Article

Application of homotopy perturbation method to the Navier-Stokes equations in cylindrical coordinates

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Abstract

This paper deals with the approximate analytical solution of the Navier-Stokes equations in cylindrical coordinates. The homotopy perturbation method is used to get the analytical approximation. Depending upon different available choices for the linear operator, we also have the advantage to choose different initial approximations to start our analysis. The analysis is done without calculating the Adomian's polynomials.

Keywords Navier-Stokes equations; homotopy perturbation method; iterative approximation; infinite series solution.

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1 Introduction

The Navier-Stokes equations describe the motion of fluids that is a substance which can be flow and it arises from Newton 2nd law applying to the fluid motion (Square, 1952). The Navier-Stokes equations are widely used in physics, they are used for modeling of weather and seas currents, designing of aircrafts and cars, for motions of stars, they are used in video games, flow of water in a pipe, blood circulations, analysis of power stations, and study of populations (Thorpe, 1997).

In fluids mechanics, the dynamics of a flowing fluid is governed and represented by the Navier- Stokes equations which are nonlinear partial differential equations. Here our case of interest is to approximate the governing equations of the flow field in a tube, since it is nonlinear in character and it is impossible to solve these equations analytically to get the exact solution. To solve these equations, we are led to adopt some restrictive assumptions and some simplifications, which involve the suppositions of weak non linearity to apply traditional perturbation methods, small parameter assumptions which restrict the wide applications of the perturbation techniques, linearization which is certainly a handy task, discretization to apply numerical techniques etc. In using the traditional numerical methods for the numerical solution of the Navier- Stokes equations are very difficult and it is due to mixing of different length scales involving in the fluid flow which results in massive out prints.

Our objective here is to find the continuous analytical solution to the governing equation in cylindrical coordinates without massive outsprints and restrictive suppositions as discussed above, which change physical problem into a mathematical problem. K. Haldar (Haldar, 1995) used Adomian's decomposition method (Adomian, 1996; Adomian, 1989) for the analytical approximation of the problem which is most transparent method for the solutions of the Navier-Stokes equation in cylindrical coordinates. However the limitations of this method involve a handy task of the calculations of the Adomian polynomials, which proved to be too difficult and cause to slow down the application. To overcome this shortcoming, we make use of the homotopy perturbation method to get analytical approximations for different choices of linear operators and the initial guesses available. Recently, the homotopy perturbation method being a powerful technique was developed by He (He, 1999, 2005) .The main advantage of this technique is to overcome the difficulties arising in the process of calculations for the nonlinear terms arising in the problem. This gives analytical approximation to the different classes of the nonlinear differential equations, system of differential equations, integral and integro-differential equation and systems of such equations. Haldar applied the Adomian's decomposition method to the Navier- stokes equations in cylindrical coordinates for two dimensional irrotational fluid flow in a tube (Hardar, 1997). Our present analysis gives the application of homotopy perturbation method without any restrictive assumptions and handy calculations of the Adomian polynomials to the Navier Stoke Equations in cylindrical coordinates, in which the steady two dimensional irrotational flow of fluid in a tube of non-uniform circular cross section can be studied.

2 The Governing Equations

Consider the governing equations of motion for the two dimensional flow field for a viscous fluid in a tube which are described by the cylindrical coordinate transformation of the Navier-Stokes equations read as;

$$u\frac{\partial u}{\partial z}+v\frac{\partial u}{\partial r}=\upsilon\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{\partial^2 u}{\partial z^2}\right)-\frac{1}{\rho}\frac{\partial P}{\partial z}, \tag{1}$$

$$u\frac{\partial v}{\partial z}+v\frac{\partial v}{\partial r}=\upsilon\left(\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}-\frac{v^2}{r^2}+\frac{\partial^2 v}{\partial z^2}\right)-\frac{1}{\rho}\frac{\partial P}{\partial r}. \tag{2}$$

It is suggested that the rotational motion of the fluid is negligible. Then the equation of continuity reads is

$$\frac{1}{r}\frac{\partial}{\partial r}(rv)+\frac{\partial u}{\partial z}=0\cdot \tag{3}$$

Where $u,$ is fluid velocity components in the axial x coordinate and v is in the radial coordinate r, and the fluid pressure is described by P, the fluid density by ρ, and the kinematic viscosity by υ for the fluid. Introducing and labeling the stream function as Φ, then we may have,

$$u=-\frac{1}{r}\frac{\partial\Phi}{\partial r}, \qquad \text{and} \qquad v=\frac{1}{r}\frac{\partial\Phi}{\partial z}\ , \tag{4}$$

The equation of continuity is satisfied identically. The dynamical equation of motion in term of the stream function Φ are obtained by eliminating P between (2) and (3), and making us of the relation (4), it is read as;

$$\frac{1}{r}\frac{\partial\left(\Im\Phi,\Phi\right)}{\partial\left(r,\mathrm{z}\right)}-\frac{2}{r^{2}}\frac{\partial\Phi\Im\Phi}{\partial\Phi}=\upsilon\Im^{2}\Phi \tag{5}$$

Introducing $\Im$ as a linear operator which is defined as;

$$\Im=\frac{\partial^{2}}{\partial r^{2}}-\left(\frac{1}{r}\right)\frac{\partial}{\partial r}+\frac{\partial^{2}}{\partial z^{2}}, \tag{6}$$

and the Jacobean defined as

$$\frac{\partial\left(L\Phi,\Phi\right)}{\partial\left(r,\mathrm{z}\right)}=\frac{\partial(\Im\Phi)}{\partial r}\cdot\frac{\partial\Phi}{\partial z}-\frac{\partial\Phi}{\partial r}\frac{\partial(\Im\Phi)}{\partial z}=\begin{vmatrix}\frac{\partial\left(\Im\Phi\right)}{\partial r} & \frac{\partial\Phi}{\partial r}\\ \frac{\partial\left(\Im\Phi\right)}{\partial z} & \frac{\partial\Phi}{\partial z}\end{vmatrix}. \tag{7}$$

Now, we here mainly discuss to forms of the linear operator $\Im$ defined by equation (5). We will split the linear operator $\Im$ in two parts and discuss the two cases. It is to note that in the homotopy perturbation method we are free to choose the linear operator. This mainly depends upon the given form of the initial or boundary condition and the problem under investigation. Therefore, depending upon our choices and the possibilities for the appearance of the auxiliary linear form of operator in the problem we consider two cases here.

