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# Modeling the effect of pollution on biological species: A socio-ecological problem

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*Received 13 December 2014*; *Accepted 20 January 2015*; *Published 1 June 2015*


**Abstract**
In this paper, a nonlinear spatial model is proposed and analyzed to study the effect of pollution on biological population. It is assumed that the pollutants enter into the environment not directly by the population but by a precursor produced by the population itself. It is further assumed that larger the population, faster the precursor is produced, and larger the precursor, faster the pollutant is produced. Criteria for nonlinear stability and instability for both spatial and non-spatial models are obtained. The various parameter ranges for stable homogeneous solutions are identified. By the simulation experiments, it is observed that by applying an appropriate effort $F$, the population density $P$ can be maintained at a higher equilibrium level. It is also shown that the equilibrium level of the concentration of precursor pollutant, concentration of pollutant in the environment and in the population decrease due to the effort $F$.

**Keywords** precursor; pollutant; biological species; stability; conservation efforts.

**Computational Ecology and Software**
**ISSN 2220-721X**
**URL: http://www.iaees.org/publications/journals/ces/online-version.asp**
**RSS: http://www.iaees.org/publications/journals/ces/rss.xml**
**E-mail: ces@iaees.org**
**Editor-in-Chief: WenJun Zhang**
**Publisher: International Academy of Ecology and Environmental Sciences**

## 1 Introduction

Our environment is getting polluted day by day due to rapid pace of urbanization, industrialization and deforestation, and we face one of the most important present day socio-ecological problems closely related to physiological and bio-spherical changes in the population. We do have several examples where the pollution is responsible for increase in death rate, decrease in birth rate and migration of population (Shukla and Dubey, 1996). The effects of pollution caused by various human factors on structure and functions of ecosystems have been studied by several researchers (Woodwell, 1970; Smith, 1981; McLaughli, 1985; Hari et al., 1986;

Woodman and Cowling, 1987; Schulze, 1989; Ghosh, 2000; Srinivasu, 2002; Ghosh et al., 2002, 2006; Naresh et al. 2006a, 2006b; Sundar, 2013; Sundar et al., 2014). In recent decades, some investigations have been made to study the effect of pollution on a single or two biological species (Hallam et al., 1983a, 1983b, Hallam and Luna, 1984; Hallam and Ma, 1986; Luna and Hallam, 1987; Freedman and Shukla, 1991; Huaping and Ma, 1991; Shukla and Dubey, 1996, 1997; Dubey, 1997; Dubey and Das, 1999; Dubey and Hussain, 2000a, 2000b, 2006; Shukla et al., 2001, 2003, 2009; Dubey et al., 2003, 2009; Naresh et al, 2006a, 2006b; Dubey and Naranayan, 2010; Sundar et al., 2014). In particular, Hallam *et al.* (1983b) studied the effect of a toxicant in the environment on a single-species population by assuming that its growth rate density decreases linearly with the uptake concentration of toxicant. Huaping and Ma (1991) proposed and analysed a mathematical model to study the effect of toxicant on naturally stable two-species communities. In these investigations, it has been assumed that carrying capacity does not depend on the concentration of toxicant present in the environment. However, in real situations the effect of toxicant is to decrease both the growth rate of species and the carrying capacity of the environment. Taking this aspect into account, Freedman and Shukla (1991) investigated the effect of a toxicant on a single-species and predator-prey system by considering the exogeneous introduction of toxicant into the environment.

Shukla and Dubey (1996) studied the simultaneous effects of two toxicants on a biological species, one being more toxic than the other. Dubey (1997) propose a mathematical model to study the depletion and conservation of forestry resources in a polluted environment. Shukla et al. (2001) studied the effect of a toxicant emitted into the environment from external sources on two competing biological species. They found that the four usual outcomes of competition between two species may be altered under certain conditions which are mainly dependent on emission rate of toxicant into the environment, uptake concentration of toxicant by the two species and their growth rates and carrying capacities. Dubey et al. (2003) studied the behaviour of a resource biomass in the presence of industrialization and pollution. They showed that in the case of small periodic influx of toxicant into the environment, the resource biomass has a periodic behaviour if the depletion rate coefficient of environmental pollution is small. However, if this coefficient increases beyond a threshold value, then the resource biomass converges towards its equilibrium. Naresh et al. (2006a) investigated the effect of an intermediate toxic product formed by uptake of a toxicant on a plant biomass. Shukla et al. (2003) proposed and analysed a mathematical model and studied effects of primary and secondary toxicants on the biomass of resources such as forestry, agricultural crops. Dubey and Hussain (2006) investigated the survival of a biological species which is dependent on a resource in a polluted environment and they showed that the diffusion plays a general role in stabilising the system.

In the above investigations, it is assumed that the pollutant enters into the environment by some manmade projects which may be population (industrialization) dependent, constant, zero or periodic. In this regard, Rescigno (1977) studied the effect of a precursor pollutant on a single species, but he did not consider the rate of uptake concentration of the pollutant on the growth of the species. Further, in the above works the effects of diffusion has not been considered. Keeping the above in view, in this paper we propose and analyse a nonlinear model to study the effect of a precursor pollutant, which is formed by various human activities in the atmosphere, on population where the effect of uptake concentration, diffusion and conservation are considered.

The paper is organized as follows. In Section 2, we discuss the model system. Under Section 3, we analyse the model system without diffusion. In Section 4, we analyze the model system with diffusion. Section 5 describes the conservation model system. In Section 6 and 7, we analyse the conversion model system without and with diffusion, respectively. Section 8 depicts the numerical results. Finally, we summarize the results in the last Section.

## 2 The Proposed Mathematical Model

Let us consider a polluted environment where a biological population is growing logistically in a closed region $D$ with smooth boundary $\partial D$. We assume that the environment is polluted by various population activities. It is further assumed that the population is affected by the pollutant formed in the atmosphere by its precursor. Let $P(x,y,t)$ be the population density, $Q(x,y,t)$ the concentration of the precursor pollutant emitted by various activities of the population, $T(x,y,t)$ the concentration of pollutant formed by $Q$ in the atmosphere and $U(x,y,t)$ uptake concentration of pollutant by the population at coordinates $(x,y)\in D$ and time $t \geq 0$. It is also assumed that the larger the population, the faster the precursor is produced. It is further assumed that the larger the precursor, the faster the pollutant is produced. Then, system may be governed by the following set of differential equations:

$$
\begin{aligned}
&\frac{\partial P}{\partial t} = r(U)P - \frac{r_0 P^2}{K(T)} + D_1 \nabla^2 P,\\
&\frac{\partial Q}{\partial t} = \gamma P - \gamma_0 Q,\\
&\frac{\partial T}{\partial t} = hQ - h_0 T + \theta_1 \delta_1 U - \alpha PT + D_2 \nabla^2 T,\\
&\frac{\partial U}{\partial t} = -\delta_1 U + \theta_0 h_0 T + \alpha PT,\\
&0 \leq \theta_0, \theta_1 \leq 1.
\end{aligned}
\tag{1}
$$

We analyse the system (1) with the following initial and boundary conditions:

$$
\begin{aligned}
&P(x,y,0) = \phi(x,y) \geq 0, Q(x,y,0) = \psi(x,y) \geq 0, T(x,y,0) = \xi(x,y) \geq 0,\\
&U(x,y,0) = \zeta(x,y) \geq 0, (x,y)\in D;\ \frac{\partial P}{\partial n} = \frac{\partial T}{\partial n} = 0, (x,y)\in \partial D, t \geq 0,
\end{aligned}
\tag{2}
$$

where $n$ is the unit outward normal to $\partial D$. We assume that the functions $P$, $Q$, $T$, $U$ belong to the class $C^2(D)$.

