^2, k_2^2 = [C \mp \sqrt{C^2 - 4BD}]/(2D)$ . Therefore, diffusive instability occurs when these conditions are satisfied. Now,  $q(k^2)$  is a quadratic in  $k^2$  and the graph of  $y = q(k^2)$ , is a parabola opening upwards. The minimum occurs at the vertex of the parabola, that is for  $k^2 = k_m^2$  where  $k^2 = k_m^2 = C/(2D)$ . Now, we numerically simulate the model system (15). The equilibrium point  $(u^*, v^*)$  is the solution of the equations (model without diffusion)

$$u^2 + (\alpha - 1)u + v - \alpha = 0, \quad \beta[(1 - \delta)u - \alpha\delta](v^2 + \eta^2) - f v(u + \alpha) = 0. \quad (18)$$

Consider the following sets of parameter values:  $\alpha = 0.3$ ,  $\beta = 2$ ,  $\delta = 0.4$ ,  $f = 0.0001$ ,  $\eta = 2.5$ . The system of nonlinear eq. (18) is solved by Newton iteration method and we obtain  $(u^*, v^*) = (0.20, 0.40)$ . For the above set of parameter values with fixed  $d_1 = 0.01$  and increasing values of  $d_2 = 1.0, 1.1, 1.2$ , we observe that for  $d_2 = 1.0$ ,  $q(k^2)$  is positive. As we increase the value of  $d_2 = 1.1, 1.2$ , the graph of the function  $q(k^2)$  crosses the line  $k^2$  (cf. Fig. 5a). Similarly, we consider another set of parameter values in which fish predation rate has been increased significantly  $\alpha = 0.3$ ,  $\beta = 2$ ,  $\delta = 0.4$ ,  $f = 0.8$ ,  $\eta = 2.5$ . We obtain  $u^* = 0.2220$ ,  $v^* = 0.4061$ . For this set of parameter values with fixed  $d_1 = 0.01$  and increasing values of  $d_2 = 1.2, 1.3, 1.4$ , we observe that for  $d_2 = 1.2$ ,  $q(k^2)$  is positive. As we increase the value of  $d_2 = 1.3, 1.4$ , the graph of the function  $q(k^2)$  crosses the line  $k^2$  (cf. Fig. 5b).



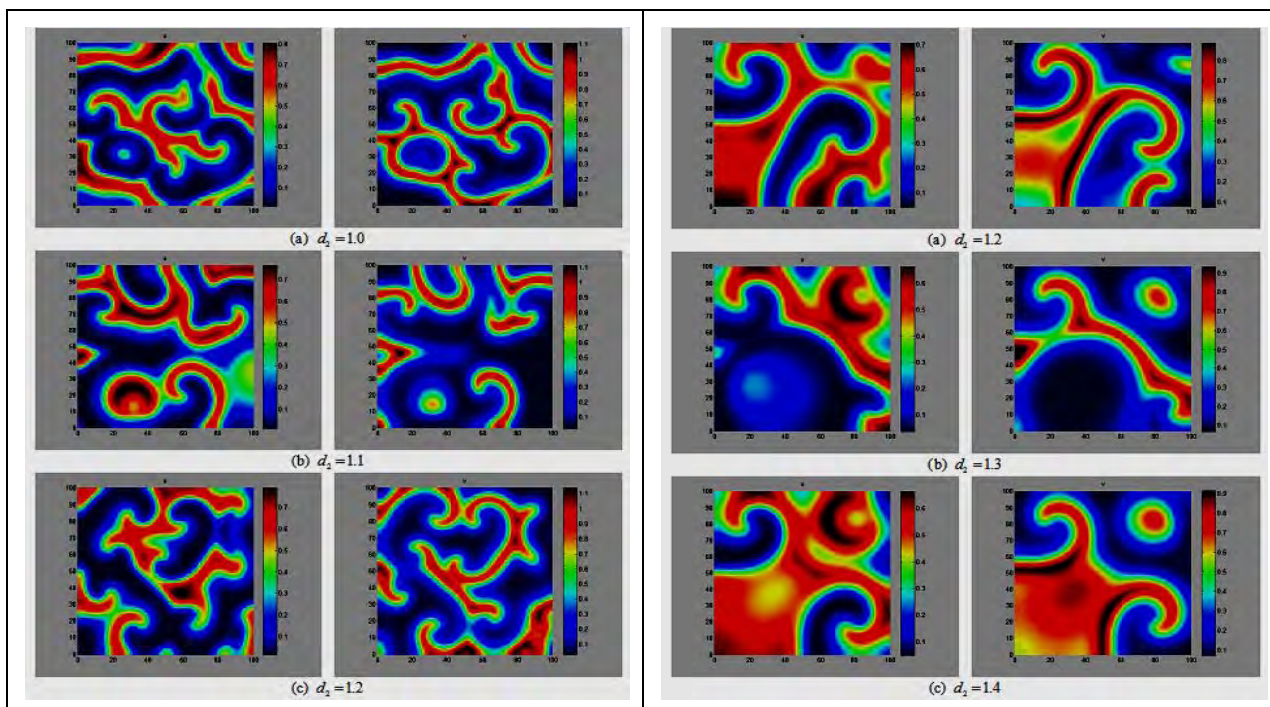
**Fig. 5** The graph of the function  $q(k^2)$  vs  $k^2$  for (a)  $f = 0.0001$  and different values of  $d_2 = 1.0, 1.1, 1.2$ , (b)  $f = 0.8$  and different values of  $d_2 = 1.2, 1.3, 1.4$ .