Case 1:

The first form of the linear operator extracted from equation (5) for the possible form of the linear operator is supposed to be;

$$\Im_{1}=\frac{\partial^{2}}{\partial r^{2}}-\frac{1}{r}\frac{\partial}{\partial r}. \tag{8}$$

Then the operator $\Im$ becomes $\Im=\Im_{1}+\frac{\partial^{2}}{\partial z^{2}}$, which implies that $\Im^{2}=\Im^{2}{}_{1}+2\frac{\partial^{2}}{\partial z^{2}}\Im_{1}+\Im^{2}{}_{1}$ then

$$\Im^{2}\Phi=\Im_{1}{}^{2}\Phi+2\frac{\partial^{2}\Phi}{\partial z^{2}}\Im_{1}\Phi+\Im^{2}{}_{1}\Phi. \tag{9}$$

Using (9) in (5), the equation (5) takes the following form

$$\frac{1}{r}\frac{\partial(\Im\Phi,\Phi)}{\partial(\mathrm{r},\mathrm{z})}-\frac{2}{r^{2}}\frac{\partial^{2}\Phi}{\partial z}\Im\Phi=\upsilon\left(\Im^{2}{}_{1}\Phi+2\frac{\partial^{2}}{\partial z^{2}}(\Im^{2}{}_{1}\Phi)+\frac{\partial^{4}\Phi}{\partial z^{4}}\right),$$

Taking υ^{-1} both sides

$$\Im^{2}{}_{1}\Phi+2\frac{\partial^{2}(\Im_{1}\Phi)}{\partial z^{2}}+\frac{\partial^{4}\Phi}{\partial z^{4}}=\upsilon^{-1}\left(\frac{1}{r}\frac{\partial(\Im\Phi,\Phi)}{\partial(\mathrm{r},\mathrm{z})}-\frac{2}{r^{2}}\frac{\partial\Phi}{\partial z}\Im\Phi\right).$$

In order to apply the proposed homotopy perturbation method to the given problem, we need to define the nonlinear term appearing in the governing equations. Therefore, we define the nonlinear term as $N\Phi$ in the above equation which is given as;

$$N\Phi=\frac{1}{r}\frac{\partial(\Im\Phi,\Phi)}{\partial(\mathrm{r},\mathrm{z})}-\frac{2}{r^{2}}\frac{\partial\Phi}{\partial z}\Im\Phi.$$

Then we get the following nonlinear form of equation for our analysis,

$$\Im_1{}^2\Phi=\left(\upsilon^{-1}\left(\frac{1}{r}\frac{\partial(\Im\Phi,\Phi)}{\partial(\mathrm{r,z})}-\frac{2}{r^2}\frac{\partial\Phi}{\partial z}\cdot\Im\Phi\right)-2\frac{\partial^2(\Im_1\Phi)}{\partial z^2}-\frac{\partial^4\Phi}{\partial z^4}\right)$$

Operating $\Im^{-1}{}_1$ on both sides of the above equation,

$$\Im_1\Phi=\Im^{-1}{}_1\left(\upsilon^{-1}\left(\left(\frac{1}{r}\frac{\partial(\Im\Phi,\Phi)}{(\mathrm{r,z})}\right)-\frac{2}{r^2}\frac{\partial\Phi}{\partial z}\cdot\Im\Phi\right)-2\frac{\partial^2\Im_1\Phi}{\partial z^2}-\frac{\partial^4\Phi}{\partial z^4}\right), \tag{10}$$

Using homotopy perturbation method (HPM) proposed by J. H. He (He, 2006), we construct a homotopy for equation (10) as; $\omega(\mathrm{r,z};\sigma):\Omega\times[0,1]\rightarrow\Re\cdot$ This satisfies $H(v,\sigma)=(1-\sigma)\left[\Im(v)-\Im(u_0)\right]+\sigma\left(A(v)-f(r)\right)=0,$ and here $\sigma\in[0,1]$ is designed to be an embedding parameter,

$$(1-\sigma)\mathrm{H}(\omega,\sigma)=\left(\Im_1\omega-\Im_1\Phi_0\right)-\sigma\left[\upsilon^{-1}\Im^{-1}{}_1\left(\frac{1}{r}\frac{\partial(\Im w,w)}{(r,z)}-\frac{2}{r^2}\frac{\partial w}{\partial z}\cdot\Im\omega\right)-2\Im^{-1}{}_1\frac{\partial\Im_1\Phi}{\partial z^2}-\Im^{-1}{}_1\frac{\partial^4\Phi}{\partial z^4}\right]=0. \tag{11}$$

Suppose the solution of (11) is of the form of

$$\omega(r,z;\sigma)=\omega_0+\sigma\omega_1+\sigma^2\omega_2+\ldots, \tag{12}$$

Using (12) in (11) we get

$$\Im_1(\omega_0+\sigma\omega_1+\ldots,)-\Im_1\Phi_0+\sigma\Im_1 w_o-\sigma\upsilon^{-1}\Im_1{}^{-1}\left[\frac{1}{r}\frac{\partial\Im(\omega_0+\sigma\omega_1+\cdots,\omega_0+\sigma\omega_1+\cdots,)}{\partial(\mathrm{r,z})}\right]+\sigma\upsilon^{-1}$$
$$\Im_1{}^{-1}\left[-\frac{2}{r^2}\frac{\partial(\omega_0+\sigma^1\omega_1+\cdots,)}{\partial z}\cdot\Im(\omega_0+\sigma^1\omega_1+\cdots,)\right]+\sigma\Im_1{}^{-1}\left[-2\frac{\partial^2\Im_1(\omega_0+\sigma\omega_1+\cdots,)}{\partial z^2}-\frac{\partial^4(\omega_0+\sigma\omega_1+\cdots,)}{\partial z^4}\right]=0. \tag{13}$$

