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Modeling and analysis of the survival of a biological species in a polluted environment: Effect of environmental tax

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Abstract
In this paper, a nonlinear dynamical model is proposed and analyzed to study the survival of biological species in a polluted environment considering the effect of environmental tax which can be used further to improve environmental quality. The environmental tax is imposed to control the emission of pollutants/toxicants only when the equilibrium concentration of pollutants go beyond its threshold level causing harm to the biological species and its ecosystem under consideration. Local and nonlinear stability conditions are obtained by considering suitable Liapunov function. Numerical simulation of the dynamical system is performed in order to illustrate the analytical findings. It is shown that the density of biological species decreases as the concentration of pollutants increases and may even become extinct if the concentration is very high. It has also been shown that the environmental tax plays an important role to control the concentration of pollutants in the atmosphere and maintaining the density of biological species at a desired level.

Keywords modeling; biological species; polluted environment; environmental tax; stability.

1 Introduction

It has been observed during last several years that various kinds of toxicants (pollutants) such as toxic gases, smoke, particulate matters, cement dust, chemicals, etc. discharged from various industries and other sources have made considerable change in the both terrestrial and aquatic environment in the form of deforestation, air pollution, water pollution, etc. The survival of biological species is threatened instantly due to polluted air, water, soil, land and vegetation, etc. caused by toxicants. Therefore, it is crucial to investigate the effect of toxicants on biological species and the reduction in concentration in the atmosphere by imposing environmental tax on emitters which may in turn reduce environmental damage and minimizing harm to economic growth.

Some investigations have been made to study the effect of toxicants released to the water bodies, gaseous pollutants and particulate matters on the environment and ecology as well as on biological species (Lovett and Kinsman, 1990; Hopke, 2009; Woo, 2009; Cambra-Lopez, 2010; Pertsev and Tsaregorodtseva, 2011). For example, Cambra-Lopez (2010) have reviewed the effect of airborne particulate matters from the livestock production systems and have shown that high concentration of particulate matters can deteriorate the environment as well as the health of human and animals.

In recent years, several studies have been conducted to comprehend the effect of toxicants on biological species living in a polluted environment (Hallam and Clark, 1981; Freedman and Shukla, 1991; Liu and Ma, 1991; Chattopadhyaya, 1996; Shukla and Dubey, 1996, 1997; Shukla et al., 2001, 2009, 2013; Mukherjee, 2002; Liu et al., 2003; Samanta and Matti, 2004; Dubey and Hussain, 2006; Naresh et al., 2006; Dubey, 2010; Samanta, 2010; Misra and Kalra, 2012; Naresh et al., 2014). In this regard, Samanta and Matti (2004) have proposed and analyzed a nonlinear mathematical model to study the effect of toxicant on a single species living in a polluted environment by considering three cases; instantaneous input of toxicant, constant input of toxicant and fluctuating emission rate of toxicant into the environment. In the analysis, it has been shown that instantaneous emission of toxicants has no significant effect on population but the population settles down to a steady state if the toxicants are emitted incessantly. Dubey and Hussain (2006) proposed a mathematical model for the survival of species dependent on a resource in polluted environment considering the effect of diffusion on the system. They have shown that the equilibrium level of the density of population decreases as the environmental concentration of the pollutant increases. Naresh et al. (2006) have studied the dynamics of the plant biomass in a polluted environment by considering the effect of intermediate toxic product formed by uptake of a toxicant on plant biomass. It has been shown that intermediate toxic product is mainly responsible for the decrease in the intrinsic growth of plant biomass and the equilibrium label of the density of plant biomass depends upon the rate of emission of toxicant into the atmosphere. Shukla et al. (2009) have studied the effect of toxicants on population emitted from extraneous sources as well as formed by its precursors by developing a nonlinear mathematical model and have shown that the densities of population and its resource decrease due to increase in the concentration of toxicants in the environment.

At present, deterioration in environmental quality due to discharge of toxicants in the atmosphere is a burning issue in India and elsewhere. Rapid growth of industries, fast population growth, increasing demand of resources and deforestation are all exacerbating problems that need to be comprehended. In this regard, some mitigation options are to be required to improve environmental quality. Environmental tax policy may be one of the most potent mitigation strategies that must be imposed to the emitters keeping in mind that it is imposed neither to enhance net additional tax revenue nor to reduce the overall energy consumptions but is implemented to get clean environment. During previous years, effect of implementation of environmental tax has been discussed by policy makers to reduce environmental damages and to get a clean environment (Symons et al., 1994; Bovenberg et al., 1996; Stern, 2006; Sterner, 2007; Braathen and Greene, 2011; Liu, 2012). Environmental tax, to be imposed to the emitters, is generally based on the following factors,

1. The quantity of the pollutants/toxicants discharged into the environment
2. The use of resources
3. The products responsible for environmental degradation
4. The vehicle excise duty

In India, about 64% policy makers have considered environmental tax as a most important significant factor making a clean environment (Kanabar, 2011). It is mentioned here that the tax is levied to the emitters only when the concentration of pollutants in the atmosphere crosses a threshold limit. Threshold means the concentration of pollutants below which there is no harm to the population and its environment. The tax is imposed on the basis of per unit emission of pollutants (beyond its threshold limit) in the environment. The study of implementation of environmental tax to reduce the concentration of pollutants in the atmosphere has less understood and received little attention using nonlinear mathematical models. In this regard, Agarwal and Devi (2010) have studied the effect of environmental tax on the survival of biological species in a polluted environment using a mathematical model but they have not considered the formation of intermediate toxic product inside the biomass.

In view of the above, in this paper, we have proposed and analyzed a nonlinear mathematical model to study the effect of environmental tax to reduce the concentration of toxicants in the atmosphere forming intermediate toxic product inside the biomass due to interaction of toxicants with sap (biomass fluid).

2 Mathematical Model

Consider a biological species living in a polluted environment affected by toxicants emitted into the atmosphere from different sources. To model the phenomenon, we have made the following assumptions,

1. The cumulative rate of emission of toxicants is constant (say Q) though it may be a function of time.

2. The growth rate of biological species is affected by an intermediate toxic product (formed inside the biomass due to the interaction of uptaken toxicants and the liquid present in the biomass).
3. The carrying capacity of biological species is affected by the concentration of toxicants emitted into the environment and it decreases with increase in the concentration of toxicants emitted into the environment.
4. Environmental tax is assumed to be imposed only when the toxicant concentration crosses a threshold and as such, it is taken to be directly proportional to the difference of toxicants concentration and its threshold level. Threshold concentration implies the level up to which there is no harmful effect on biological species.