In model (1), $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$

is the Laplacian diffusion operator. $D_1$ and $D_2$ are the diffusion rate coefficients of $P(x,y,t)$ and $T(x,y,t)$ respectively in $D$. $\gamma$ is the growth rate of $Q$ due to population $P$, $\gamma_0$ the natural depletion rate coefficient of $Q$. $h$ can be interpreted as the growth rate coefficient of $T$ due to $Q$. $h_0$ can be interpreted as the natural depletion rate coefficient of $T$, a fraction $\theta_0$ of which goes inside the body of the population. $\alpha$ is the depletion rate coefficient of $T$ due to $P$. $\delta_1$ is the natural depletion rate coefficient of $U$, a fraction $\theta_1$ of which re-enters into the environment.

In model (1), the function $r(U)$ is the specific growth rate of the population which decreases as $U$ increases, i.e.

$$r(0) = r_0 \text{ and } r'(U) < 0 \text{ for } U \geq 0. \tag{3}$$

The function $K(T)$ is the carrying capacity of the population in the presence of pollutant and it decreases as $T$ increases. Hence we assume that

$$K(0) = K_0 \text{ and } K'(T) < 0 \text{ for } T \geq 0, \text{ and } \exists \text{ a } T = T_a \text{ such that } K(T_a) = 0. \tag{4}$$

The model is analysed with and without diffusion.

## 3 Model Without Diffusion

In this section we take $D_1 = D_2 = 0$ in model (1). Then model (1) reduces to

$$\begin{aligned}
\frac{dP}{dt} &= r(U)P - \frac{r_0 P^2}{K(T)},\\
\frac{dQ}{dt} &= \gamma P - \gamma_0 Q,\\
\frac{dT}{dt} &= hQ - h_0 T + \theta_1\delta_1 U - \alpha PT,\\
\frac{dU}{dt} &= -\delta_1 U + \theta_0 h_0 T + \alpha PT,\\
P(0) &\geq 0, Q(0) \geq 0, T(0) \geq 0, U(0) \geq 0.
\end{aligned} \tag{5}$$

It can be checked that there exists two non-negative equilibria (which belong to the first orthant), namely, $E_0(0,0,0,0)$ and $\overline{E}(\overline{P},\overline{Q},\overline{T},\overline{U})$, where $\overline{P},\overline{Q},\overline{T}$ and $\overline{U}$ are the positive solutions of the following algebraic equations:

$$\begin{aligned}
r_0 P &= r(U)K(T),\\
Q &= \frac{\gamma}{\gamma_0}P,\\
T &= \frac{hQ}{h_0(1-\theta_0\theta_1) + \alpha(1-\theta_1)P} = f(P), \text{ (say)},\\
U &= \frac{1}{\delta_1}\{\theta_0 h_0 f(P) + \alpha P f(P)\} = g(P), \text{ (say)}.
\end{aligned}$$

It can be verified that the equilibrium $\overline{E}$ exists if the following inequality holds at $\overline{E}$:

$$r_0 - r'(U)g'(P)K(f(P)) - r(g(P))K'(T)f'(P) > 0. \tag{6}$$

By computing the variational matrix corresponding to the equilibrium $E_0$, it can be checked that $E_0$ is a saddle point with unstable manifold locally in the $P$ direction and with stable manifold locally in the $Q-T-U$ space.

In the following theorem, it is shown that $\overline{E}$ is locally asymptotically stable.

**Theorem 3.1** Let the following inequalities hold:

$$\left\{\frac{r_0\overline{P}}{K^2(\overline{T})}K'(\overline{T})+\alpha\overline{T}\right\}^2<\frac{4}{9}\frac{r_0}{K(\overline{T})}(h_0+\alpha\overline{P}), \quad (7)$$

$$\{\theta_1\delta_1+c_2(\theta_0h_0+\alpha\overline{P})\}^2<\frac{2}{3}c_2\delta_1(h_0+\alpha\overline{P}), \quad (8)$$

$$h^2<\frac{2}{3}c_1\gamma_0(h_0+\alpha\overline{P}), \quad (9)$$

where

$$c_1=\frac{1}{3}\frac{r_0\gamma_0}{\gamma^2K(\overline{T})} \quad \text{and} \quad c_2=-\frac{r'(\overline{U})}{\alpha\overline{T}}. \quad (10)$$

Then the equilibrium $\overline{E}$ is locally asymptotically stable.

**Proof** By taking the transformations

$$P=\overline{P}+p,\ Q=\overline{Q}+q,\ T=\overline{T}+\tau,\ U=\overline{U}+u,$$

we first linearize model (5). Then we consider the following positive definite function in the linearized form of model (5):

$$V(p,q,\tau,u\}=\frac{1}{2}\{\frac{p^2}{\overline{P}}+c_1q^2+\tau^2+c_2u^2\} \quad (11)$$

where $c_1$ and $c_2$ are positive constants given by (10). It can be checked that the derivative of $V$ with respect to $t$ is negative definite under conditions (7)-(9), proving the theorem.

In the following theorem it is shown that the equilibrium $\overline{E}$ is globally asymptotically stable. To prove this theorem, we need the following lemma which establishes a region of attraction for system (5). The proof of this lemma is easy and hence is omitted.

**Lemma 3.1** The set $\Omega_1=\{(P,Q,T,U):0\le P\le K_0,\, 0\le Q+T+U\le\frac{\gamma K_0}{\delta}\}$ is a region of attraction for all solutions initiating in the interior of the positive orthant, where

$\gamma_0>h$ and $\delta=\min\{\gamma_0-h,\ h_0(1-\theta_0),\ \delta_1(1-\theta_1)\}$.

**Theorem 3.2** In addition to the assumptions (3) and (4), let $r(U)$ and $K(T)$ satisfy in $\Omega_1$,

$$0\le -r'(U)\le\rho,\ K_m\le K(T)\le K_0 \text{ and } 0\le -K'(T)\le k, \quad (12)$$

for some positive constants $\rho, K_m$ and $k$. Let the following inequalities hold:

$$\left\{\frac{r_0 K_0\ k}{K_m^2}+\frac{\alpha\gamma K_0}{\delta}\right\}^2 < \frac{4}{9}\frac{r_0}{K(\bar{T})}(h_0+\alpha\bar{P}), \tag{13}$$

$$\left\{\rho+\frac{\alpha\gamma K_0}{\delta}\right\}^2 < \frac{2}{3}\delta_1\frac{r_0}{K(\bar{T})}, \tag{14}$$

$$h^2 < \frac{2}{3}c_1\gamma_0(h_0+\alpha\bar{P}), \tag{15}$$

$$(\theta_1\delta_1+\theta_0 h_0+\alpha\bar{P})^2 < \frac{2}{3}\delta_1(h_0+\alpha\bar{P}), \tag{16}$$

where $c_1$ is same as defined in Eq. (10).