Now we simplify the quantities enclosed in brackets,

$$\frac{\partial(\Im(\omega_0+\sigma\omega_1+\cdots,),(\omega_0+\sigma\omega_1+\cdots,))}{\partial(\mathrm{r,z})}=\begin{vmatrix}\dfrac{\partial(\Im\omega_0+\sigma\Im\omega_1+\cdots)}{\partial r} & \dfrac{\partial(\omega_0+\sigma\omega_1+\cdots)}{\partial r}\\ \dfrac{\partial(\Im\omega_0+\sigma\Im\omega_1+\cdots)}{\partial z} & \dfrac{\partial(\omega_0+\sigma\omega_1+\cdots)}{\partial z}\end{vmatrix}$$

$$=\frac{\partial(\Im\omega_0+\sigma\Im\omega_1+\sigma^2\Im\omega_2+\cdots)}{\partial r}\cdot\frac{\partial(\omega_0+\sigma\omega_1+\sigma^2\omega_2+\cdots)}{\partial z}-\frac{\partial(\Im\omega_0+\sigma\Im\omega_1+\sigma^2\Im\omega_2+\cdots)}{\partial z}\cdot\frac{\partial(\omega_0+\sigma\omega_1\sigma^2\omega_2+\cdots)}{\partial r}$$

$$+\sigma^2\left(\left((\frac{\partial\Im\omega_2}{\partial r}\cdot\frac{\partial\omega_0}{\partial z}-\frac{\partial\Im\omega_2}{\partial z}\cdot\frac{\partial\omega_0}{\partial r})+(\frac{\partial\Im\omega_0}{\partial r}\cdot\frac{\partial\omega_2}{\partial z}-\frac{\partial\Im\omega_0}{\partial z}\cdot\frac{\partial\omega_2}{\partial r})\right)+\left(\frac{\partial\Im\omega_1}{\partial r}\cdot\frac{\partial\omega_1}{\partial z}-\frac{\partial\Im\omega_1}{\partial z}\cdot\frac{\partial\omega_1}{\partial r}\right)\right)+\mathrm{O}(\sigma^3),$$

$$=\sigma^o\left(\frac{\partial(\Im\omega_0,\omega_0)}{\partial(r,z)}\right)+\sigma^1\left(\frac{\partial(\Im\omega_1,\omega_0)}{\partial(r,z)}+\frac{\partial(\Im\omega_{0,}\omega_1)}{\partial(r,z)}\right)+\sigma^2\left(\frac{\partial(\Im\omega_2,\omega_0)}{\partial(r,z)}+\frac{\partial(\Im\omega_0,\omega_2)}{\partial(r,z)}+\frac{\partial(\Im\omega_1,\omega_1)}{\partial(r,z)}\right)+\cdots \tag{14}$$

The calculations made in (14) is according to the definition of the Jacobean, and

$$\frac{\partial\left(\omega_0+\sigma^1\omega_1+\cdots\right)}{\partial z}\cdot\left(\Im\left(\omega_0+\sigma^1\omega_1+\cdots\right)\right)=\sigma^0\left(\Im\omega_0\frac{\partial\omega_0}{\partial z}\right)+\sigma^1\left(\Im\omega_1\frac{\partial\omega_0}{\partial z}+\Im\omega_0\frac{\partial\omega_1}{\partial z}\right)$$
$$+\sigma^2\left(\Im\omega_2\frac{\partial\omega_0}{\partial z}+\Im\omega_1\frac{\partial\omega_1}{\partial z}+\Im\omega_0\frac{\partial\omega_2}{\partial z}\right)+\cdots, \tag{15}$$

Combining the terms containing the equal powers of σ in equation (14) and (15)

$$C_1=\sigma^0\left(\frac{1}{r}\left(\frac{\partial\left(\Im\omega_0,\omega_0\right)}{\partial(r,z)}\right)-\Im_1^{-1}\left(\frac{-2}{r^2}\frac{\partial\omega_0}{\partial z}\cdot\Im\omega_0\right)\right), \tag{15a}$$

$$C_2=\sigma^1\left(\left(\frac{\partial\left(\Im\omega_1,\omega_0\right)}{\partial(r,z)}+\frac{\partial\left(\Im\omega_0,\omega_1\right)}{\partial(r,z)}\right)-\left(\frac{2}{r^2}\frac{\partial\omega_0}{\partial z}\cdot\Im\omega_1+\frac{2}{r^2}\frac{\partial\omega_1}{\partial z}\cdot\Im\omega_0\right)\right), \tag{15b}$$

$$C_3=\sigma^2\left(\left(\frac{\partial\left(\Im\omega_2,\omega_0\right)}{\partial(r,z)}+\frac{\partial\left(\Im\omega_0,\omega_2\right)}{\partial(r,z)}+\frac{\partial\left(\Im\omega_1,\omega_1\right)}{\partial(r,z)}\right)-\frac{2}{r^2}\frac{\partial\omega_0}{\partial z}\cdot\Im\omega_2-\frac{2}{r^2}\frac{\partial\omega_1}{\partial z}\cdot\Im\omega_1-\frac{2}{r^2}\frac{\partial\omega_2}{\partial z}\cdot\Im\omega_2\right), \tag{15c}$$

and so on. Now

$$\frac{\partial^2\left(\Im\left(\omega_0+\sigma\omega_1+\sigma^2\omega_2+\cdots\right)\right)}{\partial z^2}=\sigma^0\left(\frac{\partial^2\Im\omega_0}{\partial z^2}\right)+\sigma^1\left(\frac{\partial^2\Im\omega_1}{\partial z^2}\right)+\sigma^2\left(\frac{\partial^2\Im\omega_2}{\partial z^2}\right)+\cdots \tag{16}$$

$$\frac{\partial^4\left(\sigma^0\omega_0+\sigma^1\omega_1+\sigma^2\omega_2+\cdots\right)}{\partial z^4}=\sigma^0\left(\frac{\partial^4\omega_0}{\partial z^4}\right)+\sigma^1\left(\frac{\partial^4\omega_1}{\partial z^4}\right)+\sigma^2\left(\frac{\partial^4\omega_2}{\partial z^4}\right)+\cdots \tag{17}$$

Now combining the terms containing the equal powers of σ in equations (16) and (17)

$$D_1=\left(-2\frac{\partial^2\Im_1\omega_0}{\partial z^2}\right)-\left(\frac{\partial^4\omega_0}{\partial z^4}\right), \tag{17a}$$

$$D_2=\left(-2\frac{\partial^2\Im_1\omega_1}{\partial z^2}\right)-\left(\frac{\partial^4\omega_1}{\partial z^4}\right), \tag{17b}$$

$$D_3=\left(-2\frac{\partial^2\Im_1\omega_2}{\partial z^2}\right)-\left(\frac{\partial^4\omega_2}{\partial z^4}\right), \tag{17c}$$

and so on. Using equations (15a), (15b), (15c), (17a), (17b), (17c) in equation (13). We get from equation (13),

$$\left(\sigma^0\Im_1\omega_0+\sigma^1\Im_1\omega_1+\cdots\right)-\Im_1\Phi_0+\sigma^1\Im_1\omega_0$$
$$-\sigma\upsilon^{-1}\Im_1^{-1}\left(\sigma^0C_1+\sigma^1C_2+\sigma^2C_3+\cdots+\sigma^0D_1+\sigma^1D_2+\sigma^2D_3+\cdots\right)=0. \tag{18}$$