Let $N(t)$ be the density of biological species, $T(t), U(t)$ and $U_1(t)$ be the concentrations of toxicants emitted into the environment, the toxicants uptaken by biological species and the intermediate toxic product formed, respectively. Let $I(t)$ be the environmental tax imposed on the emitters. It is assumed that the depletion of toxicants is directly proportional to the concentration of toxicants as well as the density of biological species i.e. $\alpha N(t)T(t)$, α being the interaction rate coefficient of toxicants with biological species, θ is the uptake rate coefficient of toxicants due to biological species and $(1-\theta)\alpha$ is the rate by which biological species are directly affected by toxicants. The uptake concentration of toxicants is assumed to be depleted naturally by a rate α_0 and α_1 is the interaction rate coefficient of toxicants uptaken by the biological species. When toxicants uptaken interact with the fluid (sap) inside biological species, intermediate toxic product is formed which is mainly responsible for deterioration of biological species. Let β be the rate of formation of intermediate toxic product and the constants β_0 and β_1 are the depletion rate coefficients of intermediate toxic product due to excretion and depuration of toxicants. To control the emission of toxicants into the atmosphere, environmental tax is assumed to be imposed on the emitters when toxicant concentration crosses a threshold level and it is assumed to be proportional to the difference of toxicants concentration and its threshold value i.e. $\nu(T-T_0)$, T_0 being the threshold concentration of toxicants and ν is the tax rate coefficient. If $T \le T_0$, no tax will be imposed on the emitters. Since it is difficult to implement and maintain a foolproof tax system due to some practical problems like pilferages, natural and administrative problems, it is, therefore, obvious to consider a term $\nu_0 E$ as a depletion of environmental tax due to these factors. The constant μ is the tax repulsion rate coefficient.

$$\frac{dN(t)}{dt} = \left(r(U_1(t)) - \frac{r_0 N(t)}{K(T(t))} \right) N(t) - (1-\theta)\alpha\, N(t)T(t) \tag{1}$$

$$\frac{dT(t)}{dt} = Q(t) - \delta_0\, T(t) - \alpha\, N(t)T(t) - \mu\, E(t) \tag{2}$$

$$\frac{dU(t)}{dt} = \theta\,\alpha\, N(t)T(t) - \alpha_0 U(t) - \alpha_1 U(t)N(t) \tag{3}$$

$$\frac{dU_1(t)}{dt} = \beta U(t) - \beta_0 U_1(t) - \beta_1 U_1(t)N(t) \tag{4}$$

$$\frac{dE(t)}{dt} = \nu(T(t) - T_0) - \nu_0\, E(t) \tag{5}$$

$$N(0) \geq 0,\ T(0) = T_0 \geq 0,\ U(0) \geq 0, U_1(0) \geq 0, E(0) \geq 0$$

In the model, the function $r(U_1)$ denotes the intrinsic growth rate of biological species in the presence of intermediate toxic product formed inside it, as discussed above, and r_0 is the maximum intrinsic growth rate of biological species in the absence of intrinsic toxic product. The intrinsic growth rate $r(U_1)$ decreases as the concentration of intermediate toxic product U_1 increases and hence, we assume that,

$$r(0) = r_0 > 0\,,\ r'(U_1) < 0 \text{ for } U_1 > 0$$

The function $K(T)$ denotes the carrying capacity of species in presence of toxicants in the environment and K_0 is the maximum carrying capacity in the absence of toxicants. The carrying capacity $K(T)$ decreases as the concentration of toxicants T increases and hence,

$$K(0) = K_0 > 0\,,\ K'(T) < 0 \text{ for } T > 0$$

Remarks

1. As discussed above, the rate of discharge of toxicants (Q) into the atmosphere is assumed to be constant which is controlled by introducing a term μE (environmental tax), given in equation (2). From (2), we note that as μ (the tax repulsion coefficient) increases, the concentration of toxicant into the atmosphere decreases.

2. It is further remarked here that the cumulative concentration of toxicants/pollutants must be greater than its threshold concentration (i.e. $T > T_0$) for the physical significance of the model system and in this situation the tax will be imposed on the industrialists and would continue till $T \le T_0$. If $T \le T_0$, then $\frac{dE}{dt}$ will be negative and no tax will be imposed to the concerned industrialists. Further, if $\mu = 0$ i.e. no tax is imposed to the industrialists, the toxicants concentration would cross its harmful limit (threshold concentration) and the survival of biological species will be threatened and it might become extinct.

It is, therefore, desirable that environmental tax must be levied to keep the toxicants emission under control.

Lemma If $\frac{Q}{\delta_0} > T_0$, then the set

$$\Omega = \left\{ (N, T, U, U_1, E) : 0 \le N \le K_0, 0 \le T \le \frac{Q}{\delta_0}, 0 \le U \le U_m, 0 \le U_1 \le U_{1m}, 0 \le E \le E_m \right\}$$

is the region of attraction for all solutions of the model system (1) – (5) initiating in the interior of positive octant, where $U_m = \frac{\theta \alpha Q K_0}{\delta_0 \alpha_0}$, $U_{1m} = \frac{\beta}{\beta_0} U_m$, $E_m = \frac{\nu}{\nu_0} \left\{ \frac{Q}{\delta_0} - T_0 \right\}$.

Since $\frac{Q}{\delta_0}$ is the maximum concentration of toxicants and therefore it is remarked here that the condition $T_0 < \frac{Q}{\delta_0}$ for the existence of region of attraction Ω implies that the environmental tax can be imposed to the industrialists only when the concentration of toxicants crosses its threshold value.

