Article

Hopf bifurcation and stability analysis for a delayed logistic equation with additive Allee effect

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Abstract
In this paper the linear stability of the delayed logistic equation with additive Allee effect is investigated. We also analyze the associated characteristic transcendental equation, to show the occurrence of Hopf bifurcation at the positive equilibrium. To determine the direction of Hopf bifurcation and the stability of bifurcating periodic solution, we use the normal form approach and a center manifold theorem. Finally, a numerical example is given to demonstrate the effectiveness of the theoretical analysis.

Keywords time delay; logistic equation; stability; Hopf bifurcation; additive Allee effect.

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1 Introduction

Considering the fact that the environment has limited resources, the Belgium mathematician Pierre-Francois Verhulst (Verhulst, 1838) proposed one of the most famous equation that used to model a lot of applications in ecology and biology. The logistic equation - also known as Verhulst model- is a model of population growth first proposed by Verhulst (1845, 1847). Verhulst (Agarwal et al., 2014) argued that the unlimited growth in the exponential growth model $\dot{N}(t) = rN(t)$ must be restricted by the Malthusian "struggle for existence" and he proposed the model

$$\dot{N}(t) = rN(t)\left(1 - \frac{N(t)}{k}\right) \qquad (1)$$

Equation (1) is called logistic growth in a population where $r > 0$ is the intrinsic growth rate and $k > 0$ is the carrying capacity (the maximum number of individuals that the environment can support). We can see that $N = k$ is globally stable steady state for equation (1) with any initial condition. If the initial condition is more than (less than) k then the population decreases (increase) approaching k as t tends to ∞ .

The logistic equation has a lot of applications in many fields like economy (Shone, 2002), ecology (Pastor, 2008), biology (Murray, 2002), medicine (Forys and Marciniak-Czochra, 2003) and neurosciences (Gershenfeld, 1999). To know more about the history of the logistic equation see Kingsland (1982).

In the last few years the importance of embedding the time delay into dynamical systems was increased, especially in ecological and biological systems because in these systems the reproduction is not instantaneous. Incorporating the time delay into system allow the system rate of change to depend on his own past history. Also by using time delay in equations that model eco-systems or bio-systems, phenomena as feeding time, reaction time, maturation periods, etc., can be represented.

Time delays have been incorporated into biological and ecological models to fix the deficiencies of ordinary differential equation that ignored important phenomena. Furthermore, so many of the processes, both natural and man-made, in medicine, diseases, physics, chemistry, bio-systems, eco-systems, economics, etc., involve time delays. In general, delay differential equations exhibit much more complicated dynamics than ordinary differential equations. For these reasons, the researchers in many fields pay great attention for studying delayed systems (Agarwal et al., 2014; Ding et al., 2013; Engelborghs et al., 2002; Ruan, 2006; Kuang, 1993; Braddock, 1983; Bi and Xiao, 2014; Hu and Li, 2012).

To make the logistic equation more realistic, Hutchinson [Hutchinson, 1948] proposed incorporating the effect of delay and he introduced the delayed logistic equation

$$\dot{N}(t) = rN(t)\left(1 - \frac{N(t-\tau)}{k}\right) \tag{2}$$

where $\tau > 0$ is time delay. For other formula of delayed logistic equation see (Arino et al., 2006).

Hutchinson suggested that the equation (2) can be used to model the dynamics of a single species population growing towards a saturation level $\mathbf{k}$ with a constant reproduction rate r (Kuang, 1993; Gobalsamy, 1992; Cuching, 1977).

More interesting topological changes in the population size as limit cycles, chaos and damped oscillations are produced in the existence of delay (Storgaz, 1994).

Noticing the behavior of species one can see that some species often help each other in their search for food or habitat and to escape from their predators. For example, some social species such as ants, bees, etc., have developed complex cooperative behavior involving division of labor, altruism, etc. Such cooperative processes have a positive feedback influence since individuals have been provided a greater chance to survive and reproduce as density increase.

The ecologist Warder Clyde Allee (Allee, 1931) paid a lot of attention to aggregation and associated cooperative and social characteristics among members of a species in animal populations, and his work has been among the most influential for animal behavioral research.

In numerous writings (Allee, 1931; Allee, 1941; Allee et al., 1949) Allee shows that for a variety of biological reasons positive (negative) feedback effects can happen at low (high) population density. The positive feedback is called Allee effects (Dennis, 1989; Stephens et al., 1999). In population dynamics, the Allee effect refers to a process that reduces the growth rate for small population densities.