Equating the coefficients of equal powers of σ we have the zeroth order problem as:

Zeroth Order problem: $\Im_1\omega_0-\Im_1\phi_0=0,$ which implies $\omega_0=\Phi_0(r,z)\cdot$ (19)

Here Φ_0 is defined as the solution of homogenous equation $\Im_1^2\Phi_0=0,$ (20)

subject to the pre-prescribed boundary conditions. Now to find the approximation for ω_1 for which we first find the inverse operator $\Im_1^{-2}$ and for it we consider equation (3), $\Im_1\Phi = 0\cdot$ (21)

We now define $\Im_{rr} = \dfrac{\partial^2}{\partial r^2}$ & $\Im_r = \dfrac{\partial}{\partial r}$, then operator $\Im_1$ takes the following form as $\Im_1 = \Im_{rr} - \dfrac{1}{r}\Im_r$, $\Im_1\Phi = \Im_{rr}\Phi - \dfrac{1}{r}\Im_r\Phi$. Using equation (21) we get $\Im_{rr}\Phi - \dfrac{1}{r}\Im_r\Phi = 0\cdot$ Solving for $\Im_{rr}$ and $\Im_r$ that is for linear terms

$$\Im_{rr}\Phi = \frac{1}{r}\Im_r\Phi, \tag{22}$$

$$\Im_r\Phi = r\Im_{rr}\Phi, \tag{23}$$

Operating $\Im_r^{-1}$ on (24) and $\Im_{rr}^{-1}$ on (24) we get

$$\Phi = \varphi_1 + \Im_{rr}^{-1}\left(\frac{1}{r}\Im_r\Phi\right), \tag{24}$$

$$\Phi = \varphi_2 + \Im_r^{-1}\left(r\Im_{rr}\Phi\right), \tag{25}$$

φ_1 and φ_2 are the solutions of two homogenous equations $\Im_{rr}\Phi = 0$, and $\Im_r\Phi = 0$, respectively. The inverse linear operators $\Im_{rr}^{-1}$ and $\Im_r^{-1}$ are defined as

$$\begin{aligned}\Im_{rr}^{-1} &= \iint(\cdot)\,drdr,\\ \Im_r^{-1} &= \int(\cdot)\,dr.\end{aligned} \tag{26}$$

Adding (25) and (26) we get $2\Phi = \varphi_1 + \varphi_2 + \Im_{rr}^{-1}\left(\dfrac{1}{r}\Im_r\Phi\right) + \Im^{-1}{}_r\left(r\Im_{rr}\Phi\right)$, and dividing both sides by 2, to get

$$\Phi = \frac{\varphi_1 + \varphi_2}{2} + \left[\Im_{rr}^{-1}\left(\frac{1}{r}\Im_r\Phi\right) + \Im_r^{-1}\left(r\Im_{rr}\Phi\right)\right], \tag{27}$$

$$\Phi = \varphi_0 + \frac{1}{2}\left[\Im_{rr}^{-1}\left(\frac{1}{r}\Im_r\Phi\right) + \Im_r^{-1}\left(r\Im_{rr}\Phi\right)\right], \tag{28}$$

where $\Phi_0 = \dfrac{\varphi_1 + \varphi_2}{2}$, then

$$\begin{aligned}\Phi_1 &= \frac{1}{2}\left[\Im_{rr}^{-1}\left(\frac{1}{r}\Im_r\right) + \Im_r^{-1}\left(r\Im_{rr}\right)\right]\Phi_0,\\ \Phi_2 &= \frac{1}{2}\left[\Im_{rr}^{-1}\left(\frac{1}{r}\Im_r\right) + \Im_r^{-1}\left(r\Im_{rr}\right)\right]\Phi_1,\\ \vdots &= \vdots\\ \Phi_{n+1} &= \frac{1}{2}\left[\Im_{rr}^{-1}\left(\frac{1}{r}\Im_r\right) + \Im_r^{-1}\left(r\Im_{rr}\right)\right]\Phi_n.\end{aligned} \tag{29}$$

Let the quantity in brackets is denoted by β

$$\begin{bmatrix} \Phi_1 = \frac{1}{2}\beta\Phi_0 \\ \Phi_2 = \frac{1}{2}\beta\Phi_1 = \frac{1}{2}\beta\frac{1}{2}\beta\Phi_0 = \frac{1}{2^2}\beta^2\Phi_0 \\ \vdots = \vdots \\ \Phi_{n+1} = \frac{1}{2^{n+1}}\beta^{n+1}\Phi_0 \end{bmatrix}, \tag{30}$$

$$\Phi = \sum_{n=0}^{\infty}\frac{1}{2^n}\beta^n\Phi_0 = \sum_{n=0}^{\infty}\frac{1}{2^n}\left[\Im_{rr}{}^{-1}\left(\frac{1}{r}\Im_r\right) + \Im_r{}^{-1}\left(r\Im_{rr}\right)\right]^n\Phi_0\,\cdot \tag{31}$$

Now the inverse of the linear operator is defined $\Im_1^{-2} = \left(\Im_{rr} - \frac{1}{r}\Im_r\right)^{-2}$ as

$$\Im_1^{-2} = \sum_{n=0}^{\infty}\frac{1}{2^n}\left[\Im^{-1}{}_{rr}\left(\frac{1}{r}\Im_r\right) + \Im^{-1}{}_r\left(r\Im_{rr}\right)\right]^n\cdot \tag{32}$$

Now we come to equation (19) and define the zeroth order problem as, $\omega_0 = \Phi_0\cdot$ And the 1st order problem

as $\Im_1\omega_1 + \Im_1\omega_0 = \Im_1^{-1}\left(\upsilon^{-1}C_1 + D_1\right).$

Substituting values from equation (15a) (17a) we get,

$$\Im_1\omega_1 + \Im_1\omega_0 = \Im_1^{-1}\left[\upsilon^{-1}\left(\frac{1}{r}\frac{\partial\left(\Im\omega_0,\omega_0\right)}{\partial\left(r,z\right)} - \Im\omega_0\frac{2}{r^2}\frac{\partial\omega_0}{\partial z}\right) - 2\frac{\partial^2\Im_1\omega_0}{\partial z^2} - \frac{\partial^4\omega_0}{\partial z^4}\right].$$