3 Equilibrium Analysis

The model under consideration has following two nonnegative equilibria,

(i) $E_0(0, \bar{T}, 0, 0, \bar{E})$

where $\bar{T} = \dfrac{Q + \frac{\mu\nu}{\nu_0} T_0}{\delta_0 + \frac{\mu\nu}{\nu_0}}$ and $\bar{E} = \dfrac{\nu}{\nu_0} \dfrac{Q - \delta_0 T_0}{\delta_0 + \frac{\mu\nu}{\nu_0}}$

(ii) $E^*(N^*, T^*, U^*, U_1^*, E^*)$

The positive solution of E^* is given by the following system of algebraic equations,

$$r(U_1) - \frac{r_0 N}{K(T)} - (1-\theta)\alpha T = 0 \qquad (6)$$

$$Q - \delta_0 T - \alpha N T - \mu E = 0 \qquad (7)$$

$$\theta \alpha NT - \alpha_0 U - \alpha_1 UN = 0 \tag{8}$$

$$\beta U - \beta_0 U_1 - \beta_1 U_1 N = 0 \tag{9}$$

$$\nu(T - T_0) - \nu_0 E = 0 \tag{10}$$

From equations (7) and (10), we have

$$Q - \delta_0 T - \alpha NT - \mu \frac{\nu}{\nu_0}(T - T_0) = 0 \tag{11}$$

Equation (11), (8) and (9) can also be respectively written as,

$$T = \frac{Q + \frac{\mu\nu}{\nu_0} T_0}{\delta_0 + \frac{\mu\nu}{\nu_0} + \alpha N} = f(N) \tag{12}$$

$$U = \frac{\theta \alpha N f(N)}{\alpha_0 + \alpha_1 N} = g(N) \tag{13}$$

$$U_1 = \frac{\beta g(N)}{\beta_0 + \beta_1 N} = h(N) \tag{14}$$

Now, from (6), we have

$$r(h(N)) - \frac{r_0 N}{K(T)} - (1-\theta)\alpha T = 0 \tag{15}$$

To show the existence of nontrivial equilibrium E^*, we plot the isoclines given by equations (11) and (15) in $N - T$ plane as follows,
From equation (11) we note the following,

(i) $N = 0 \Rightarrow T = \dfrac{Q + \frac{\mu\nu}{\nu_0} T_0}{\delta_0 + \frac{\mu\nu}{\nu_0}} > 0$

(ii) $\dfrac{dN}{dT} = -\dfrac{\delta_0 + \frac{\mu\nu}{\nu_0} + \alpha N}{\alpha T} < 0$ in first quadrant.

(iii) $N = -\dfrac{\delta_0 + \frac{\mu\nu}{\nu_0}}{\alpha}$ and $T = 0$ are the asymptotes.

From equation (15), we also note the following,

(i) $T = 0 \Rightarrow N = K_0$

(ii) $N = 0 \Rightarrow T = \dfrac{r_0}{(1-\theta)\alpha} > 0$

(iii) $\dfrac{dN}{dT} = \dfrac{-\dfrac{r_0 N}{K^2(T)}K'(T) + (1-\theta)\alpha}{r'(g(N))g'(N) - \dfrac{r_0}{K(T)}} < 0$ in first quadrant, provided $N^2 > \dfrac{\alpha_0\beta_0}{\alpha_1\beta_1}$

In view of the above, it is shown in figure 1 that the isoclines given by (11) and (15) intersect at a unique point (N^*, T^*) in the interior of first quadrant in $N-T$ plane showing that the steady-state values of N^* and T^* are within the invariant region.

Knowing the values of N^* and T^*, we can find the values of U^*, U_1^* and E^* from the equations (13), (14) and (10) respectively.

It is noted that,

$$U = \frac{\theta\alpha N T}{\alpha_0 + \alpha_1 N}$$

$$\leq \frac{\theta\alpha N T}{\alpha_0}$$

$$\leq \frac{\theta\alpha K_0 Q}{\alpha_0\delta_0} = U_m$$

This shows that the steady-state value of U^* is within the invariant region.

We also note that,

$$U_1 = \frac{\beta U}{\beta_0 + \beta_1 N}$$

$$\leq \frac{\beta U}{\beta_0}$$

$$\leq \frac{\beta U_m}{\beta_0} = U_{1m}$$

This shows that the steady-state value of U_1^* is within the invariant region.

Further,

$$E = \frac{\nu(T - T_0)}{\nu_0}$$

$$\leq \frac{\nu}{\nu_0}\left(\frac{Q}{\delta_0} - T_0\right) = E_m$$

which shows that the steady-state value of E^* is within the invariant region.

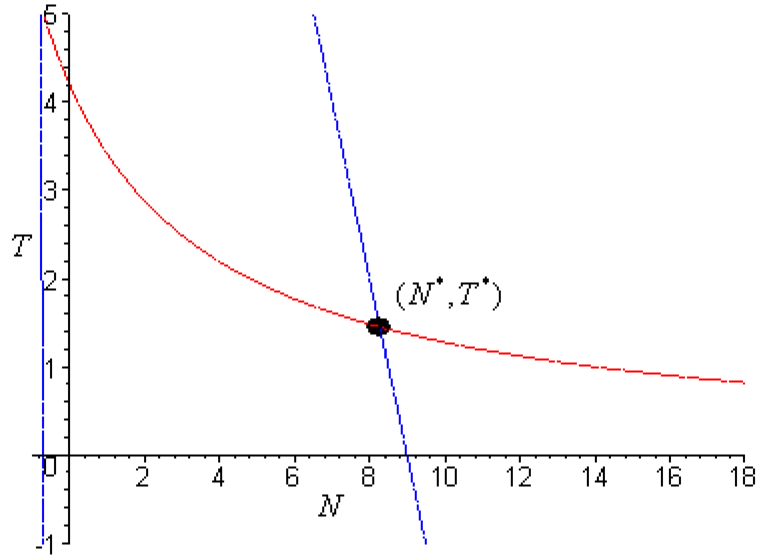

Fig. 1 Existence of E^* in $N-T$ plane.

4 Stability Analysis

4.1 Local stability of equilibria

The local stability analysis of an equilibrium point determines the behaviour of the dynamical system. It characterizes whether or not the system settles down to the equilibrium point if it initiates very close to equilibrium point. The local stability of an equilibrium point can be determined by computing the eigenvalues of variational matrix corresponding to that equilibrium point.

To establish the local stability behaviour of equilibria, we compute the following Jacobian matrix M for model system (1) – (5),

$$M = \begin{bmatrix} r(U_1) - \frac{2r_0 N}{K(T)} - (1-\theta)\alpha T & \frac{r_0 N^2}{K^2(T)} K'(T) - (1-\theta)\alpha N & 0 & r'(U_1)N & 0 \\ -\alpha T & -(\delta_0 + \alpha N) & 0 & 0 & -\mu \\ (\theta\alpha T - \alpha_1 U) & \theta\alpha N & -(\alpha_0 + \alpha_1 N) & 0 & 0 \\ -\beta_1 U_1 & 0 & \beta & -(\beta_0 + \beta_1 N) & 0 \\ 0 & \nu & 0 & 0 & -\nu_0 \end{bmatrix}$$

It can easily be checked that the equilibrium $E_0(0, \overline{T}, 0, 0, \overline{E})$ is stable or unstable according as $\overline{T} > \frac{r_0}{(1-\theta)\alpha}$ or $\overline{T} < \frac{r_0}{(1-\theta)\alpha}$ (necessary condition for the existence of E^*) respectively. This implies that E_0 is saddle point if E^* exist otherwise it is locally asymptotically stable.