For the same set of parameter values as we have taken in Fig. 5(a) and 5(b), we have plotted the spatiotemporal patterns to see the behavior of the system after increasing the nonlinearity in the model system with different set of initial conditions

(i)  $u_0 = 0.2 * (X / 100) + 0.19, v_0 = 0.4 * ((100 - Y) / 100) + 0.31$ , (see Fig. 6(a))

(ii)  $u_0 = 0.2220 * (X / 100) + 0.19, v_0 = 0.4061 * ((100 - Y) / 100) + 0.31$ , (see Fig. 6(b)).

In Fig. 6, we have studied the spatiotemporal patterns to see the effect of increasing nonlinearity outside and inside the Turing domain for the model system (15) for different values of fish predation rate  $f = 0.0001$  and  $0.8$ . We also observed that for  $f = 0.8$ , model have more regular patchy pattern than for  $f = 0.0001$ . It shows that fish predation acts as regularizing factor for the plankton dynamics but there is no drastic change in the patterns just outside the Turing domain.



**Fig. 6(a)** Snapshots of phytoplankton [first column figures] and zooplankton population [second column figures] is plotted at fixed time  $t = 1000$  at the different values of diffusivity constant (a)  $d_2 = 1.0$ , (b)  $d_2 = 1.1$ , and (c)  $d_2 = 1.2$ .

**Fig. 6(b)** Snapshots of phytoplankton [first column figures] and zooplankton population [second column figures] is plotted at fixed time  $t = 1000$  at the different values of diffusivity constant (a)  $d_2 = 1.2$ , (b)  $d_2 = 1.3$ , and (c)  $d_2 = 1.4$ .

## 6 Discussions and Conclusions

In this paper, we have considered a minimal model for phytoplankton-zooplankton interaction with Holling type II functional responses. We have studied the reaction-diffusion model both analytically and numerically. We obtained the stable and unstable region in parametric domain for temporal model. We also obtained the conditions for Turing instability in terms of parameters. We have also validated the idea that resulting spatial pattern lies outside the Turing domain. With the help of numerical simulation, we observed that there are no major changes in the pattern just outside the Turing domain but very far away from Turing domain no pattern has been observed. For a particular set of parameter values for  $d_2 = 1.0$  (see Fig. 3),  $H(k^2) = 0.024 > 0$ , therefore no Turing instability but from Fig. 4a, we observed that there is no major change in the pattern for  $d_2 = 1.0$ . We have also seen that the choice of parameter values is important to study the effect of diffusion. Also it would depend more on the nonlinearity of the system. To understand the effect of nonlinearity we have included the fish predation rate in the model system which follows the Holling Type III functional response. With the help of Fig. 5 and 6, we conclude that there is no major change in the propagation of spatiotemporal pattern just outside the Turing domain for different values of fish predation rate (e. g.  $f = 0.0001, 0.8$ ).

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