Operating $\Im_1$ on both sides of the above equation yields,

$$\Im_1^2\omega_1 + \Im_1^2\omega_0 = \upsilon^{-1}\left(\frac{1}{r}\frac{\partial\left(\Im\omega_0,\omega_0\right)}{\partial\left(r,z\right)} - \Im\omega_0\frac{2}{r^2}\frac{\partial\omega_0}{\partial z}\right) - 2\frac{\partial^2\Im_1\omega_0}{\partial z^2} - \frac{\partial^4\omega_0}{\partial z^4}.$$

Making use of $\omega_0 = \Phi_0$, for the initial guess of HPM methodology, $\Im_1^2\Phi_0 = 0,$

$$\Im_1^2\omega_1 = \upsilon^{-1}\left(\frac{1}{r}\frac{\partial\left(\Im\omega_0,\omega_0\right)}{\partial\left(r,z\right)} - \Im\omega_0\frac{2}{r^2}\frac{\partial\omega_0}{\partial z}\right) - 2\frac{\partial^2\Im_1\omega_0}{\partial z^2} - \frac{\partial^4\omega_0}{\partial z^4}.$$

Operating with $\Im_1^{-2}$ on both sides of the above expression,

$$\omega_1 = \upsilon^{-1}\Im_1^{-2}\left(\frac{1}{r}\frac{\partial\left(\Im\omega_0,\omega_0\right)}{\partial\left(r,z\right)} - \Im\omega_0\frac{2}{r^2}\frac{\partial\omega_0}{\partial z}\right) - 2\Im_1^{-2}\frac{\partial^2\Im_1\omega_0}{\partial z^2} - \Im_1^{-2}\frac{\partial^4\omega_0}{\partial z^4}, \tag{33}$$

where $\Im_1^{-2}$ is given in equation (32). The 2nd order problem is given as, $\Im_1\omega_2 = \Im^{-1}{}_1\left(\upsilon^{-1}C_2 + D_2\right).$ Using the values of C_2 and D_2 from equations (15b) and (17b), we get

$$\Im_1\omega_2=\Im_1^{-1}\left[\upsilon^{-1}\left[\frac{1}{r}\left(\frac{\partial\left(\Im\omega_1,\omega_0\right)}{\partial\left(r,z\right)}+\frac{\partial\left(\Im\omega_0,\omega_1\right)}{\partial\left(r,z\right)}\right)\right]-\left(\Im\omega_1\frac{2}{r^2}\frac{\partial\omega_0}{\partial z}+\Im\omega_0\frac{2}{r^2}\frac{\partial\omega_1}{\partial z}\right)-\left(2\frac{\partial^2\Im_1\omega_1}{\partial z^2}\right)-\left(\frac{\partial^4\omega_1}{\partial z^4}\right)\right].$$

Operating with $\Im_1^{-1}$ both sides of the above equation to get,

$$\begin{aligned}\omega_2&=\upsilon^{-1}\Im_1^{-2}\left[\left(\frac{1}{r}\frac{\partial\left(\Im\Phi_1,\Phi_0\right)}{\partial\left(r,z\right)}-\Im\Phi_1\frac{2}{r^2}\frac{\partial\Phi_0}{\partial z}\right)+\left(\frac{1}{r}\frac{\partial\left(\Im\Phi_0,\Phi_1\right)}{\partial\left(r,z\right)}-\Im\Phi_0\frac{2}{r^2}\frac{\partial\Phi_1}{\partial z}\right)\right]\\&\quad-2\Im_1^{-2}\frac{\partial^2\Im_1\omega_1}{\partial z^2}-\Im_1^{-2}\frac{\partial^4\omega_1}{\partial z^4}+\cdots,\end{aligned}\tag{34}$$

Now since in the methodology of HPM, we suppose the following expression for the approximate solution of the problem,

$$\Phi=\lim_{\sigma\to1}\omega=\omega_0+\omega_1+\omega_2+\omega_3+\cdots,\tag{35}$$

where the components of the series solution are defined to be as; $\omega_0=\Phi_0$,

$$\Phi_1=\upsilon^{-1}\Im_1^{-2}\left(\frac{1}{r}\frac{\partial\left(\Im\Phi_0,\Phi_0\right)}{\partial\left(r,z\right)}-\Im\Phi_0\frac{2}{r^2}\frac{\partial\Phi_0}{\partial z}\right)-2\Im_1^{-2}\frac{\partial^2\Im_1\Phi_0}{\partial z^2}-\Im_1^{-2}\frac{\partial^4\Phi_0}{\partial z^4},$$

$$\begin{aligned}\Phi_2&=\upsilon^{-1}\Im_1^{-2}\left[\left(\frac{1}{r}\frac{\partial\left(\Im\Phi_1,\Phi_0\right)}{\partial\left(r,z\right)}-\Im\Phi_1\frac{2}{r^2}\frac{\partial\Phi_0}{\partial z}\right)+\left(\frac{1}{r}\frac{\partial\left(\Im\Phi_0,\Phi_1\right)}{\partial\left(r,z\right)}-\Im\Phi_0\frac{2}{r^2}\frac{\partial\Phi_1}{\partial z}\right)\right]\\&\quad-2\Im_1^{-2}\frac{\partial^2\Im_1\Phi_1}{\partial z^2}-\Im_1^{-2}\frac{\partial^4\Phi_1}{\partial z^4},\end{aligned}$$

and so on. If once Φ_0 is obtained, which can be easily obtained by constructing the homotopy for the given problem and equating the coefficients for the zeroth order problem, then we can find Φ_1 in terms of Φ_0 and in the similar fashion Φ_2 can be evaluated in terms of Φ_1 and Φ_0 .The other higher order components can be easily obtained having the all other lower order values. Thus all the components of Φ can be calculated. The series solution $\Phi=\sum_{n=0}^{\infty}\Phi_n$, thus can be given the following form,

$$\begin{aligned}\Phi&=\Phi_0+\upsilon^{-1}\Im_1^{-2}\left(\frac{1}{r}\frac{\partial\left(\Im\Phi_0,\Phi_0\right)}{\partial\left(r,z\right)}-\Im\Phi_0\frac{2}{r^2}\frac{\partial\Phi_0}{\partial z}\right)-2\Im_1^{-2}\frac{\partial^2\Im_1\Phi_0}{\partial z^2}-\Im_1^{-2}\frac{\partial^4\Phi_0}{\partial z^4}\\&\quad+\upsilon^{-1}\Im_1^{-2}\left[\left(\frac{1}{r}\frac{\partial\left(\Im\Phi_1,\Phi_0\right)}{\partial\left(r,z\right)}-\Im\Phi_1\frac{2}{r^2}\frac{\partial\Phi_0}{\partial z}\right)+\left(\frac{1}{r}\frac{\partial\left(\Im\Phi_0,\Phi_1\right)}{\partial\left(r,z\right)}-\Im\Phi_0\frac{2}{r^2}\frac{\partial\Phi_1}{\partial z}\right)\right]\\&\quad-2\Im_1^{-2}\frac{\partial^2\Im_1\Phi_1}{\partial z^2}-\Im_1^{-2}\frac{\partial^4\Phi_1}{\partial z^4}+\cdots\end{aligned}\tag{36}$$