To study the local stability behaviour of the model system about $E^*(N^*, T^*, U^*, U_1{}^*, E^*)$, we define following positive definite function E^*,

$$V = \frac{1}{2}k_1 n^2 + \frac{1}{2}k_2\tau^2 + \frac{1}{2}k_3 u^2 + \frac{1}{2}k_4 u_1{}^2 + \frac{1}{2}k_5 e^2 \quad (16)$$

where n, τ, u, u_1 and e are small perturbations about E^* and k_i $(i = 1, 2, ..., 5)$ are positive constants to be chosen appropriately.

Differentiating (16) with respect to $'t'$ and using linearized system of (1) – (5), we get,

$$\frac{dV}{dt} = -k_1\frac{r_0 N^*}{K(T^*)}n^2 - k_2(\delta_0 + \alpha N^*)\tau^2 - k_3(\alpha_0 + \alpha_1 N^*)u^2$$

$$-k_4(\beta_0 + \beta_1 N^*)u_1{}^2 - k_5\nu_0 e^2$$

$$+\left[k_1\left(\frac{r_0 N^{*2}}{K^2(T^*)}K'(T^*) - (1-\theta)\alpha N^*\right) - k_2\alpha T^*\right]n\tau + k_3(\theta\alpha T^* - \alpha_1 U^*)nu$$

$$+(k_1 r'(U_1{}^*)N^* - k_4\beta_1 U_1{}^*)nu_1 + k_3(\theta\alpha N^*)\tau u$$

$$+(-k_2\mu + k_5\nu)\tau e + (k_4\beta)u u_1$$

Now, $\frac{dV}{dt}$ will be negative definite under the following conditions,

$$\left[k_1\left(\frac{r_0 N^{*2}}{K^2(T^*)}K'(T^*) - (1-\theta)\alpha N^*\right) - k_2\alpha T^*\right]^2 < \frac{4}{9}k_1 k_2\frac{r_0 N^*}{K(T^*)}(\delta_0 + \alpha N^*)$$

$$k_3(\theta\alpha T^* - \alpha_1 U^*)^2 < \frac{4}{9}k_1\frac{r_0 N^*}{K(T^*)}(\alpha_0 + \alpha_1 N^*)$$

$$(k_1 r'(U_1{}^*)N^* - k_4\beta_1 U_1{}^*)^2 < \frac{2}{3}k_1 k_4\frac{r_0 N^*}{K(T^*)}(\beta_0 + \beta_1 N^*)$$

$$k_3(\theta\alpha N^*)^2 < \frac{4}{9}k_2(\delta_0 + \alpha N^*)(\alpha_0 + \alpha_1 N^*)$$

$$(-k_2\mu + k_5\nu)^2 < \frac{4}{3}k_2 k_5(\delta_0 + \alpha N^*)\nu_0$$

$$k_4\beta^2 < \frac{2}{3}k_3(\alpha_0 + \alpha_1 N^*)(\beta_0 + \beta_1 N^*)$$

Choosing $k_1 = k_2 = 1,\ k_3 < \frac{4}{9}(\alpha_0 + \alpha_1 N^*)\min\left\{\frac{r_0 N^*}{K(T^*)}\frac{1}{(\theta\alpha T^* - \alpha_1 U^*)^2}, \frac{(\delta_0 + \alpha N^*)}{(\theta\alpha N^*)^2}\right\}$,

$k_4 < \frac{2}{3}\frac{(\alpha_0 + \alpha_1 N^*)(\beta_0 + \beta_1 N^*)}{\beta^2}k_3$ and $k_5 < \frac{\mu}{\nu}$, $\frac{dV}{dt}$ will be negative definite provided the conditions (17) and (18) are satisfied implying that $E^*(N^*, T^*, U^*, U_1^*, E^*)$ is locally asymptotically stable.

Theorem 4.1 The interior equilibrium $E^*(N^*, T^*, U^*, U_1^*, E^*)$ is locally asymptotically stable if the following conditions hold,

$$\left[\left(\frac{r_0 N^{*2}}{K^2(T^*)}K'(T^*) - (1-\theta)\alpha N^*\right) - \alpha T^*\right]^2 < \frac{4}{9}\frac{r_0 N^*}{K(T^*)}(\delta_0 + \alpha N^*) \tag{17}$$

$$(r'(U_1^*)N^* - k_4\beta_1 U_1^*)^2 < \frac{2}{3}k_4\frac{r_0 N^*}{K(T^*)}(\beta_0 + \beta_1 N^*) \tag{18}$$

This theorem implies that if the interaction rate coefficient of toxicants with biological species (i.e. α) and $r'(U_1^*)$ are large then the conditions (17) and (18) may not be satisfied. This implies that these parameters destabilize the system.

4.2 Nonlinear stability of equilibrium $E^*(N^*, T^*, U^*, U_1^*, E^*)$

In this section we discuss the nonlinear stability character of an interior equilibrium $E^*(N^*, T^*, U^*, U_1^*, E^*)$ inside the region of attraction by using Liapunov second method.

To establish the nonlinear stability behaviour of $E^*(N^*, T^*, U^*, U_1^*, E^*)$, we consider the following positive definite function,

$$W = m_1\left(N - N^* - N^*\log\frac{N}{N^*}\right) + \frac{m_2}{2}(T - T^*)^2 + \frac{m_3}{2}(U - U^*)^2 + \frac{m_4}{2}(U_1 - U_1^*)^2 + \frac{m_5}{2}(E - E^*)^2$$

where m_i $(i = 1, 2..., 5)$ are positive constants to be chosen appropriately.