The so-called Allee effect (Elabbasy et al., 2007) refers to a population that has a maximal per capita growth rate at intermediate density. This occurs when the per capita growth rate increases as density increases and decreases after the density passes a certain value.

Modelling Allee effects in population dynamics and fields that related to it as a multi-species interactions in eco-systems, disease dynamics and the spread of epidemics, etc., has great interest in mathematical literature. (Dennis, 1989; Elaydi and Sacker, 2010; Courchamp et al., 2008; Schreiber, 2003; Cushing and Hudsona, 2012; Lewis and Kareiva, 1993)

The equation

$$\dot{N}(t) = rN(t)\left(1 - \frac{N(t)}{k}\right)(N(t) - m) \tag{3}$$

is the prototypical model for a multiplicative Allee effect where r is the intrinsic growth rate and k is the carrying capacity. If $-k < m < 0$ it shows the weak Allee effect, while if $0 < m < k$, it shows the strong Allee effect.

The strong Allee effect introduces a population threshold (the minimal size of the population required to survive), and the population must surpass this threshold to grow. In contrast, a population with a weak Allee effect does not have a threshold [Wang et al, 2011; Wang and Kot, 2001].

Dennis (Dennis, 1989) who first introduced the equation that modeled the additive Allee effect in the form

$$\dot{N}(t) = rN(t)\left(1 - \frac{N(t)}{k}\right) - \frac{mN(t)}{N(t)+b} \quad , \qquad (4)$$

and then it used in [Aguirre et al, 2009].

The term $\frac{mN(t)}{N(t)+b}$ is called the additive Allee effect where $0 < m < 1$ and $0 < b < 1$ are called Allee constants with $k > b$. If $m < br$ then the equation (4) exhibits a weak Allee effect and if $m > br$ then it exhibits a strong Allee effect (Wang and Kot, 2001).

In our paper, we study the delayed logistic equation with additive Allee effect in the form

$$\dot{N}(t) = rN(t)\left(1 - \frac{N(t-\tau)}{k}\right) - \frac{mN(t)}{N(t)+b} \qquad (5)$$

2 Local Stability and Existence of Hopf Bifurcation

The model (5) – at $br > m$ - has a trivial equilibrium $N_0^* = 0$, and positive equilibrium $N_1^* = \frac{(k-b)+\sqrt{(k-b)^2+\frac{4k}{r}(br-m)}}{2}$. And at $br < m$ the model (5) has a trivial equilibrium $N_0^{**} = 0$ and two positive equilibrium

$N_1^{**} = \frac{(k-b)+\sqrt{(k-b)^2+\frac{4k}{r}(br-m)}}{2}$ and $N_2^{**} = \frac{(k-b)-\sqrt{(k-b)^2+\frac{4k}{r}(br-m)}}{2}$.

For convenience, we indicate to the next lemma which consolidates our stability analysis.

Lemma 1 (Hale and Lunel, 1993)

All roots of the characteristic equation $\lambda + c + be^{\lambda} = 0$, where c **and** b are real, have negative real parts if and only if

$c > -1$,

$c + b > 0$ and

$$b < \sqrt{c^2 + \xi^2}$$

where ξ is the root of $\xi = -c\tan\xi$, $0 < \xi < \pi$ if $c \neq 0$ and $\xi = \frac{\pi}{2}$ if $c = 0$.

Theorem 1

(I) At $br - m > 0$

1. The equilibrium $N_0^* = 0$ of equation (5) is unstable.

2. The equilibrium $N_1^* = \frac{(k-b)+\sqrt{(k-b)^2+\frac{4k}{r}(br-m)}}{2}$ of Eq. (5) is stable if $\tau < \tau_j$ and is unstable if $\tau > \tau_j$.

(II) At $br - m < 0$

1. The equilibrium $N_0^{**} = 0$ of equation (5) is stable.

2. The two equilibrium $\boldsymbol{N_1^{**}} = \frac{(\mathbf{k-b})+\sqrt{(\mathbf{k-b})^2+\frac{4\mathbf{k}}{r}(\mathbf{br-m})}}{\mathbf{2}}$ and $\boldsymbol{N_2^{**}} = \frac{(\mathbf{k-b})-\sqrt{(\mathbf{k-b})^2+\frac{4\mathbf{k}}{r}(\mathbf{br-m})}}{\mathbf{2}}$ of equation (5) are stable if $\boldsymbol{\tau < \tau_j}$ and are unstable if $\boldsymbol{\tau > \tau_j}$.