Case 2:

We may have the other available or chosen form of the linear operator as;

$\Im = \Im_{rr} - \frac{1}{r}\Im_r + \Im_{zz}$. Whereas $\Im_r = \frac{\partial}{\partial r}, \Im_{rr} = \frac{\partial^2}{\partial r^2}$ and $\Im_{zz} = \frac{\partial^2}{\partial z^2}$, then from equation (5) we have,

$$\upsilon \Im^2 \Phi = \frac{1}{r}\frac{\partial(\Im\Phi,\Phi)}{\partial(r,\mathrm{z})} - \frac{2}{r^2}\frac{\partial\Phi\Im\Phi}{\partial\Phi}.$$

Taking υ^{-1} both sides of the above expression, $\Im^2\Phi = \upsilon^{-1}\left(\frac{1}{r}\frac{\partial(\Im\Phi,\Phi)}{\partial(r,\mathrm{z})} - \Im\Phi\frac{2}{r^2}\frac{\partial\Phi}{\partial z}\right).$

$$\Im^2\Phi - \upsilon^{-1}\left(\frac{1}{r}\frac{\partial(\Im\Phi,\Phi)}{\partial(r,\mathrm{z})} - \Im\Phi\frac{2}{r^2}\frac{\partial\Phi}{\partial z}\right) = 0. \tag{37}$$

Using the methodology of HPM, we may construct a homotopy for equation (37) as; $\varpi(r,z;\mathrm{q}):\Omega[0,1]\to\Re\cdot$ $H(\varpi,q) = (1-q)(\Im\varpi - \Im\Phi_0) - q\Im^{-1}\left(\upsilon^{-1}\left(\frac{1}{r}\frac{\partial(\Im\varpi,\varpi)}{\partial(r,\mathrm{z})} - \Im\varpi\frac{2}{r^2}\frac{\partial\Im\varpi}{\partial z}\right)\right)$ (38)

Suppose the solution for equation (38) is of the following form

$$\varpi(r,z;q) = q^0\varpi_0(r,z) + q^1\varpi_1(r,z) + q^2\varpi_2(r,z) + \dots \tag{39}$$

Where as

$$\begin{aligned}\frac{\partial(\Im\varpi,\varpi)}{\partial(r,\mathrm{z})} &= \begin{vmatrix}\frac{\partial\Im(\varpi_0+q\varpi_1+\cdots,)}{\partial r} & \frac{\partial(\varpi_0+q\varpi_1+\cdots),}{\partial r}\\ \frac{\partial\Im(\varpi_0+q\varpi_1+\cdots,)}{\partial z} & \frac{\partial(\varpi_0+q\varpi_1+\cdots,)}{\partial z}\end{vmatrix}\\
&= \frac{\partial\Im(\varpi_0+q\varpi_1+\cdots,)}{\partial r}\cdot\frac{\partial(\varpi_0+q\varpi_1+\cdots,)}{\partial z} - \frac{\partial(\varpi_0+q\varpi_1+\cdots,)}{\partial r}\cdot\frac{\partial\Im(\varpi_0+q\varpi_1+\cdots,)}{\partial z}\\
&= \left(\frac{\partial(\Im\varpi_0,\varpi_0)}{\partial(r,z)}\right) + q^1\left(\frac{\partial(\Im\varpi_1,\varpi_0)}{\partial(r,z)} + \frac{\partial(\Im\varpi_0,\varpi_1)}{\partial(r,z)}\right)\\
&\quad + q^2\left(\frac{\partial(\Im\varpi_2,\varpi_0)}{\partial(r,z)} + \frac{\partial(\Im\varpi_0,\varpi_2)}{\partial(r,z)} + \frac{\partial(\Im\varpi_1,\varpi_1)}{\partial(r,z)}\right) + \dots,\end{aligned} \tag{40}$$

$$\begin{aligned}\frac{\partial(\varpi_0+\sigma^1\varpi_1+\cdots,)}{\partial z}\cdot\Im(\varpi_0+\sigma^1\varpi_1+\cdots,) &= q^0\left(\Im\varpi_0\frac{\partial\varpi_0}{\partial z}\right) + q^1\left(\Im\varpi\frac{\partial\varpi_0}{\partial z} + \Im\varpi_0\frac{\partial\varpi_1}{\partial z}\right)\\
&\quad + q^2\left(\Im\varpi_2\frac{\partial\omega_0}{\partial z} + \Im\varpi_1\frac{\partial\varpi_1}{\partial z} + \Im\varpi_0\frac{\partial\varpi_2}{\partial z}\right) + \cdots,\end{aligned} \tag{41}$$

Using (40) and (41) in (38) we get

$$\Im\varpi - \Im\Phi_0 + \Im\varpi_0 - q\Im^{-1}\Bigg[\left(\frac{\partial(\Im\varpi_0,\varpi_0)}{\partial(r,z)}\right) + q^1\left(\frac{\partial(\Im\varpi_1,\varpi_0)}{\partial(r,z)} + \frac{\partial(\Im\varpi_0,\varpi_1)}{\partial(r,z)}\right) +$$
$$q^2\left(\frac{\partial(\Im\varpi_2,\varpi_0)}{\partial(r,z)} + \frac{\partial(\Im\varpi_0,\varpi_2)}{\partial(r,z)} + \frac{\partial(\Im\varpi_1,\varpi_1)}{\partial(r,z)}\right) + \ldots$$
$$-q^0\left(\Im\omega_0 \frac{-2}{r^2}\frac{\partial\omega_0}{\partial z}\right) - q^1\left(\Im\omega_1 \frac{-2}{r^2}\frac{\partial\omega_0}{\partial z} + \Im\omega_0 \frac{-2}{r^2}\frac{\partial\omega_1}{\partial z}\right) + \cdots,\Bigg] = 0$$