Differentiating it, we get

$$\begin{aligned}\frac{dW}{dt} &= m_1(N - N^*)\frac{1}{N}\frac{dN}{dt} + m_2(T - T^*)\frac{dT}{dt} + m_3(U - U^*)\frac{dU}{dt} \\ &\quad + m_4(U_1 - U_1^*)\frac{dU_1}{dt} + m_5(E - E^*)\frac{dE}{dt} \\ &= -m_2\alpha N(T - T^*)^2 - m_3\alpha_1 N(U - U^*)^2\end{aligned}$$

$$-m_1\frac{r_0}{K(T^*)}(N-N^*)^2-m_2\delta_0(T-T^*)^2-m_3\alpha_0(U-U^*)^2$$

$$-m_4(\beta_0+\beta_1 N)(U_1-U_1^{\ *})^2-m_5\nu_0(E-E^*)^2$$

$$-[\{(1-\theta)\alpha+r_0N\eta(T)m_1\}+m_2\alpha T^*](N-N^*)(T-T^*)$$

$$+m_3(\theta\alpha T^*-\alpha_1U^*)(N-N^*)(U-U^*)-\{-m_1\xi(U_1)+m_4\beta_1U_1^{\ *}\}(N-N^*)(U_1-U_1^{\ *})$$

$$-m_3\theta\alpha N(T-T^*)(U-U^*)+\{-\mu m_2+m_5\nu\}(T-T^*)(E-E^*)-m_4\beta(U-U^*)(U_1-U_1^{\ *})$$

where $\xi(U_1)=\begin{cases}\dfrac{r(U_1)-r(U_1^{\ *})}{U_1-U_1^{\ *}}, & U_1\neq U_1^{\ *}\\ r'(U_1^{\ *}) & U_1=U_1^{\ *}\end{cases}$ and $\eta(T)=\begin{cases}\dfrac{\dfrac{1}{K(T)}-\dfrac{1}{K(T^*)}}{T-T^*}, & T\neq T^*\\ -\dfrac{K'(T)}{K^2(T^*)} & T=T^*\end{cases}$

Now $\dfrac{dW}{dt}$ will be negative definite provided the following conditions are satisfied,

$$[\{(1-\theta)\alpha+r_0N\eta(T)m_1\}+m_2\alpha T^*]^2<\frac{4}{9}m_1m_2\frac{r_0}{K(T^*)}\delta_0$$

$$m_3(\theta\alpha T^*-\alpha_1U^*)^2<\frac{4}{9}m_1\frac{r_0}{K(T^*)}\alpha_0$$

$$\{-m_1\xi(U_1)+m_4\beta_1U_1^{\ *}\}^2<\frac{2}{3}m_1\frac{r_0}{K(T^*)}\beta_0$$

$$m_3(\theta\alpha N)^2<\frac{4}{9}m_2\delta_0\alpha_0$$

$$(-\mu m_2+m_5\nu)^2<\frac{4}{3}m_2m_5\delta_0\nu_0$$

$$m_4\beta^2<\frac{2}{3}m_3\alpha_0\beta_0$$

Let $r(U_1)$ and $K(T)$ satisfying in Ω such that $K_m\le K(T)\le K_0$, $0\le -r'(U_1)\le p$, $0\le K'(T)\le q$, where K_m, p, q are some positive constants.

Using mean value theorem, we get $|\xi(U_1)|\le p$ and $|\eta(T)|\le q$.

Now maximizing LHS and choosing $m_1=m_2=1$,

$m_3 < \frac{4}{9}\alpha_0 \min\left\{\frac{r_0}{K(T^*)}\frac{1}{(\theta\alpha T^* - \alpha_1 U^*)^2}, \frac{\delta_0}{(\theta\alpha K_0)^2}\right\}$, $m_5 = \frac{\mu}{\nu}$, $m_4 < \frac{2}{3}\frac{\alpha_0\beta_0}{\beta^2}m_3$, $\frac{dV}{dt}$ will be negative definite inside the region of attraction Ω, provided the conditions (19) and (20) are satisfied implying that $E^*(N^*, T^*, U^*, U_1^*, E^*)$ is nonlinearly asymptotically stable.

Theorem 4.2 Let $r(U_1)$ and $K(T)$ satisfying in Ω such that $K_m \le K(T) \le K_0$, $0 \le -r'(U_1) \le p$, $0 \le K'(T) \le q$, where K_m, p, q are some positive constants then equilibrium $E^*(N^*, T^*, U^*, U_1^*, E^*)$ will be nonlinearly asymptotically stable provided the inequalities are satisfied,

$$\left[\left\{(1-\theta)\alpha + r_0 K_0 \frac{q}{K_m{}^2}\right\} + \alpha T^*\right]^2 < \frac{4}{9}\frac{r_0\delta_0}{K(T^*)} \qquad (19)$$

$$(m_4\beta_1 U_1{}^* + p)^2 < \frac{2}{3}m_1\frac{r_0\beta_0}{K(T^*)} \qquad (20)$$

where $m_3 < \frac{4}{9}\alpha_0 \min\left\{\frac{r_0}{K(T^*)}\frac{1}{(\theta\alpha T^* - \alpha_1 U^*)^2}, \frac{\delta_0}{(\theta\alpha K_0)^2}\right\}$ and $m_4 < \frac{2}{3}\frac{\alpha_0\beta_0}{\beta^2}m_3$

This theorem implies that if the interaction rate coefficient of toxicants with biological species (i.e. α) and p are large then the conditions (19) and (20) may not be satisfied. This implies that these parameters have destabilizing effect on the model system.

5 Permanence of Solution

From a biological point of view, permanence (persistence) is defined as the long-term survival of all interacting populations in an ecosystem. It also deals with the growth of biological species as well as other components of the system. It is noted that the steady state level of all species settles asymptotically above a certain threshold. Mathematically, persistence is defined as,

Let $N(t)$ be the population density at any time 't' then it is said to be persistent (Freedman and Waltman, 1984), if

$$\liminf_{t\to\infty} N(t) > 0$$

provided $N(0) > 0$. If there exists $\varepsilon > 0$ such that

$$\liminf_{t\to\infty} N(t) \ge \varepsilon$$

then the population is said to be uniformly persistent in an ecological system. Thus, the population is said to be permanent, if it is uniformly persistent and if the bound of population size does not depend on initial conditions as $t \to \infty$.

Theorem 5

If $r\left(\frac{Q}{\delta_0}\right) > (1-\theta)\alpha\frac{Q}{\delta_0}$ then the system (1)-(5) is uniformly persistent.