Proof.

(I) Linearizing equation $(\mathbf{5})$ about equilibrium $\boldsymbol{N^*}_{\mathbf{0}} = \mathbf{0}$ using $\boldsymbol{N(t) = N^* + p(t)}$, it becomes

$$\dot{\boldsymbol{p}}(\boldsymbol{t}) = \frac{\boldsymbol{br-m}}{\boldsymbol{b}}\boldsymbol{p}(\boldsymbol{t}) \qquad (6)$$

It is easy to show that equation $(\mathbf{6})$ has the characteristic equation in the form

$$\lambda = \frac{br-m}{b} \qquad (7)$$

Since $br - m > 0$ then $\frac{br-m}{b} > 0$

Then the model is unstable.

Again, by linearizing about $N_1^* = \frac{(\mathrm{k-b})+\sqrt{(\mathrm{k-b})^2-\frac{4\mathrm{k}}{r}(\mathrm{br-m})}}{2}$ equation (5) will be

$$\dot{p}(t) = \left[r\left(1 - \frac{N_1^*}{k}\right) - \frac{bm}{(b+N_1^*)^2}\right]p(t) - \frac{rN_1^*}{k}p(t-\tau) \qquad (8)$$

Or

$$\dot{p}(t) = a_1 p(t) - a_2 p(t-\tau) \qquad (9)$$

Where $a_1 = \left[r\left(1 - \frac{N_1^*}{k}\right) - \frac{bm}{(b+N_1^*)^2}\right]$ and $a_2 = \frac{rN_1^*}{k}$

Equation (9) has the characteristic equation

$$\lambda + a_2 e^{-\lambda\tau} - a_1 = 0 \qquad (10)$$

Let $i\omega$ be the root of equation (10) ; then:

$$i\omega + a_2(\cos\omega\tau - i\sin\omega\tau) - a_1 = 0$$

$$(a_2\cos\omega\tau - a_1) + i(\omega - a_2\sin\omega\tau) = 0 \qquad (11)$$

Then, by separating and equating real parts and imaginary parts

$$a_2\cos\omega\tau - a_1 = 0 \qquad (12.a)$$

$$\omega - a_2\sin\omega\tau = 0 \qquad (12.b)$$

Then $\omega = \pm\sqrt{a_2^2 - a_1^2}$ (13)

From (13) and using $(12.a, 12.b)$

$$\tau_j = \frac{1}{\sqrt{a_2^2-a_1^2}}\tan^{-1}\frac{\sqrt{a_2^2-a_1^2}}{a_1} \qquad (14)$$

By the same way, we can prove part (II).

Theorem 2

If $\lambda_j(\tau) = \alpha_j(\tau) + i\,\omega_j(\tau)$ denote a root of Eq. (10) near $\tau = \tau_j$, such that $\omega_j(\tau_j) = \omega_0$ and $\alpha_j(\tau_j) = 0$ then

$$\left.\frac{d\alpha_j(\tau)}{d\tau}\right|_{\tau=\tau_j} > 0$$

Proof.

By differentiating the characteristic equation (10) with respect to τ we get

$$\frac{d\lambda}{d\tau} = \frac{a_2\lambda e^{-\lambda\tau}}{1 - a_2\tau e^{-\lambda\tau}}$$

This gives

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{e^{\lambda\tau}}{a_2\lambda} - \frac{\tau}{\lambda}$$

Then

$$\text{Sign}\left\{\frac{d\alpha_j(\tau)}{d\tau}\Big|_{\tau=\tau_j}\right\} = \text{sign}\left\{Re\left(\frac{d\lambda}{d\tau}\right)^{-1}\Big|_{\tau=\tau_j}\right\} = \text{sign}\left\{Re\left(\frac{e^{\lambda\tau}}{a_2\lambda} - \frac{\tau}{\lambda}\right)\Big|_{\tau=\tau_j}\right\}$$

$$= \text{sign}\left\{\frac{\sin\omega_0\tau_j}{a_2\omega_0}\right\} = \text{sign}\left\{\frac{1}{a_2{}^2}\right\} > 0 \quad ■$$

Theorem 2 stated the last condition for the occurrence of Hopf bifurcations and the results can be introduced as follows.

Theorem 3.

(I) in the case if $br - m > 0$, When the parameter τ passes through the critical value $\tau = \tau_j{}^*$, there are Hopf bifurcations at the equilibrium

$N_1^* = \frac{(k-b)+\sqrt{(k-b)^2 - \frac{4k}{r}(br-m)}}{2}$ to a periodic orbit.