Then, $\bar{E}$ is globally asymptotically stable with respect to all solutions initiating in the interior of the positive orthant $\Omega_1$.

**Proof** Consider the following positive definite function around $\bar{E}$,

$$V_1(P,Q,T,U) = P-\bar{P}-\bar{P}\ln\frac{P}{\bar{P}}+\frac{c_1}{2}(Q-\bar{Q})^2+\frac{1}{2}(T-\bar{T})^2+\frac{1}{2}(U-\bar{U})^2 \ . \tag{17}$$

Now differentiating $V_1$ with respect to $t$ along the solutions of (5), we get

$$\begin{aligned}\frac{dV_1}{dt} &= -\frac{r_0}{K(\bar{T})}(P-\bar{P})^2 - c_1\gamma_0(Q-\bar{Q})^2-(h_0+\alpha\bar{P})(T-\bar{T})^2-\delta_1(U-\bar{U})^2\\ &\quad + c_1\gamma(P-\bar{P})(Q-\bar{Q})-(r_0P\xi(T)+\alpha T)(P-\bar{P})(T-\bar{T})+(\eta(U)+\alpha T)(P-\bar{P})(U-\bar{U})\\ &\quad + h(Q-\bar{Q})(T-\bar{T})+(\theta_1\delta_1+\theta_0h_0+\alpha\bar{P})(T-\bar{T})(U-\bar{U}),\end{aligned} \tag{18}$$

where

$$\eta(U)=\begin{cases}\dfrac{r(U)-r(\bar{U})}{U-\bar{U}}, & U\neq\bar{U}\\ r'(\bar{U}), & U=\bar{U}\end{cases},$$

$$\xi(T)=\begin{cases}\left\{\dfrac{1}{K(T)}-\dfrac{1}{K(\bar{T})}\right\}/(T-\bar{T}), & T\neq\bar{T}\\ -\dfrac{K'(\bar{T})}{K^2(\bar{T})}, & T=\bar{T}\end{cases}.$$

From (12) and the mean value theorem, we note that

$$|\eta(U)|\le\rho \text{ and } |\xi(T)|\le\frac{k}{K_m^2}.$$

Now Eq. (18) can be rewritten as the sum of the quadratics:

$$\begin{aligned}\frac{dV_1}{dt} = &-\frac{1}{2}a_{11}(P-\overline{P})^2 + a_{12}(P-\overline{P})(Q-\overline{Q}) - \frac{1}{2}a_{22}(Q-\overline{Q})^2 \\ &-\frac{1}{2}a_{11}(P-\overline{P})^2 + a_{13}(P-\overline{P})(T-\overline{T}) - \frac{1}{2}a_{33}(T-\overline{T})^2 \\ &-\frac{1}{2}a_{11}(P-\overline{P})^2 + a_{14}(P-\overline{P})(U-\overline{U}) - \frac{1}{2}a_{44}(U-\overline{U})^2 \\ &-\frac{1}{2}a_{22}(Q-\overline{Q})^2 + a_{23}(Q-\overline{Q})(T-\overline{T}) - \frac{1}{2}a_{33}(T-\overline{T})^2 \\ &-\frac{1}{2}a_{33}(T-\overline{T})^2 + a_{34}(T-\overline{T})(U-\overline{U}) - \frac{1}{2}a_{44}(U-\overline{U})^2,\end{aligned}$$

where

$$a_{11} = \frac{2}{3}\frac{r_0}{K(\overline{T})},\ a_{22} = c_1\gamma_0,\ a_{33} = \frac{2}{3}(h_0 + \alpha\overline{P}),\ a_{44} = \delta_1,\quad a_{12} = c_1\gamma,\quad a_{13} = -(r_0 P\xi(T) + \alpha T),$$

$$a_{14} = \eta(U) + \alpha T,\ a_{23} = h,\ a_{34} = \theta_1\delta_1 + \theta_0 h_0 + \alpha\overline{P}.$$

Sufficient conditions for $\frac{dV_1}{dt}$ to be negative definite are that the following conditions hold:

$$a_{12}^2 < a_{11}a_{22}, \tag{19}$$

$$a_{13}^2 < a_{11}a_{33}, \tag{20}$$

$$a_{14}^2 < a_{11}a_{44}, \tag{21}$$

$$a_{23}^2 < a_{22}a_{33}, \tag{22}$$

$$a_{34}^2 < a_{33}a_{44}. \tag{23}$$

We note that inequality (19) is satisfied automatically for the chosen value of $c_1$ in the theorem. We also note that that $(13) \Rightarrow (20)$, $(14) \Rightarrow (21)$, $(15) \Rightarrow (22)$ and $(16) \Rightarrow (23)$. Hence $V_1$ is a Liapunov function with respect to $\overline{E}$ whose domain contains the region of attraction $\Omega_1$, proving the theorem.

The above theorem implies that the population living in a polluted environment attains an equilibrium level under certain conditions. The equilibrium level of the precursor pollutant is crucial in affecting the equilibrium level of population which decreases as the equilibrium level of precursor pollutant increases. We also note that if $\gamma$ and $h$ are kept at small level, then possibility of satisfying conditions (13)-(15) increases. This implies that the stability of the system can be maintained by lowering the rate of formations of precursor and environmental pollutants.

**4 Model With Diffusion**

In this section, we consider the complete model (1)-(2) and state the main results in the form of the following theorem.

**Theorem 4.1** (i) If the equilibrium $\overline{E}$ of model (5) is globally asymptotically stable, then the corresponding uniform steady state of the initial-boundary value problems (1)-(2) is also globally asymptotically stable.

(ii) If the equilibrium $\overline{E}$ of model (5) is unstable even then the uniform steady state of the initial-boundary value problems (1)-(2) can be made stable by increasing diffusion coefficients to sufficiently large values.

**Proof** Let us consider the following positive definite function

$$V_2(P(t),Q(t),T(t),U(t)) = \iint_D V_1(P,Q,T,U)\,dA$$

where $V_1$ is given by Eq. (17).

We assume that $V_1$ is differentiable and the functions *P, Q, T, U* belong to the class $C^2(D)$.

Then we have,

$$\frac{dV_2}{dt} = \iint_D \left\{ \frac{\partial V_1}{\partial P}\frac{\partial P}{\partial t} + \frac{\partial V_1}{\partial Q}\frac{\partial Q}{\partial t} + \frac{\partial V_1}{\partial T}\frac{\partial T}{\partial t} + \frac{\partial V_1}{\partial U}\frac{\partial U}{\partial t} \right\} dA$$

$$= I_1 + I_2, \tag{24}$$

where

$$I_1 = \iint_D \frac{dV_1}{dt}\,dA \text{ and } I_2 = \iint_D \left\{ D_1 \frac{\partial V_1}{\partial P}\nabla^2 P + D_2 \frac{\partial V_1}{\partial T}\nabla^2 T \right\} dA. \tag{25}$$

We note the following properties of $V_1$, namely,

$$\left.\frac{\partial V_1}{\partial P}\right]_{\partial D} = \left.\frac{\partial V_1}{\partial T}\right]_{\partial D} = 0,$$

and for all points of $D$,

$$\frac{\partial^2 V_1}{\partial P \partial Q} = \frac{\partial^2 V_1}{\partial P \partial T} = \frac{\partial^2 V_1}{\partial P \partial U} = \frac{\partial^2 V_1}{\partial Q \partial T} = \frac{\partial^2 V_1}{\partial Q \partial U} = \frac{\partial^2 V_1}{\partial T \partial U} = 0,$$

$$\frac{\partial^2 V_1}{\partial P^2} > 0,\ \frac{\partial^2 V_1}{\partial Q^2} > 0,\ \frac{\partial^2 V_1}{\partial T^2} > 0 \text{ and } \frac{\partial^2 V_1}{\partial U^2} > 0.$$

Now we consider $I_2$ and determine the sign of each term. We utilize the following formula known as Green's first identity in the plane:

$$\iint_D F\nabla^2 G \; dA = \oint_{\partial D} F \frac{\partial G}{\partial n} ds - \iint_D (\nabla F . \nabla G) dA,$$

where $\frac{\partial G}{\partial n}$ is the directional derivative in the direction of the unit outward normal to $\partial D$ and $s$ is the arc length.