We first define the inverse linear operator $\Im^{-2}$, consider equation (3), $\Im\Phi = 0.$ Then

$$\Im\Phi = \Im_{rr}\Phi - \frac{1}{r}\Im_r\Phi + \Im_{zz}\Phi, \qquad \Rightarrow \Im_{rr}\Phi - \frac{1}{r}\Im_r\Phi + \Im_{zz}\Phi = 0, \qquad \Im_{rr}\Phi + \Im_{zz}\Phi = \frac{1}{r}\Im_r\Phi,$$

Multiplying both sides of the above equation by r , we get

$$\Im_r\Phi = r\Im_{rr}\Phi + r\Im_{zz}\Phi \ . \tag{42}$$

In similar way, we get

$$\Im_{zz}\Phi = \frac{1}{r}\Im_r\Phi - \Im_{rr}\Phi, \tag{43}$$

$$\Im_{rr}\Phi = \frac{1}{r}\Im_r\Phi - \Im_{zz}\Phi. \tag{44}$$

$$\begin{pmatrix} \Phi = \psi_1 + \Im^{-1}{}_r\left(r\Im_{rr}\Phi + r\Im_{zz}\Phi\right) \\ \Phi = \psi_2 + \Im^{-1}{}_{zz}\left(\frac{1}{r}\Im_r\Phi - \Im_{rr}\Phi\right) \\ \Phi = \psi_3 + \Im^{-1}{}_{rr}\left(\frac{1}{r}\Im_r\Phi - \Im_{zz}\Phi\right) \end{pmatrix} \tag{45}$$

$$\Phi = \frac{\psi_1 + \psi_2 + \psi_3}{3} + \frac{1}{3}\left(\Im^{-1}{}_r\left(r\Im_{rr}\Phi + r\Im_{zz}\Phi\right) + \Im^{-1}{}_{zz}\left(\frac{1}{r}\Im_r\Phi - \Im_{rr}\Phi\right) + \Im^{-1}{}_{rr}\left(\frac{1}{r}\Im_r\Phi - \Im_{zz}\Phi\right)\right),$$

(46)

where $\Phi_0 = \frac{\psi_1 + \psi_2 + \psi_3}{3}$, and ψ_1, ψ_2 and ψ_3 are the solutions of homogenous equations $\Im_r\Phi = 0$, $\Im_{rr}\Phi = 0$ and $\Im_{zz}\Phi = 0$, then

$$\Phi = \Phi_0 + \frac{1}{3}\left(\Im^{-1}{}_r\left(r\Im_{rr}\Phi + r\Im_{zz}\Phi\right) + \Im^{-1}{}_{zz}\left(\frac{1}{r}\Im_r\Phi - \Im_{rr}\Phi\right) + \Im^{-1}{}_{rr}\left(\frac{1}{r}\Im_r\Phi - \Im_{zz}\Phi\right)\right). \tag{47}$$

The inverse linear operators $\Im_r{}^{-1}$, $\Im_{zz}{}^{-1}$, and $\Im_{rr}{}^{-1}$ are defined by

$$\Im_r{}^{-1} = \int(\cdot)\,dr, \qquad \Im_{zz}{}^{-1} = \int(\cdot)\,dzdz, \qquad \Im_{rr}{}^{-1} = \int(\cdot)\,drdr \,\cdot \tag{48}$$

Then we have,

$$\begin{aligned}
\Phi_1 &= \frac{1}{3}\left(\Im_r^{-1}\left(r\Im_{rr}+r\Im_{zz}\right)+\Im_{zz}^{-1}\left(\frac{1}{r}\Im_r-\Im_{rr}\right)+\Im_{rr}^{-1}\left(\frac{1}{r}\Im_r-\Im_{zz}\right)\right)\Phi_0,\\
\Phi_2 &= \frac{1}{3}\left(\Im_r^{-1}\left(r\Im_{rr}+r\Im_{zz}\right)+\Im_{zz}^{-1}\left(\frac{1}{r}\Im_r-\Im_{rr}\right)+\Im_{rr}^{-1}\left(\frac{1}{r}\Im_r-\Im_{zz}\right)\right)\Phi_1,\\
\vdots &= \vdots\\
\Phi_{n+1} &= \frac{1}{3}\left(\Im_r^{-1}\left(r\Im_{rr}+r\Im_{zz}\right)+\Im_{zz}^{-1}\left(\frac{1}{r}\Im_r-\Im_{rr}\right)+\Im_{rr}^{-1}\left(\frac{1}{r}\Im_r-\Im_{zz}\right)\right)\Phi_n.
\end{aligned} \tag{49}$$

Let us denote the quantity within the brackets by ℓ, then the following expression is obtained,

$$\left[\begin{aligned}
\Phi_1 &= \frac{1}{3}\ell\Phi_0,\\
\Phi_2 &= \frac{1}{3}\ell\Phi_1 = \frac{1}{3}\ell\frac{1}{3}\ell\Phi_0 = \frac{1}{3^2}\ell^2\Phi_0,\\
\Phi_3 &= \frac{1}{3}\ell\Phi_2 = \frac{1}{3^3}\ell^3\Phi_0,\\
\vdots &= \vdots\\
\Phi_{n+1} &= \frac{1}{3^{n+1}}\ell^{n+1}\Phi_0,
\end{aligned}\right] \tag{50}$$

$$\Phi = \sum_{n=0}^{\infty}\frac{1}{3^n}\ell^n\Phi_0 = \sum_{n=0}^{\infty}\frac{1}{3^n}\left(\Im^{-1}{}_r\left(r\Im_{rr}+r\Im_{zz}\right)+\Im^{-1}{}_{zz}\left(\frac{1}{r}\Im_r-\Im_{rr}\right)+\Im^{-1}{}_{rr}\left(\frac{1}{r}\Im_r-\Im_{zz}\right)\right)^n\Phi_n.$$

Thus the inverse linear operator can be easily identified as;

$$\Im^{-2} = \left(\Im_{rr}-\frac{1}{r}\Im_r+\Im_{zz}\right)^{-2} = \sum_{n=0}^{\infty}\frac{1}{3^n}\left(\Im_r^{-1}\left(r\Im_{rr}+r\Im_{zz}\right)+\Im_{zz}^{-1}\left(\frac{1}{r}\Im_r-\Im_{rr}\right)+\Im_{rr}^{-1}\left(\frac{1}{r}\Im_r-\Im_{zz}\right)\right)^n\cdot \tag{51}$$

Now the zeroth order problem is $\Im\varpi_0 = \Im\Phi_0, \qquad \Rightarrow \varpi_0 = \Phi_0.$ (52)