(II) In the case if $br - m < 0$, when the parameter τ passes through the critical value $\tau = \tau_j{}^{**}$, there are Hopf bifurcations at the equilibriums $N_1^{**} = \frac{(k-b)+\sqrt{(k-b)^2 + \frac{4k}{r}(br-m)}}{2}$ and $N_2^{**} = \frac{(k-b)-\sqrt{(k-b)^2 + \frac{4k}{r}(br-m)}}{2}$ to a periodic orbit.

3 Stability and Direction of the Hopf Bifurcation

Let $y(t) = p(\tau t)$, then the equation (5) written as

$$\dot{y}(t) = a_1\tau y(t) - a_2\tau y(t-1)$$
$$-\frac{r}{k}\tau y(t)y(t-1) + \frac{bm}{(b+N^*)^3}\tau y^2(t) - \frac{m}{(b+N^*)^3}\tau y^2(t) \quad (15)$$

In $C = C([-1,0],\mathbb{R})$ equation (15) written as

$$\dot{y}(t) = L_\mu(y_t) + F(\mu, y_t) \quad (16)$$

Where

$$L_\mu(\varphi) = (\tau_j + \mu)(a_1\varphi(0) - a_2\varphi(-1))$$

$$F(\mu, \varphi) = (\tau_j + \mu)\left\{\frac{-r}{k}\varphi(0)\varphi(-1) + \frac{bm}{(b+N^*)^3}\varphi^2(0)\right\} \quad (17)$$

Using the Riesz representation theorem, there exists a function $\eta(\theta, \mu)$ of bounded variation for $\theta \in [-1,0]$ such that

$$L_\mu(\varphi) = \int_{-1}^{0} d\eta(\theta, 0)\varphi(\theta) \quad \text{for } \varphi \epsilon c \quad (18)$$

We can choose

$$\eta(\theta, \mu) = (\tau_j + \mu)(a_1\delta(\theta) - a_2\delta(\theta+1)) \quad (19)$$

where $\delta(\theta)$ is the Dirac delta function.

For $\varphi \epsilon C^1([-1,0],\mathbb{R})$; define

$$A(\mu)\varphi = \begin{cases} \frac{d\varphi(\theta)}{d\theta} & if\ \theta \in [-1,0) \\ \int_{-1}^{0} d\eta(\mu, s)\varphi(s) & if\ \theta = 0 \end{cases} \quad (20)$$

and

$$R(\mu)\varphi = \begin{cases} 0 & if\ \theta \in [-1,0) \\ F(\mu,\varphi) & if\ \theta = 0 \end{cases} \quad (21)$$

Then the system (16) can be written as operation equation

$$\dot{y}_t = A(\mu)y_t + R(\mu)y_t \quad (22)$$

where

$$y_t(\theta) = y(t+\theta) \qquad for \quad \theta \in [-1,0]$$

The equation (22) is more mathematically pleased because this equation involves a single unknown variable y_t.

For $\psi \epsilon C[0,-1]$, the adjoint operator A^* of A is defined as

$$A^*(\mu)\psi(s) = \begin{cases} \dfrac{d\psi(s)}{ds} & if \quad s \in [-1,0) \\ \displaystyle\int_{-1}^{0} d\eta\,(\mu,s)\psi(s) & if \quad s = 0 \end{cases} \quad (23)$$

For $\varphi \epsilon C([-1,0]$ and $\psi \epsilon C[0,-1]$, the bilinear inner product defined as

$$\langle \psi(s), \varphi(\theta)\rangle = \bar{\psi}(0)\varphi(0) - \int_{-1}^{0}\int_{\xi=0}^{\theta} \bar{\psi}(\xi-\theta)\,d\eta\,(\theta)\varphi(\xi)d\xi \quad (24)$$

To determine the Poincare normal form of the operator A we need to calculate the eigenvector $q(\theta)$ and $q^*(\theta)$ of A and A^* that corresponding to the eigenvalues $i\omega_0\tau_j$ and $-i\omega_0\tau_j$ respectively.

It is easy to be verified that $q(\theta) = e^{i\omega_0\tau_j\theta}$and $q^*(\theta) = De^{i\omega_0\tau_j\theta}$.