Then with $F = \frac{\partial V_1}{\partial P}$ and $G = P$ we obtain

$$\iint_D \left\{ \frac{\partial V_1}{\partial P} \nabla^2 P \right\} dA = \oint_{\partial D} \frac{\partial V_1}{\partial P} \frac{\partial P}{\partial n} ds - \iint_D \left\{ \nabla \left( \frac{\partial V_1}{\partial P} \right) . \nabla P \right\} dA$$

$$= -\iint_D \left\{ \nabla \left( \frac{\partial V_1}{\partial P} \right) . \nabla P \right\} dA, \text{ since } \frac{\partial P}{\partial n} = 0.$$

Now $\nabla \left( \frac{\partial V_1}{\partial P} \right) = \frac{\partial^2 V_1}{\partial P^2} \frac{\partial P}{\partial x} \hat{i} + \frac{\partial^2 V_1}{\partial P^2} \frac{\partial P}{\partial y} \hat{j}.$

Hence $\iint_D \left\{ \frac{\partial V_1}{\partial P} \nabla^2 P \right\} dA = -\iint_D \frac{\partial^2 V_1}{\partial P^2} \left\{ \left( \frac{\partial P}{\partial x} \right)^2 + \left( \frac{\partial P}{\partial y} \right)^2 \right\} dA \le 0,$

Similarly $\iint_D \frac{\partial V_1}{\partial T} \nabla^2 T \; dA \le 0.$

i.e., $I_2 \le 0.$ (26)

Thus we note that if $I_1 \le 0,$ then $\frac{dV_1}{dt} = I_1 + I_2 \le 0.$ This shows that if $\overline{E}$ is globally asymptotically stable in the absence of diffusion, then the uniform steady state of the initial-boundary value problems (1)-(2) also must be globally asymptotically stable. This proves the first part of Theorem 4.1.

We further note that if $\frac{dV_1}{dt} > 0,$ i.e., if $I_1 > 0,$ then $\overline{E}$ may become unstable in the absence of diffusion. But, Eqs. (24) and (26) show that by increasing diffusion coefficients $D_1$ and $D_2$ to sufficiently large values, $\frac{dV_2}{dt}$ can be made negative even if $I_1 > 0.$ This proves the second part of Theorem 4.1.

The above theorem implies that diffusion with reservoir boundary conditions may stabilize a system which is otherwise unstable.

We shall explain the above theorem for a rectangular habitat $D$ defined by

$$D = \{(x, y) : 0 \le x \le a, 0 \le y \le b\} \tag{27}$$

in the form of the following theorem.

**Theorem 4.2** In addition to assumptions (3) and (4), let $r(U),\ K(T)$ satisfy the inequalities in (12). Let the following inequalities hold:

$$\left\{\frac{r_0K_0k}{K_m^2}+\frac{\alpha\gamma K_0}{\delta}\right\}^2<\frac{4}{9}\left\{\frac{r_0}{K(\overline{T})}+\frac{D_1\overline{P}\pi^2(a^2+b^2)}{K_0{}^2a^2b^2}\right\}\left\{h_0+\alpha\overline{P}+\frac{D_2\pi^2(a^2+b^2)}{a^2b^2}\right\} \tag{28}$$

$$\left\{\rho+\frac{\alpha\gamma K_0}{\delta}\right\}^2<\frac{2}{3}\delta_1\left\{\frac{r_0}{K(\overline{T})}+\frac{D_1\overline{P}\pi^2(a^2+b^2)}{K_0{}^2a^2b^2}\right\}, \tag{29}$$

$$h^2<\frac{2}{3}c_3\gamma_0\left\{h_0+\alpha\overline{P}+\frac{D_2\pi^2(a^2+b^2)}{a^2b^2}\right\}, \tag{30}$$

$$\{\theta_0h_0+\theta_1\delta_1+\alpha\overline{P}\}^2<\frac{2}{3}\delta_1\left\{h_0+\alpha\overline{P}+\frac{D_2\pi^2(a^2+b^2)}{a^2b^2}\right\}, \tag{31}$$

where

$$c_3=\frac{\gamma_0}{3\gamma^2}\left\{\frac{r_0}{K(\overline{T})}+\frac{D_1\overline{P}\pi^2(a^2+b^2)}{K_0{}^2a^2b^2}\right\}. \tag{32}$$

Then the uniform steady state of the initial-boundary value problems (1)-(2) is globally asymptotically stable with respect to all solutions initiating in the interior of the positive orthant.

**Proof** Let us consider the rectangular region D given by equation (27). In this case $I_2$, which is defined in Theorem (4.1), can be written as

$$I_2=-D_1\iint_D\left(\frac{\partial^2V_1}{\partial P^2}\right)\left\{\left(\frac{\partial P}{\partial x}\right)^2+\left(\frac{\partial P}{\partial y}\right)^2\right\}dA-D_2\iint_D\left(\frac{\partial^2V_1}{\partial T^2}\right)\left\{\left(\frac{\partial T}{\partial x}\right)^2+\left(\frac{\partial T}{\partial y}\right)^2\right\}. \tag{33}$$

From Eq. (17), we obtain

$$\frac{\partial^2V_1}{\partial P^2}=\frac{\overline{P}}{P^2}\ \text{and}\ \frac{\partial^2V_1}{\partial T^2}=1.$$

Hence

$$I_2\le-\frac{D_1\overline{P}}{K_0^2}\iint_D\left\{\left(\frac{\partial P}{\partial x}\right)^2+\left(\frac{\partial P}{\partial y}\right)^2\right\}dA-D_2\iint_D\left\{\left(\frac{\partial T}{\partial x}\right)^2+\left(\frac{\partial T}{\partial y}\right)^2\right\}dA.$$

Now

$$\iint_D\left(\frac{\partial P}{\partial x}\right)^2dA=\iint_D\left\{\frac{\partial(P-\overline{P})}{\partial x}\right\}^2dA=\int_0^b\int_0^a\left\{\frac{\partial(P-\overline{P})}{\partial x}\right\}^2dxdy.$$

Let $z = \dfrac{x}{a}$, then

$$\iint_D \left(\frac{\partial P}{\partial x}\right)^2 dA = \frac{1}{a}\int_0^b\int_0^1 \left\{\frac{\partial (P-\overline{P})}{\partial z}\right\}^2 dzdy\,.$$