Proof

From equation (1), we have

$$\frac{dN(t)}{dt} \geq N(t)\left[\left(r(U_1(t)) - \frac{r_0 N(t)}{K(T(t))}\right) - (1-\theta)\alpha\,\frac{Q}{\delta_0}\right]$$

$$\geq N(t)\left[\left(r\left(\frac{Q}{\delta_0}\right) - \frac{r_0 N(t)}{K_0}\right) - (1-\theta)\alpha\,\frac{Q}{\delta_0}\right]$$

$$= \frac{r_0 N(t)}{K_0}\left[\frac{K_0}{r_0}\left(r\left(\frac{Q}{\delta_0}\right) - (1-\theta)\alpha\,\frac{Q}{\delta_0}\right) - N(t)\right]$$

This implies that

$$\liminf_{t\to\infty} N(t) \geq \frac{K_0}{r_0}\left(r\left(\frac{Q}{\delta_0}\right) - (1-\theta)\alpha\,\frac{Q}{\delta_0}\right) = N_{\inf} \text{ (let)}$$

From equation (2), we have

$$\frac{dT(t)}{dt} \geq Q(t) - \delta_0\,T(t) - \alpha\,K_0\,T(t) - \mu\frac{\nu}{\nu_0}\left(\frac{Q}{\delta_0} - T_0\right)$$

$$= Q - \mu\frac{\nu}{\nu_0}\left(\frac{Q}{\delta_0} - T_0\right) - (\delta_0 + \alpha\,K_0)T(t)$$

This implies that

$$\liminf_{t\to\infty} T(t) \geq \frac{Q - \mu\frac{\nu}{\nu_0}\left(\frac{Q}{\delta_0} - T_0\right)}{(\delta_0 + \alpha\,K_0)} = T_{\inf} \text{ (let)}$$

From equation (3), we have

$$\frac{dU(t)}{dt} \geq \theta\,\alpha\,N_1 T_1 - (\alpha_0 + \alpha_1 K_0)U(t)$$

This implies that

$$\liminf_{t\to\infty} U(t) \geq \frac{\theta\,\alpha\,N_{\inf}T_{\inf}}{(\alpha_0 + \alpha_1 K_0)} = U_{\inf} \text{ (let)}$$

From equation (4), we have

$$\frac{dU_1(t)}{dt} \geq \beta U_{\inf}(t) - (\beta_0 + \beta_1 K_0)U_1(t)$$

This implies that

$$\liminf_{t\to\infty} U_1(t) \geq \frac{\beta U_{\inf}}{(\beta_0 + \beta_1 K_0)}$$

Similarly, we can find from equation (5), that

$$\liminf_{t\to\infty} E(t) \geq \frac{\nu(T_{\inf} - T_0)}{\nu_0}$$

Thus, we have,

$$\frac{K_0}{r_0}\left(r\left(\frac{Q}{\delta_0}\right) - (1-\theta)\alpha \frac{Q}{\delta_0}\right) \leq \liminf_{t\to\infty} N(t) \leq \limsup_{t\to\infty} N(t) \leq K_0$$

$$\frac{Q - \mu \dfrac{\nu}{\nu_0}\left(\dfrac{Q}{\delta_0} - T_0\right)}{(\delta_0 + \alpha K_0)} \leq \liminf_{t\to\infty} T(t) \leq \limsup_{t\to\infty} T(t) \leq \frac{Q}{\delta_0}$$

$$\frac{\theta \alpha N_{\inf} T_{\inf}}{(\alpha_0 + \alpha_1 K_0)} \leq \liminf_{t\to\infty} U(t) \leq \limsup_{t\to\infty} U(t) \leq \frac{Q}{\delta_0}$$

$$\frac{\beta U_{\inf}}{(\beta_0 + \beta_1 K_0)} \leq \liminf_{t\to\infty} U_1(t) \leq \limsup_{t\to\infty} U_1(t) \leq \frac{Q}{\delta_0}$$

$$\frac{\nu(T_{\inf} - T_0)}{\nu_0} \leq \liminf_{t\to\infty} E(t) \leq \limsup_{t\to\infty} E(t) \leq \frac{\nu}{\nu_0}\left(\frac{Q}{\delta_0} - T_0\right)$$

Hence the theorem.

6 Numerical Simulations

In this section, we have performed some numerical simulations using software Maple7 in the presence and absence of environmental tax considering the effect of intermediate toxic product on biological species. For this, we have assumed the following set of parameters,

$$Q = 5, a = 0.01, b = 0.01, r_0 = 4, K = 9, \delta = 0.5, \alpha = 0.3, \alpha_1 = 0.1, \alpha_0 = 0.2, \theta = 0.3$$

$$\beta = 0.2, \beta_0 = 0.02, \beta_1 = 0.03, \mu = 0.4, \nu = 0.4, \nu_0 = 0.2, T_0 = 0.6$$

The equilibrium values corresponding to $E^*(N^*, T^*, U^*, U_1^*, E^*)$ are given by,

$$N^* = 8.285064, T^* = 1.447621, U^* = 0.781606, U_1^* = 1.049509, E^* = 1.695243$$

Eigenvalues corresponding to $E^*(N^*, T^*, U^*, U_1^*, E^*)$ are given by, $-4.2706, -2.3388, -1.0300, -0.2656 + 0.0073i, -0.2656 - 0.0073i$. Since all the eigenvalues are either negative or have negative real parts and therefore equilibrium $E^*(N^*, T^*, U^*, U_1^*, E^*)$ is locally asymptotically stable.

The existence of $E^*(N^*, T^*, U^*, U_1^*, E^*)$ in $N-T$ plane is shown in figure 1. The nonlinear stability behavior of $E^*(N^*, T^*, U^*, U_1^*, E^*)$ is shown in figure 2 where the trajectories with different initial starts have been plotted. It is noted that all the trajectories with different initial starts approach to the equilibrium point E^*. The variation of density of biological species, concentration of toxicants, uptaken concentration, concentration of intermediate toxic product and the amount of environmental tax with time 't' for different values of rate of emission of toxicants is shown in figures 3 – 7 respectively. From these figures, it can easily be observed that the density of biological species decreases while the concentrations of toxicants, uptaken toxicants and intermediate toxic product increase as the rate of emission of toxicants increases. Further, it has also been shown in figure 7 that the environmental tax increases as the rate of emission of toxicants in the environment increases beyond its threshold level.