In order to assure that $\langle q^*(s),\ q(\theta)\rangle = 1$, we need to determine the value. From (24)

$$\langle q^*(s), q(\theta)\rangle = \bar{q}^*(0)q(0) - \int_{-1}^{0}\int_{\xi=0}^{\theta} \bar{D}e^{-i\omega_0\tau_j(\xi-\theta)}\,d\eta\,(\theta)e^{i\omega_0\tau_j\xi}\,d\xi$$

$$\langle q^*(s), q(\theta)\rangle = \bar{D}\left[1 - \int_{-1}^{0}\theta\, e^{i\omega_0\tau_j\theta} d\eta\,(\theta)\right] = 1$$

$$\bar{D} = \frac{1}{1-\frac{rN^*}{k}e^{-i\omega_0\tau_j}} \quad (25)$$

Hassard et al. (1981) introduced a method to compute the co-ordinates that describe the center manifold c_0 at $\mu = 0$.

Tracking Hassard method, for y_t, a solution of (22) at $\mu = 0$, we define:

$$z(t) = \langle q^*, y_t\rangle \quad \text{and} \quad w(t,\theta) = y_t(\theta) - 2Re\{z(t)q(\theta)\} \quad (26)$$

On the center manifold c_0 we have:

$w(t,\theta) = w(z(t),\bar{z}(t),\theta)$, Where

$$w(t,\theta) = w_{20}(\theta)\frac{z^2}{2} + w_{11}(\theta)z\bar{z} + w_{02}(\theta)\frac{\bar{z}^2}{2} + \cdots, \quad (27)$$

where $z\ and\ \bar{z}$ are local co-ordinates for center manifold c_0 in C in the direction of $q^*$and $\bar{q}^*$. Note that , w is real if y_t is real. We shall deal with real solution only.

Now, for solution $y_t \epsilon c_0$ of equation (22)
$\langle q^*,\dot{y}_t\rangle = \langle q^*, A(\mu)y_t + R(\mu)y_t\rangle$,
Then since $\mu = 0$

$$\dot{z}(t) = \langle \dot{q}^*,y_t\rangle + \langle q^*, \ A(\mu)y_t + R(\mu)y_t\rangle$$

$$\dot{z}(t) = i\omega_0\tau_j\, z(t) + q^*(0)f\big(z(t),\bar{z}(t)\big) \qquad (28)$$

Equation (28) can be written in abbreviated form as

$$\dot{z}(t) = i\omega_0\tau_{\mathrm{j}}z(t) + g(z,\bar{z}) \qquad (29)$$

Where

$$g(z,\bar{z}) = q^*(0)f\big(z(t),\bar{z}(t)\big)$$

$$g(z,\bar{z}) = g_{20}(\theta)\frac{z^2}{2} + g_{11}(\theta)z\bar{z} + g_{02}(\theta)\frac{\bar{z}^2}{2} + \cdots \qquad (30)$$

Since from (26)

$$\dot{w} = \dot{y}_t - \dot{z}q - \dot{\bar{z}}\bar{q}$$

Then

$$\dot{w} = \begin{cases} Aw - 2Re\big[q^*(0)f\big(z(t),\bar{z}(t)\big)q\big] & if\ \theta \in [-1,0) \\ Aw - 2Re\big[q^*(0)f\big(z(t),\bar{z}(t)\big)q\big] + f(z,\bar{z}) & if\ \theta = 0 \end{cases} \qquad (31)$$

This can be written as

$$\dot{w} = Aw + H(z(t),\bar{z}(t),\theta) \qquad (32)$$

where

$$H(z(t),\bar{z}(t),\theta) = H_{20}(\theta)\frac{z^2}{2} + H_{11}(\theta)z\bar{z} + H_{02}(\theta)\frac{\bar{z}^2}{2} + \cdots \qquad (33)$$

Since

$$\begin{aligned} g(z,\bar{z}) &= q^*(0)f\big(z(t),\bar{z}(t)\big) \\ &= q^*(0)f(0,y_t) \\ &= \bar{D}\tau_{\mathrm{j}}\left\{\frac{-r}{k}y_t(0)y_t(-1) + \frac{mb}{(b+N^*)^2}{y_t}^2(0)\right\} \end{aligned} \qquad (34)$$