Now using the inequality (Denn, 1972),

$$\int_0^1 \left(\frac{\partial P}{\partial x}\right)^2 dx \ge \pi^2 \int_0^1 P^2 dx,$$

we obtain

$$\iint_D \left(\frac{\partial P}{\partial x}\right)^2 dA \ge \frac{\pi^2}{a}\int_0^b\int_0^1 (P-\overline{P})^2 dzdy$$

$$= \frac{\pi^2}{a}\int_0^b\int_0^a (P-\overline{P})^2 dxdy$$

$$= \frac{\pi^2}{a^2}\iint_D (P-\overline{P})^2 dA.$$

Similarly, $\iint_D (\frac{\partial P}{\partial y})^2 dA \ge \frac{\pi^2}{b^2}\iint_D (P-\overline{P})^2 dA.$

Thus, $I_2 \le -\dfrac{D_1\overline{P}\pi^2(a^2+b^2)}{K_0{}^2a^2b^2}\iint_D (P-\overline{P})^2 dA - \dfrac{D_2\pi^2(a^2+b^2)}{a^2b^2}\iint_D (T-\overline{T})^2 dA.$

Now from (18) and (24), we obtain

$$\begin{aligned}
\frac{dV_2}{dt} \le \iint_D [ & -\{\frac{r_0}{K(\overline{T})} + \frac{D_1\overline{P}\pi^2(a^2+b^2)}{K_0{}^2a^2b^2}\}(P-\overline{P})^2 - c_1\gamma_0(Q-\overline{Q})^2 \\
& -\{h_0 + \alpha\overline{P} + \frac{D_2\pi^2(a^2+b^2)}{a^2b^2}\}(T-\overline{T})^2 - \delta_1(U-\overline{U})^2 \\
& + c_3\gamma(P-\overline{P})(Q-\overline{Q}) - \{r_0P\xi(T) + \alpha T\}(P-\overline{P})(T-\overline{T}) \\
& + \{\eta(U) + \alpha T\}(P-\overline{P})(U-\overline{U}) \\
& + \{\theta_0h_0 + \theta_1\delta_1 + \alpha\overline{P}\}(T-\overline{T})(U-\overline{U})]dA, \qquad (34)
\end{aligned}$$

where $\xi(T)$ and $\eta(U)$ are defined in Eq. (18).

Now Eq. (34) can be written as the sum of the quadratics

$$\frac{dV_2}{dt} \le \iint_D \{-\frac{1}{2}b_{11}(P-\overline{P})^2 + b_{12}(P-\overline{P})(Q-\overline{Q}) - \frac{1}{2}b_{22}(Q-\overline{Q})^2$$

$$-\frac{1}{2}b_{11}(P-\overline{P})^2+b_{13}(P-\overline{P})(T-\overline{T})-\frac{1}{2}b_{33}(T-\overline{T})^2$$
$$-\frac{1}{2}b_{11}(P-\overline{P})^2+b_{14}(P-\overline{P})(U-\overline{U})-\frac{1}{2}b_{44}(U-\overline{U})^2$$
$$-\frac{1}{2}b_{22}(Q-\overline{Q})^2+b_{23}(Q-\overline{Q})(T-\overline{T})-\frac{1}{2}b_{33}(T-\overline{T})^2$$
$$-\frac{1}{2}b_{33}(T-\overline{T})^2+b_{34}(T-\overline{T})(U-\overline{U})-\frac{1}{2}b_{44}(U-\overline{U})^2\}dA,$$

where

$$b_{11}=\frac{2}{3}\{\frac{r_0}{K(\overline{T})}+\frac{D_1\overline{P}\pi^2(a^2+b^2)}{K_0{}^2a^2b^2}\},\ b_{22}=c_1\gamma_0,\ b_{33}=\frac{2}{3}\{h_0+\alpha\overline{P}+\frac{D_2\pi^2(a^2+b^2)}{a^2b^2}\},\ \mathrm{b}_{44}=\delta_1,$$

$$b_{12}=c_3\gamma,\ b_{13}=-(r_0P\xi(T)+\alpha T),\ b_{14}=\eta(U)+\alpha T,\ b_{23}=h,\ b_{34}=\theta_1\delta_1+\theta_0h_0+\alpha\overline{P}.$$

Sufficient conditions for $\frac{dV_2}{dt}$ to be negative definite are that the following conditions hold:

$$b_{12}^2<b_{11}b_{22}, \tag{35}$$

$$b_{13}^2<b_{11}b_{33}, \tag{36}$$

$$b_{14}^2<b_{11}b_{44}, \tag{37}$$

$$b_{23}^2<b_{22}b_{33}, \tag{38}$$

$$b_{34}^2<b_{33}b_{44}. \tag{39}$$

We note that inequality (35) is automatically satisfied for the value of $c_3$ given in (32). We further note (28)$\Rightarrow$(36), (29)$\Rightarrow$(37), (30)$\Rightarrow$(38) and (31)$\Rightarrow$(39). Hence $V_2$ is a Liapunov function with respect to $\overline{E}$ whose domain contains the region of attraction $\Omega_1$, proving the theorem.

From the above theorem we note that inequalities (28)-(31) may be satisfied by increasing $D_1$ and $D_2$ to sufficiently large values. This implies that in the case of diffusion stability is more plausible than the case of no diffusion. Thus, in the case of diffusion the population converges towards its carrying capacity faster than the case of no diffusion, and hence the survival of the population may be ensured.

## 5 Conservation Model

In the previous section, it has been noted that uncontrolled human activities that are polluting the environment may harm itself considerably. Therefore, some kind of efforts must be adopted to stop further deterioration of the environment. In this section a mathematical model is proposed and analysed to control the undesired level of precursor pollutant by some mechanisms. It is assumed that the effort applied to control the precursor pollutant is proportional to the undesired level of the precursor pollutant. Then the dynamics of the system is assumed to be governed by the system of differential equations given below:

$$\frac{\partial P}{\partial t} = r(U)P - \frac{r_0 P^2}{K(T)} + D_1 \nabla^2 P,$$
$$\frac{\partial Q}{\partial t} = \gamma P - \gamma_0 Q - r_1 F,$$
$$\frac{\partial T}{\partial t} = hQ - h_0 T + \theta_1 \delta_1 U - \alpha PT + D_2 \nabla^2 T, \qquad (40)$$
$$\frac{\partial U}{\partial t} = -\delta_1 U + \theta_0 h_0 T + \alpha PT,$$
$$\frac{\partial F}{\partial t} = \mu_1 (Q - Q_c) H(Q - Q_c) - \upsilon_1 F,$$
$$0 \le \theta_0, \theta_1 \le 1.$$

The above model (40) is to be analysed with following initial and boundary conditions:

$$P(x, y, 0) = \phi(x, y) \ge 0, Q(x, y, 0) = \psi(x, y) \ge 0, T(x, y, 0) = \xi(x, y) \ge 0,$$

$$U(x, y, 0) = \zeta(x, y) \ge 0, F(x, y, 0) = \zeta_1(x, y) \ge 0, \ (x, y) \in D;$$

$$\frac{\partial P}{\partial n} = \frac{\partial T}{\partial n} = 0, (x, y) \in \partial D, t \ge 0, \qquad (41)$$

where $n$ is the unit outward normal to $\partial D$. Again we assume that the functions *P, Q, T, U, F* belong to the class $C^2(D)$.