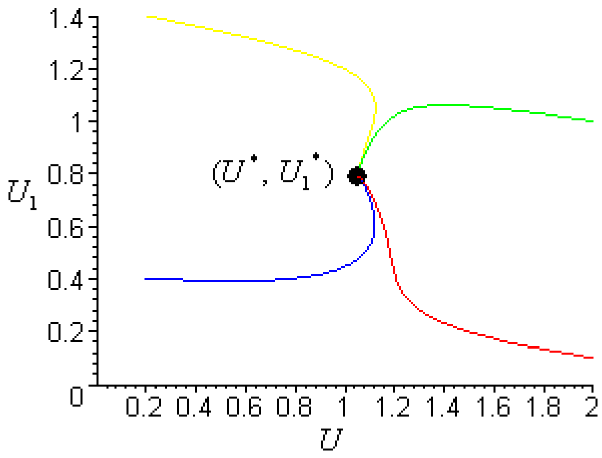

Fig. 2 Nonlinear stability in $U-U_1$ plane.

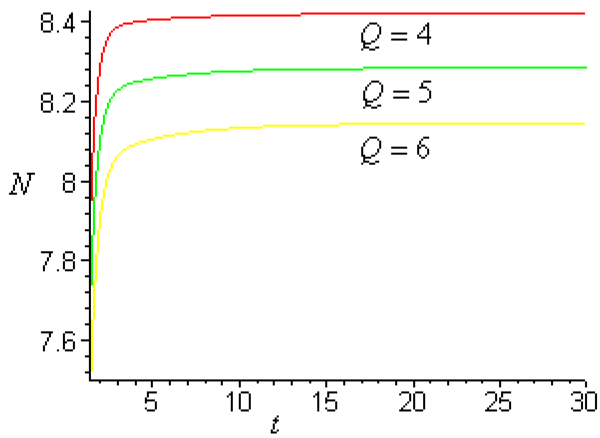

Fig. 3 Variation of N with time 't' for different values of Q

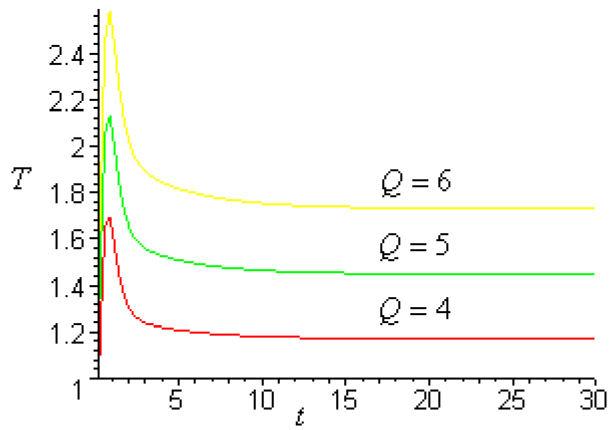

Fig. 4 Variation of T with time $'t'$ for different values of Q

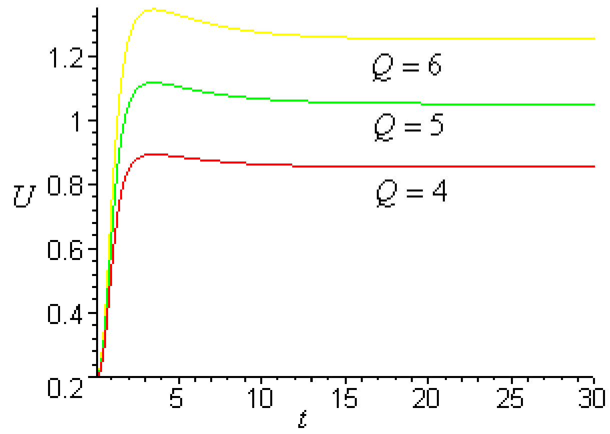

Fig. 5 Variation of U with time $'t'$ for different values of Q

The effect of intermediate toxic product on biological species with time $'t'$ has been shown in figure 8. It can be seen from figure 8 that as the rate of formation of intermediate toxic product inside the biological species increases the density of biomass decreases. The variation of density of biological species (N) and the concentration of toxicants (T) with time $'t'$ in the presence and absence of environmental tax has been shown in figures 9 and 10 respectively. From these figures, it is noted that in the absence of environmental tax, density of biological species decreases as a result of increase in the concentration of toxicants while in the presence of environmental tax, density of biological species increases due to decrease in the concentration of toxicants. The variation of density of biological species (N) and the concentration of toxicants (T) with time $'t'$ for different values of tax rate coefficient is shown in figures 11 and 12 respectively. It is shown that as the tax rate coefficient increases, the equilibrium level of density of biological species increases and the concentration of toxicants decreases.