Using equations (26) in (34) and comparing coefficient with (30), we find

$$g_{20} = 2\bar{D}\tau_{\mathrm{j}}\left(\frac{-r}{k}e^{-i\omega_0} + \frac{mb}{(b+N^*)^2}\right)(\qquad 35.a)$$

$$g_{11} = \bar{D}\tau_{\mathrm{j}}\left(\frac{-r}{k}(e^{i\omega_0} + e^{-i\omega_0}) + \frac{2mb}{(b+N^*)^2}\right) \qquad (35.b)$$

$$g_{02} = 2\bar{D}\tau_{\mathrm{j}}\left(\frac{-r}{k}e^{i\omega_0} + \frac{mb}{(b+N^*)^2}\right)(\qquad 35.c)$$

$$g_{21} = \bar{D}\tau_{\mathrm{j}}\left\{\frac{-r}{k}\left(e^{-i\omega_0}w_{11}(0) + \frac{1}{2}e^{i\omega_0}w_{20}(0) + \frac{1}{2}w_{20}(-1) + w_{11}(-1)\right) + \frac{mb}{(b+N^*)^2}\big(2w_{11}(0) + w_{20}(0)\big)\right\} \qquad (35.d)$$

Since $\dot{w}(z,\bar{z}) = w_z\dot{z} + w_{\bar{z}}\dot{\bar{z}}$ (36)

Using (32) and substitution by the expansions of previous functions and comparing coefficients we find that

$$H_{20}(\theta) = (2i\omega_0 - A)w_{20}(\theta) \qquad (37)$$

$$H_{11}(\theta) = -Aw_{11}(\theta) \quad (38)$$

$$H_{02}(\theta) = (2i\omega_0 - A)w_{02}(\theta) \quad (39)$$

Since from (31), (32) and (33) we find

$$H_{20}(\theta) = -q(\theta)g_{20}(\theta) - \bar{q}(\theta)\bar{g}_{02}(\theta) \quad (40)$$

$$H_{11}(\theta) = -q(\theta)g_{11}(\theta) - \bar{q}(\theta)\bar{g}_{11}(\theta) \quad (41)$$

$$H_{02}(\theta) = -q(\theta)g_{20}(\theta) - \bar{q}(\theta)\bar{g}_{20}(\theta) \quad (42)$$

From (37) and (40)

$$Aw_{20}(\theta) = 2i\omega_0 w_{20} - H_{20}(\theta)$$

$$\dot{w}_{20}(\theta) = 2i\omega_0 w_{20} + q(\theta)g_{20}(\theta) + \bar{q}(\theta)\bar{g}_{02}(\theta) \quad (43)$$

This equation has the solution

$$w_{20}(\theta) = \frac{ig_{20}}{\omega_0}e^{i\omega_0\theta} + \frac{i\bar{g}_{02}}{3\omega_0}e^{-i\omega_0\tau_j\theta} + E_1 e^{2i\omega_0\theta} \quad (44)$$

By the same way

$$w_{11}(\theta) = \frac{g_{11}}{i\omega_0}e^{i\omega_0\theta} - \frac{\bar{g}_{11}}{i\omega_0}e^{-i\omega_0\theta} + E_2 \quad (45)$$

Where E_1 and E_2 are constants and they are evaluated from the formulas

$$E_1 = \frac{2\bar{D}\tau[\frac{-r}{k}e^{-i\omega_0} + \frac{mb}{(b+N^*)^2}]}{2i\omega_0 - a_1\tau + \frac{rN^*}{k}\tau e^{-2i\omega_0}} \quad (46)$$

$$E_2 = \bar{D}[\frac{-r}{k}(e^{-i\omega_0} + e^{i\omega_0}) + \frac{2mb}{(b+N^*)^2}]/\frac{rN^*}{k} \quad (47)$$