In model (40), $F(x, y, t)$ is the density of effort applied to control the undesired level of precursor pollutant formed by the population. $r_1 > 0$ is depletion rate coefficient of $Q(x, y, t)$ due to the effort $F$. $\mu_1$ is the growth rate coefficient of $F$ and $\upsilon_1$ its natural depreciation rate coefficient. $Q_c$ is the critical level of precursor pollutant which is assumed to be harmless to the population. In the last equation of system (40), $H(t)$ denotes the unit step function which takes into account the case for which $Q \le Q_c$.

**6 Conservation Model Without Diffusion**

In this section we take $D_1 = D_2 = 0$ in model (40). Then model (40) has only one interior equilibrium, namely, $E^*(P^*, Q^*, T^*, U^*, F^*)$, where $P^*, Q^*, T^*, U^*$ and $F^*$ are the positive solutions of the system of algebraic equations given below:

$$r_0 P = r(U)K(T),$$
$$Q = \frac{\gamma \nu_1 P + r_1 \mu_1 Q_c}{\nu_1 \gamma_0 + r_1 \mu_1} = f_1(P), \text{ (say)}$$
$$T = \frac{h f_1(P)}{h_0(1 - \theta_0 \theta_1) + \alpha(1 - \theta_1)P} = f_2(P), \text{ (say)}$$

$$U=\frac{1}{\delta_1}(\theta_0 h_0+\alpha P)f_2(P)=f_3(P),\ \text{(say)}$$

$$F=\begin{cases}0, & Q\le Q_c\\ \frac{\mu_1}{\nu_1}(Q-Q_c), & Q>Q_c\end{cases}.$$

As earlier, it is easy to check that $E^*$ exists if the following inequality holds at $E^*$:

$$r_0-r'(U)\,f_3'(P)K(f_2(P))-K'(T)\,f_2'(P)r(f_3(P))>0. \tag{42}$$

In the following theorem, it is shown that $E^*$ is locally asymptotically stable. The proof is similar to Theorem 3.1 and hence is omitted.

**Theorem 6.1** Let the following inequalities hold:

$$\left\{\frac{r_0P^*}{K^2(T^*)}K'(T^*)+\alpha T^*\right\}^2<\frac{4}{9}\frac{r_0}{K(T^*)}(h_0+\alpha P^*), \tag{43}$$

$$h^2<\frac{4}{9}c_1\gamma_0(h_0+\alpha P^*), \tag{44}$$

$$\{\theta_1\delta_1+c_2(\theta_0h_0+\alpha P^*)\}^2<\frac{2}{3}c_2\delta_1(h_0+\alpha P^*), \tag{45}$$

where

$$c_1=\frac{r_0\gamma_0}{3\gamma^2K(T^*)}\ \text{and}\ c_2=-\frac{r'(U^*)}{\alpha T^*}.$$

Then $E^*$ is locally asymptotically stable.

In the following lemma a region of attraction for system (40) without diffusion is established. The proof of this lemma is easy and hence is omitted.

**Lemma 6.1** The set $\Omega_2=\{(P,Q,T,U,F):0\le P\le K_0,\ 0\le Q+T+U\le\frac{\gamma K_0}{\delta},0\le F\le\frac{\gamma\mu_1K_0}{\nu_1\delta}\}$ is a region of attraction for all solutions initiating in the interior of the positive orthant, where

$$\gamma_0>h\ \text{and}\ \delta=\min\{(\gamma_0-h),h_0(1-\theta_0),\delta_1(1-\theta_1)\}.$$

The following theorem gives criteria for global stability of $E^*$, whose proof is similar to Theorem 3.2 and hence is omitted.

**Theorem 6.2** In addition to the assumptions (3) and (4), let $r(U)$ and $K(T)$ satisfy in $\Omega_2$,

$$0\le -r'(U)\le\rho^*,\ K_m^*\le K(T)\le K_0\ \text{and}\ 0\le -K'(T)\le k^*,$$

for some positive constants $\rho^*, K_m^*$ and $k^*$. Let the following inequalities hold:

$$\left\{\frac{r_0 K_0 k^*}{K_m^{*2}} + \frac{\alpha\gamma K}{\delta}\right\}^2 < \frac{4}{9}\frac{r_0}{K(T^*)}(h_0 + \alpha P^*), \tag{46}$$

$$\left\{\rho^* + \frac{\alpha\gamma K_0}{\delta}\right\}^2 < \frac{2}{3}\delta_1 \frac{r_0}{K(T^*)}, \tag{47}$$

$$h^2 < \frac{4}{9}c_1\gamma_0(h_0 + \alpha P^*), \tag{48}$$

$$(\theta_1\delta_1 + \theta_0 h_0 + \alpha P^*)^2 < \frac{2}{3}\delta_1(h_0 + \alpha P^*), \tag{49}$$

where $c_1 = \dfrac{r_0\gamma_0}{3\gamma^2 K(T^*)}$.

Then $E^*$ is globally asymptotically stable with respect to all solutions initiating in the positive orthant.

Theorems (6.1) and (6.2) show that if suitable efforts are made to control the undesired level of precursor pollutant formed by the activities of populations in the environment, the population density may be maintained at a desired level under certain conditions.

**7 Conservation Model With Diffusion**

We now consider the case when $D_i > 0\ (i = 1, 2)$ in model (40). Under an analysis similar to Section 4, it can be established that if the interior equilibrium $E^*$ of model (40) with no diffusion is globally asymptotically stable, then the corresponding uniform steady state of system (40)-(41) is also globally asymptotically stable with respect to solutions such that

$$\phi(x,y) > 0, \psi(x,y) > 0, \xi(x,y) > 0, \varsigma(x,y) > 0, \zeta_1(x,y) > 0,\ (x,y) \in D.$$

Further, it should be noted that if system (40) with no diffusion is unstable even then the corresponding uniform steady state of system (40)-(41) can be made stable by increasing diffusion coefficients to sufficiently large values.

Thus, we conclude that diffusion in our model plays the general role of stabilizing the system.

**8 Numerical Simulations**

In this section, numerical simulation results are presented to illustrate the results of previous sections. Matlab 7.5 is used for numerical simulation to study the dynamical behaviour of the model system (5). Model (5) is integrated numerically using the fourth order Runge-Kutta method. We consider the following particular form of the function in model (5):

$$\begin{aligned} r(U) &= r_0 - r_{10}U, \\ K(T) &= K_0 - K_1 T. \end{aligned} \tag{50}$$

The model system (5) displays stable focus for the following set of parameter values given in Eq. (51):

$$r_0 = 20.0,\ r_{10} = 0.07,\ K_0 = 60.0,\ K_1 = 0.08,\ K_m = 50.0,\ \gamma = 0.05,\ \gamma_0 = 0.04,$$
$$h = 0.30,\ h_0 = 0.20,\ \delta_1 = 7.0,\ \theta_0 = 0.01,\ \theta_1 = 0.02,\ \alpha = 0.06, \quad (51)$$
with initial conditions $(P_0, Q_0, T_0, U_0) = (5, 5, 5, 5)$.