The 1st order problem is: $\Im\varpi_1+\Im\varpi_0 = \upsilon^{-1}\Im^{-1}\left(\frac{1}{r}\frac{\partial\left(\Im\varpi_0,\varpi_0\right)}{\partial\left(r,z\right)}-\frac{2}{r^2}\frac{\partial\varpi_0}{\partial z}\cdot\Im\varpi_0\right).$

Operating with $\Im$ both sides of the above equations;

$$\Im^2\varpi_1+\Im^2\varpi_0 = \upsilon^{-1}\left(\frac{1}{r}\frac{\partial\left(\Im\varpi_0,\varpi_0\right)}{\partial\left(r,z\right)}-\frac{2}{r^2}\frac{\partial\varpi_0}{\partial z}\cdot\Im\varpi_0\right)$$

In order to find the initial guess of HPM, we make use of the zeroth order problem as: $\Im^2\varpi_0 = 0.$

$$\Im^2\varpi_1 = \upsilon^{-1}\left(\frac{1}{r}\frac{\partial\left(\Im\varpi_0,\varpi_0\right)}{\partial\left(r,z\right)}-\frac{2}{r^2}\frac{\partial\varpi_0}{\partial z}\cdot\Im\varpi_0\right),$$

Operating $\Im^{-2}$ on both sides to get, $\varpi_1 = \upsilon^{-1}\Im^{-2}\left(\frac{1}{r}\frac{\partial\left(\Im\varpi_0,\varpi_0\right)}{\partial\left(r,z\right)}-\frac{2}{r^2}\frac{\partial\varpi_0}{\partial z}\cdot\Im\varpi_0\right),$ (53)

The 2nd order problem is:

$$\varpi_2 = \upsilon^{-1}\Im^{-2}\left(\frac{\partial(\Im\omega_1,\omega_0)}{\partial(r,z)} - \frac{2}{r^2}\frac{\partial\varpi_0}{\partial z}\cdot\Im\varpi_1 + \frac{\partial(\Im\omega_0,\omega_1)}{\partial(r,z)} - \frac{2}{r^2}\frac{\partial\varpi_1}{\partial z}\cdot\Im\varpi_0\right),$$

and so on. The series solution form of the problem reads as, $\Phi = \lim_{\vartheta\to 1}\varpi(r,z;\mathrm{q}) = \varpi_0 + \varpi_2 + \varpi_3 + \cdots,$

Where the following quantities are defined, $\Phi_0 = \dfrac{\psi_1 + \psi_2 + \psi_3}{3}$

$$\Phi_1 = \upsilon^{-1}\Im^{-2}\left(\frac{1}{r}\frac{\partial(\Im\Phi_0,\Phi_0)}{\partial(r,z)} - \frac{2}{r^2}\frac{\partial\Phi_0}{\partial z}\cdot\Im\Phi_0\right),$$

$$\Phi_2 = \upsilon^{-1}\Im^{-2}\left(\frac{\partial(\Im\Phi_1,\Phi_0)}{\partial(r,z)} - \frac{2}{r^2}\frac{\partial\Phi_0}{\partial z}\cdot\Im\Phi_1 + \frac{\partial(\Im\Phi_0,\Phi_1)}{\partial(r,z)} - \frac{2}{r^2}\frac{\partial\Phi_1}{\partial z}\cdot\Im\Phi_0\right), \tag{54}$$

and so on. The series form of the solution can be written as; $\Phi = \sum_{n=0}^{\infty}\Phi_n,$

$$\begin{aligned}\Phi &= \Phi_0 + \upsilon^{-1}\Im_1^{-2}\left(\frac{1}{r}\frac{\partial(\Im\Phi_0,\Phi_0)}{\partial(r,z)} - \Im\Phi_0\frac{2}{r^2}\frac{\partial\Phi_0}{\partial z}\right) - 2\Im_1^{-2}\frac{\partial^2\Im_1\Phi_0}{\partial z^2} - \Im_1^{-2}\frac{\partial^4\Phi_0}{\partial z^4}\\ &+ \upsilon^{-1}\Im_1^{-2}\left[\left(\frac{1}{r}\frac{\partial(\Im\Phi_1,\Phi_0)}{\partial(r,z)} - \Im\Phi_1\frac{2}{r^2}\frac{\partial\Phi_0}{\partial z}\right) + \left(\frac{1}{r}\frac{\partial(\Im\Phi_0,\Phi_1)}{\partial(r,z)} - \Im\Phi_0\frac{2}{r^2}\frac{\partial\Phi_1}{\partial z}\right)\right]\\ &+ \cdots - 2\Im_1^{-2}\frac{\partial^2\Im_1\Phi_1}{\partial z^2} - \Im_1^{-2}\frac{\partial^4\Phi_1}{\partial z^4} + \cdots\end{aligned}$$

3 Conclusion

We have considered two cases for the available linear operators and obtained the approximation for our problem. Of course, the selection of the linear operators mainly depends upon the given initial or boundary conditions. We can see that for the first case, the available linear operator was split in two parts and for the second case, we considered the full linear form of the operator without splitting it into parts. Thus, on the basis on methodology of the Adomian decomposition and the homotopy perturbation method (Haldar, 1995), the present analysis can be applied to a wide range of the physical and engineering problems (Shakil et al., 2013; Wahab et al., 2013; Wahab et al., 2014; Siddiqui et al., 2014).

As compared to the Adomian decomposition method for the analysis of the problem (Haldar, 1995), we have the great advantage of the selection of the initial guess which can be chosen on the basis of the previous knowledge, and most importantly, the initial approximation should satisfy the given initial or boundary conditions, which leads us to the uniformly valid approximately series solution. While, the Adomian decomposition method does not have such advantage, because we have to select the initial guess based on the recursive relation produced by the method. But this initial approximations sometimes, may lead to non-uniformly valid series solution which also may contain the secular terms in the series. In homotopy perturbation method, the initial guess satisfying the given conditions may give a uniformly valid series solution.

On the other hand, the calculation of the Adomian polynomials is not an easy task for the nonlinear terms appearing in the problems. However, there are some computer programs which can calculate the Adomian polynomials, but they are for some specific cases. In our analysis, we avoid such handy calculation because homotopy perturbation method transforms a non-linear problem into a small number or sub-linear problems with prescribed conditions. No matter of concern with the existence of the parameter small or large. This is again a dominating advantage of the method over Adomian decomposition method (Shakil et al., 2013; Wahab et al., 2013; Wahab et al., 2014; Siddiqui et al., 2014).

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