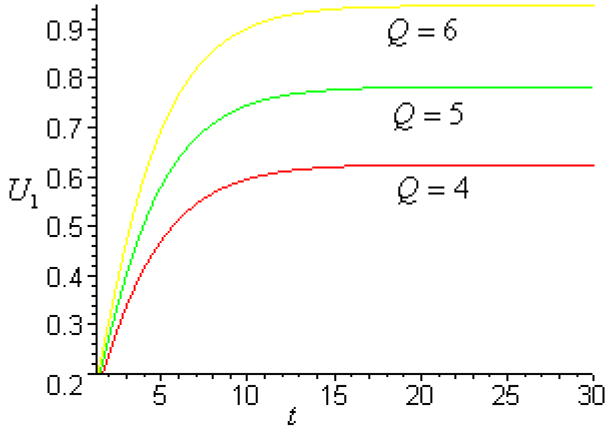

Fig. 6 Variation of U_1 with time $'t'$ for different values of Q

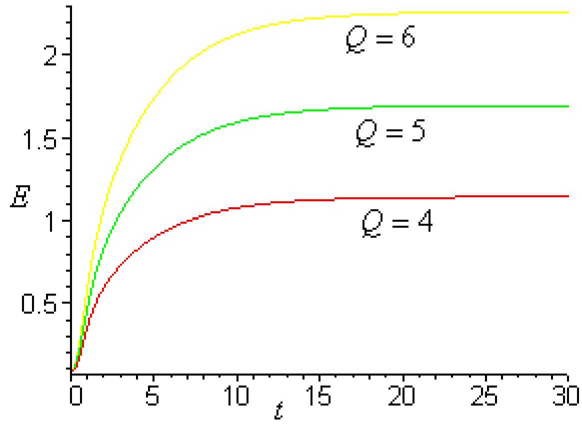

Fig. 7 Variation of E with time $'t'$ for different values of Q

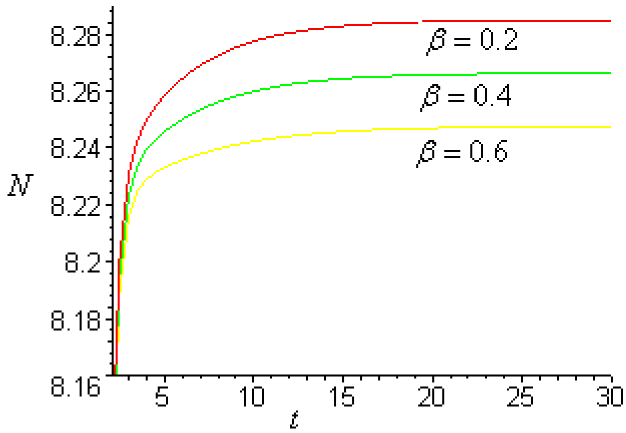

Fig. 8 Variation of N with time $'t'$ for different values of β

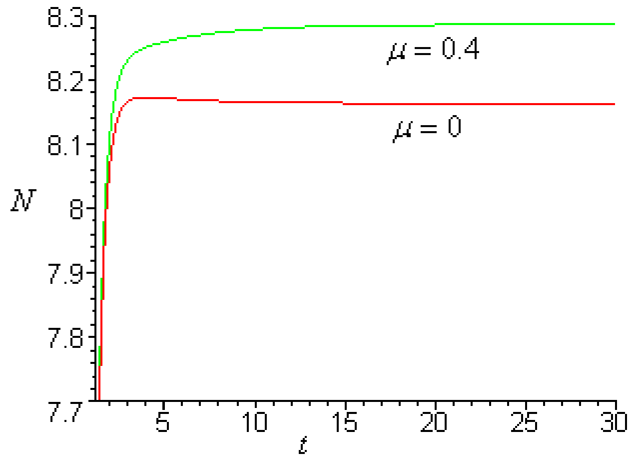

Fig. 9 Variation of N with time 't' in the presence $(\mu = 0.4)$ and absence $(\mu = 0)$ of environmental tax.

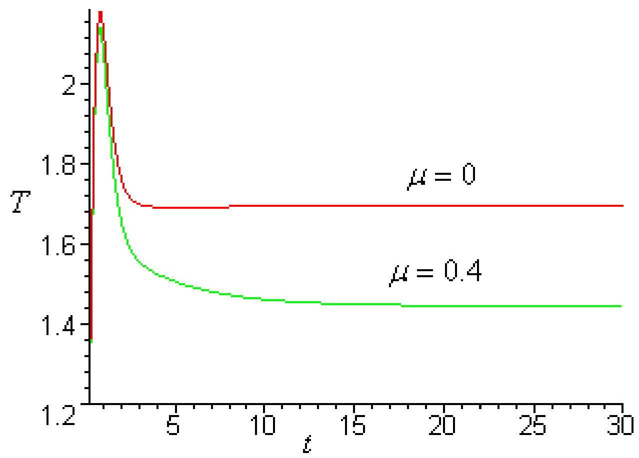

Fig. 10 Variation of T with time 't' in the presence $(\mu = 0.4)$ and absence $(\mu = 0)$ of environmental tax

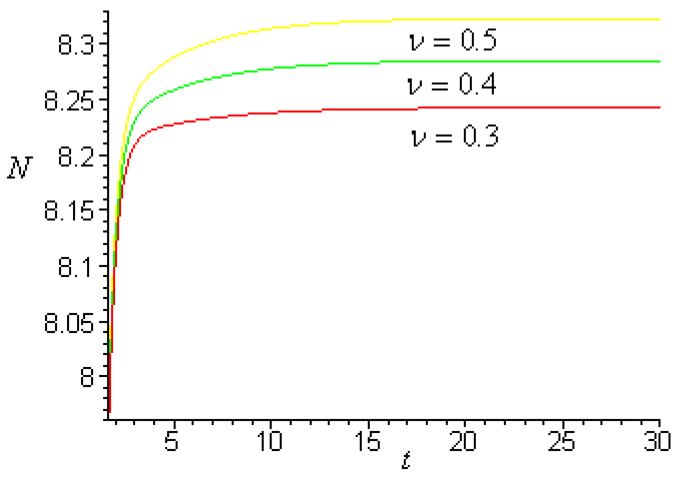

Fig. 11 Variation of N with time 't' for the different values of tax rate coefficient ν

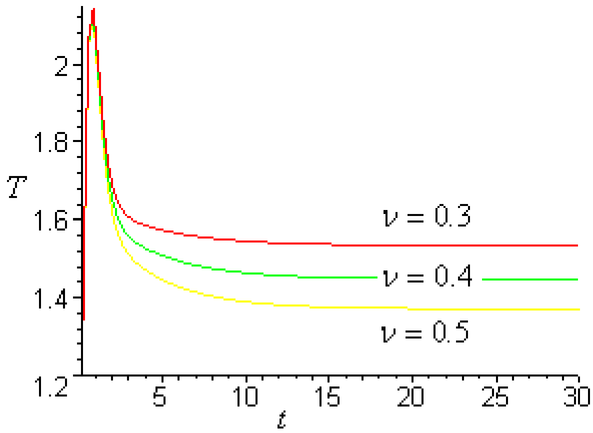

Fig. 12 Variation of T with time $'t'$ for the different values of tax rate coefficient ν

7 Conclusions

In this paper, our main aim is to study the survival of biological species living in a polluted environment and to reduce the concentration of toxicants into the atmosphere by imposing environmental tax on emitters. The concentration of toxicants into the atmosphere can only be reduced by reducing the cumulative emission rate of toxicants. It is assumed that the environmental tax is levied on emitters only when the concentration of pollutants goes beyond a threshold level, as discussed earlier. Existence of equilibria and their stability behavior has been obtained. It is shown that the first equilibrium (i.e. $\overline{E}$) corresponding to the extinction of biological species is unstable. The nontrivial equilibrium (i.e. E^*) is locally and globally stable under certain conditions within the region of attraction. It is shown that the equilibrium density of biological population decreases while the concentration of toxicants, uptaken concentration and the concentration of intermediate toxic product increase as the emission rate of toxicants increases. It is further shown that as the rate of emission of toxicants increases, the environmental tax to be imposed on emitters, also increases. This implies that it is advantageous to levy tax to reduce the discharge of toxicants and to improve the environmental quality. It is also noted here that the tax revenue, thus generated, can be used for environmental protection so that we have a clean environment and the biological species may survive potentially.

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