With above values of parameters, it is found that condition (6) for the existence of interior equilibrium $\bar{E}$ is satisfied and it is given by

$$(\bar{P}, \bar{Q}, \bar{T}, \bar{U}) = (58.8821, 73.5844, 6.0276, 3.0440). \quad (52)$$

We also note that for the values of parameters given above, all conditions of Theorem (3.1) and (3.2) are satisfied. This shows that $\bar{E}$ is locally as well as globally asymptotically stable. The time series analysis of model system (5) is presented in Fig.1 which shows that the positive equilibrium $\bar{E}$ is a stable focus.

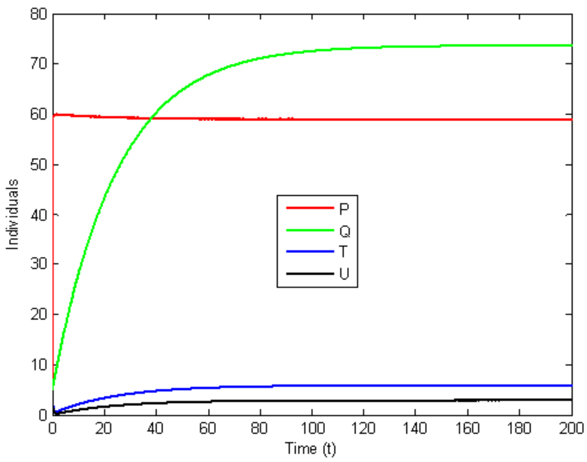
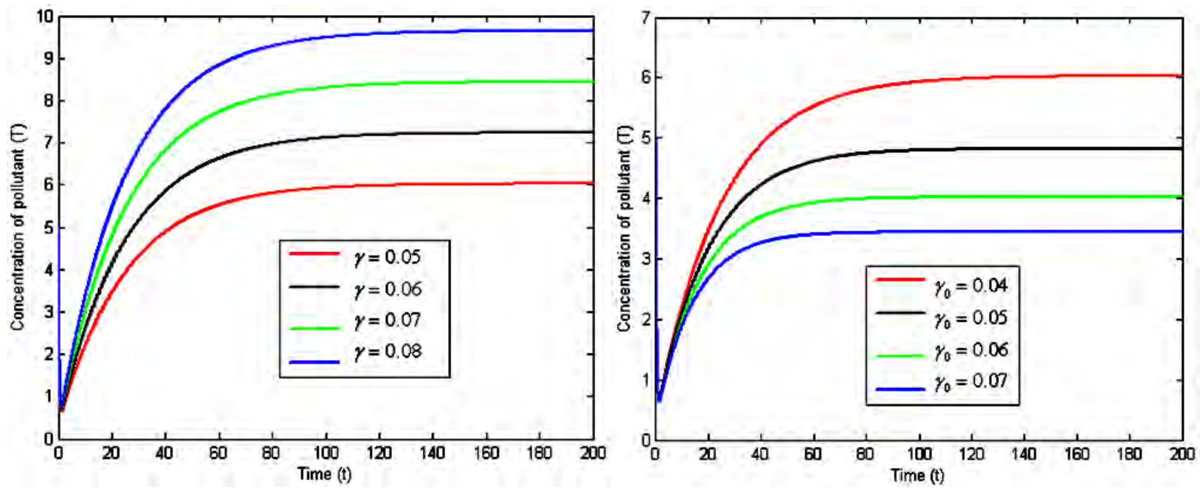



**Fig. 1** Time-series corresponding to the individuals of the model system (5) with parameter values given in Eq. (51).

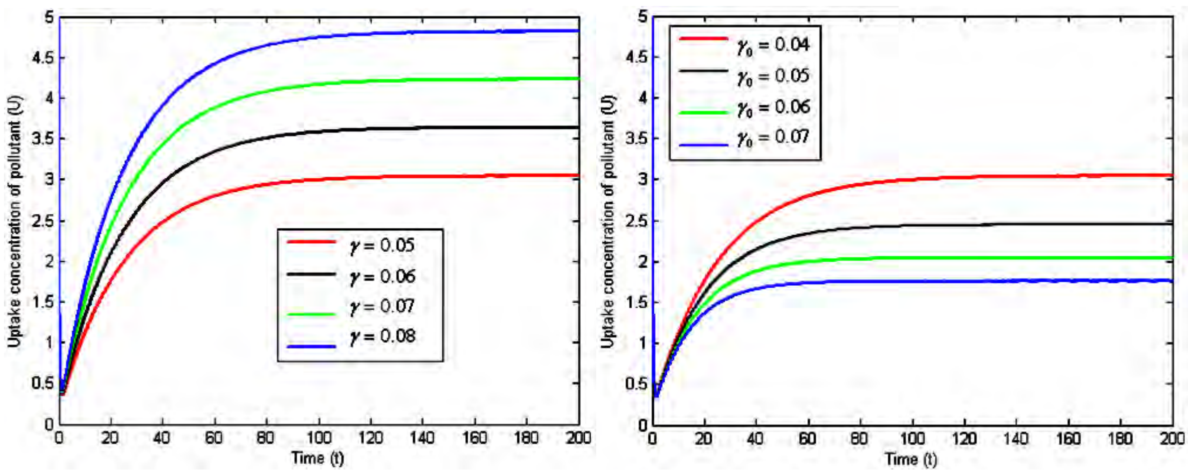
To study the dynamical behaviour of the model system (5), the temporal evolution of $T$ and $U$ are observed for different values of control parameters. We observe the temporal dynamics of the concentration of pollutant $T$ in the atmosphere formed by $Q$ for different control parameters and found that it increases for the increasing value of the growth rate parameter of $Q$ due to $P$ (i.e., $\gamma$) but it is of decline nature as we increase the value of the parameter $\gamma_0$, the natural depletion rate coefficient of $Q$. We have presented the increasing and decreasing nature of $T$ in Fig. 2.



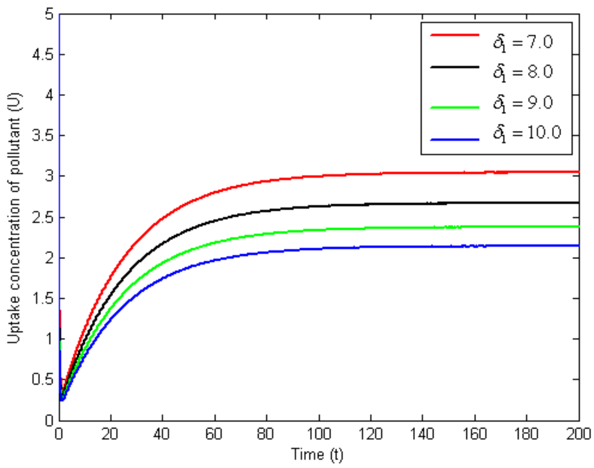

(a) (b)

**Fig. 2** Temporal evolution (*t* vs. *T*) for the model system (5) with variation of parameters (a) $\gamma$, (b) $\gamma_0$, and other parameters are given in Eq. (51).

We have also studied the temporal dynamics of uptake concentration of pollutant by the population due to the variation of parameters $\gamma$, $\gamma_0$ and $\delta_1$. It is found that the uptake concentration of pollutant by population increases if we increase the growth rate coefficient of *Q* due to *P* (i.e. $\gamma$), and it decreases if we increase the values of natural depletion rate coefficient of *Q* and *U* respectively (see Fig. 3).