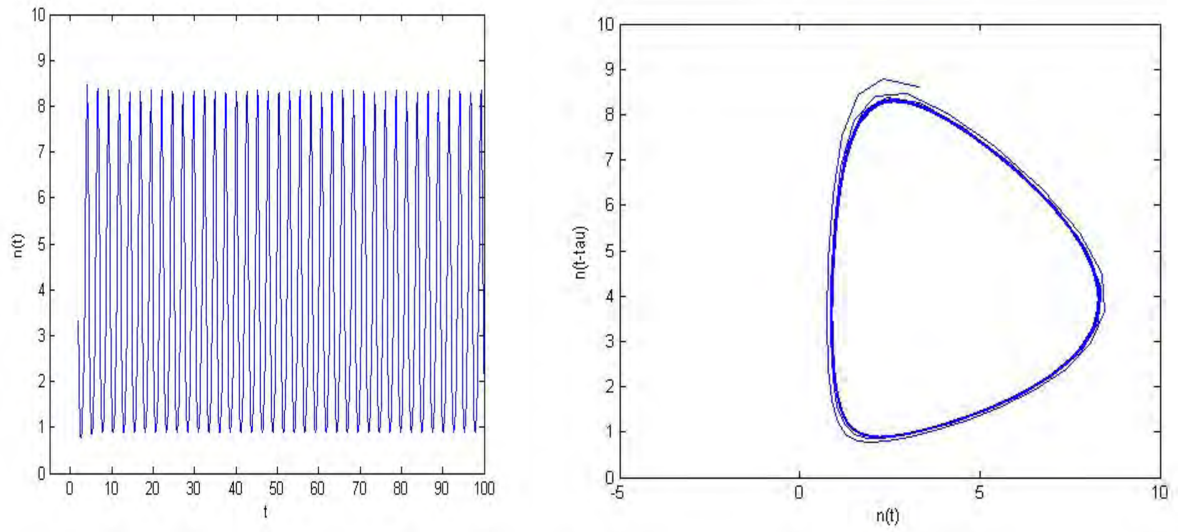
We can also compute

$$\begin{cases} c_1(0) = \dfrac{i}{2\omega_0}\left(g_{20}g_{11} - 2|g_{11}|^2 - \dfrac{|g_{02}|^2}{3}\right) + \dfrac{g_{21}}{2} \\ \mu_2 = \dfrac{-\Re e\{c_1(0)\}}{\Re e\{\tau_j\dot{\alpha}(\tau_j)\}} \\ \beta_2 = 2\Re e\{c_1(0)\} \end{cases} \quad (48)$$

Theorem (Hassard et al., 1981): In (48), the directions of Hopf bifurcation are determined by the sign of μ_2 and the stability of bifurcating periodic solutions by the sign of β_2 . In this case, if $\mu_2 > 0 (< 0)$, then the Hopf bifurcation is supercritical (subcritical) and if $\beta_2 < 0 (> 0)$ the bifurcating periodic solutions are orbitally stable (unstable).

4 Numerical Example

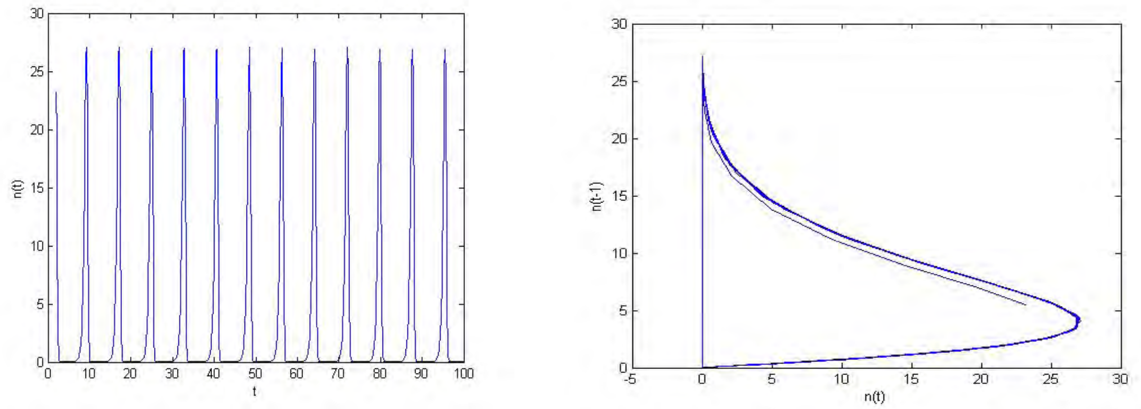
In this section, we give some numerical simulations supporting our theoretical analysis. In the first case ($br - m > 0$), by choosing $r = 3$, $k = 4$, $m = 0.5$; $b = 0.75$ and $N(t) = 0.5$ for $t \in [-\tau, 0]$, fig. 1 at $\tau = 0.6$ shows the existence of Hopf bifurcation and limit cycle behavior for model (5).



(a) (b)

Fig. 1 Wave form plot (a) and phase plot (b) at $\tau = 0.6$ for the case $(br - m > 0)$ for model (5).

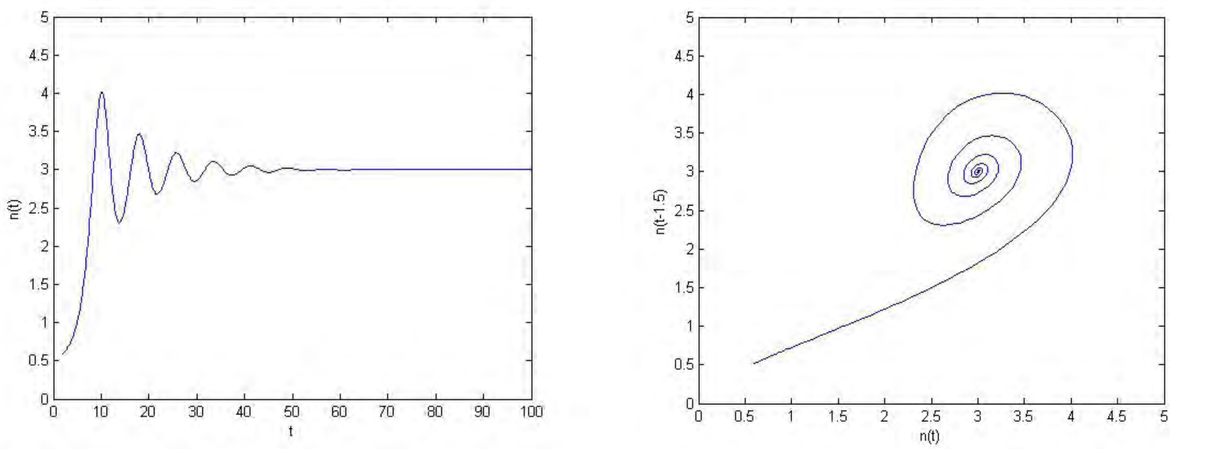
The wave form plot and the phase plot in Fig. 2 show the periodicity of the solution and existence of an attractor for model (5) at $\tau = 1$.



(a) (b)

Fig. 2 Wave form plot (a) and phase plot (b) at $\tau = 1$ for the case $(br - m > 0)$ for model (5).

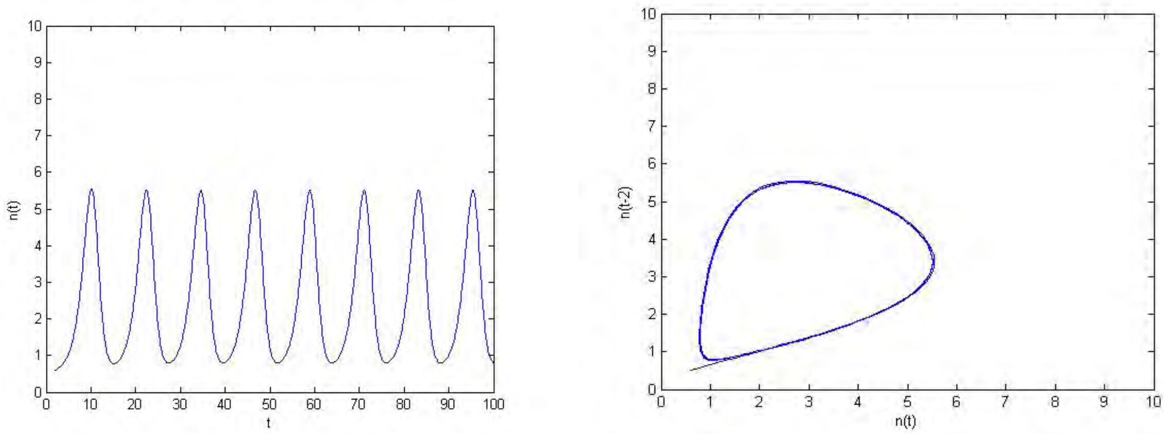
In the second case $(br - m < 0)$ - we will choose $r = 1$, $k = 4$, $m = 0.9$; $b = 0.6$ and $N(t) = 0.5$ for , $t \in [-\tau, 0]$-Fig. 3 shows that the equilibrium point for the model (5) is asymptotically stable at τ=1.5.



(a) (b)

Fig. 3 Wave form plot (a) and phase plot (b) at $\tau = 1.5$for the case ($br - m < 0$) for model (5).

In Fig. 4, wave form plot and phase plot at τ=2 show the existence of Hopf bifurcation and limit cycle behavior for model (5).



(a) (b)

Fig. 4 Wave form plot (a) and phase plot (b) at τ=2 for the case ($br - m < 0$) for model (5).

5 Conclusions

In this paper, we have investigated the stability and Hopf bifurcation of a delayed logistic equation with additive Allee effect. Also we have obtained stability conditions and we showed that a Hopf Bifurcation will occur when the time delay parameter pass through critical values; that is, a family of periodic orbits bifurcates from the equilibrium. The direction of Hopf bifurcation and the stability of the bifurcating periodic orbits are discussed by applying the normal form approach and the center manifold theorem. Numerical simulations have shown that the analytical results are correct.

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