(a) (b)




(c)

**Fig. 3** Temporal evolution (*t* vs. *U*) for the model system (5) with variation of parameters (a) $\gamma$, (b) $\gamma_0$, (c) $\delta_1$, and other parameters are given in Eq. (51).

Now if we compare the nature of Fig. 2 with Figs.3(a)-(b), it is found that the growth rate and the natural depletion rate of *Q* have almost same impact on the dynamics of the pollutant formed by *Q* in the atmosphere and on the uptake concentration of pollutant by the population.

To study the dynamical behaviour of model system (40) without diffusion, we select the same particular form of the function as given in Eq. (50) and values of parameters are given below in Eq. (53):

$$\begin{aligned} &r_0 = 20.0,\ r_{10} = 0.07,\ K_0 = 60.0,\ K_1 = 0.08,\ \gamma = 0.05,\ \gamma_0 = 0.04, \\ &h = 0.30,\ h_0 = 0.20,\ \delta_1 = 7.0,\ \theta_0 = 0.01,\ \theta_1 = 0.02,\ \alpha = 0.06, \\ &r_1 = 0.09,\ \mu_1 = 12.0,\ \upsilon_1 = 0.9,\ Q_c = 0.14, \end{aligned} \tag{53}$$

with initial conditions $(P_0, Q_0, T_0, U_0, F_0) = (5.0, 5.0, 5.0, 5.0, 5.0)$.

With above values of parameters, it is found that condition (42) for the existence of interior equilibrium $E^*$ is satisfied and is given by

$$(P^*, Q^*, T^*, U^*, F^*) = (59.9511,\ \ 2.9680,\ \ 0.2390,\ \ 0.1229,\ \ 31.9893). \tag{54}$$

By choosing $K_m^* = 50.0$, we note that all conditions of Theorem (6.1) and (6.2) are satisfied. This shows that equilibrium $E^*$ is locally as well as globally asymptotically stable. The time series of model (40) without diffusion is presented in Fig.4 which shows that the positive equilibrium $E^*$ is a stable focus.

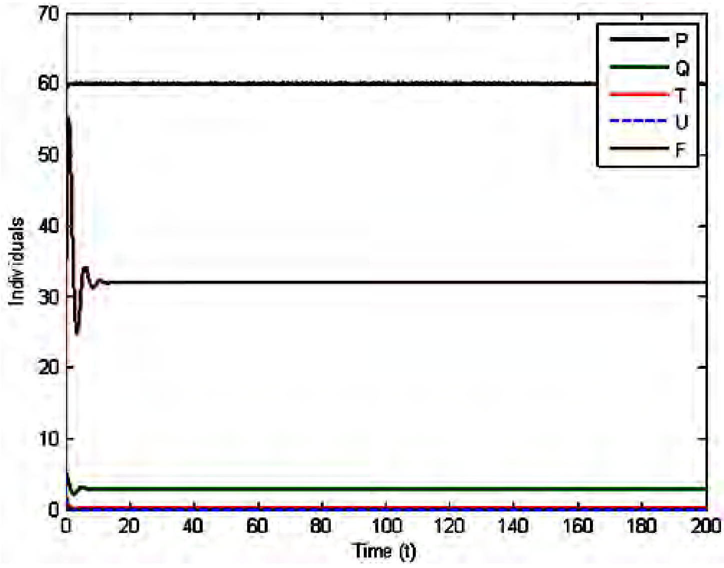



**Fig. 4** Time-series corresponding to the individuals of the model system (40) without diffusion with parameter values given in eq. (53).

By comparing Figs.1 and 4, we note that due to effort *F*, the equilibrium level of the population has increased whereas equilibrium level of the concentration of precursor pollutant, concentration of pollutant in the environment and population have decreased.

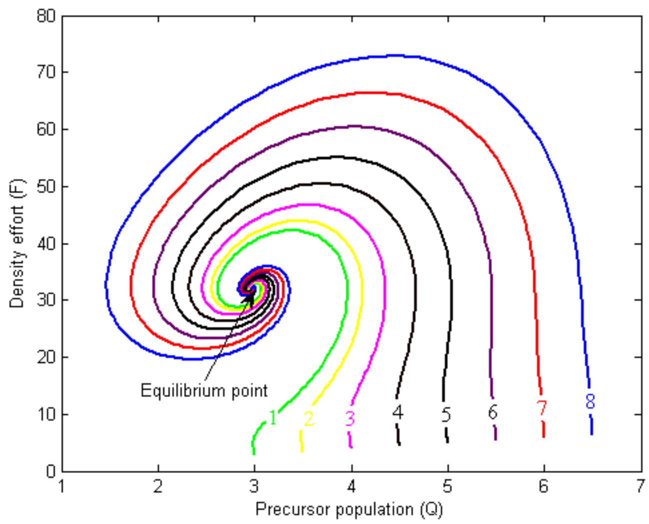
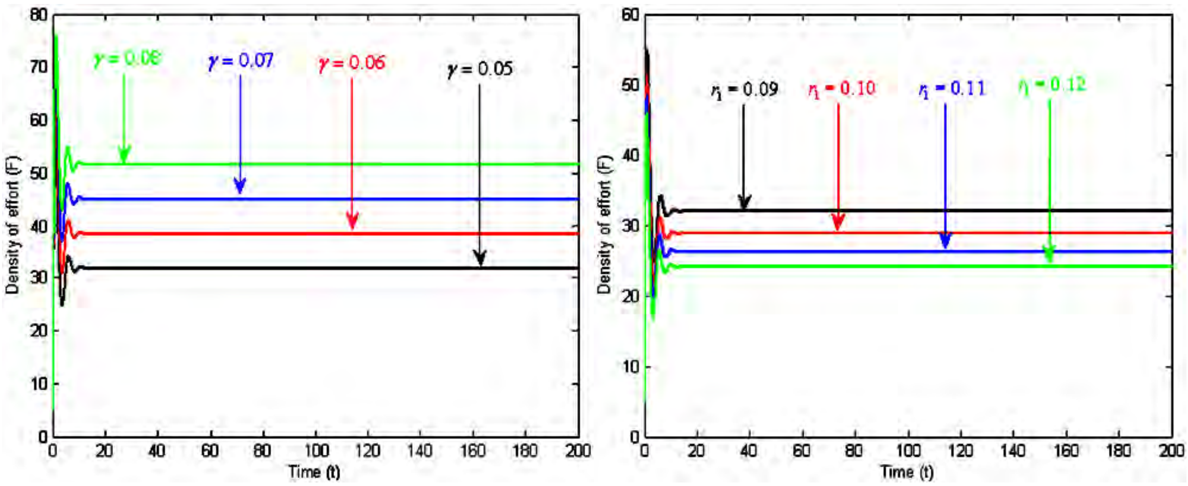



**Fig. 5** Graph of $F$ versus $Q$ for the different initial starts for the set of parameter value given in Eq. (53).

The phase plane analysis of the model system (40) without diffusion in the (*Q*, *F*) plane is shown in Fig. 5 which also shows that the positive equilibrium is a stable focus.

The time series analysis of *F*, the effort applied to control the undesired level of precursor pollutant formed by the population is shown in Fig.6. It shows the positive and negative impact as we increase the value of growth rate of *Q* due to *P* (i.e., $\gamma$ ) and the depletion rate of *Q* due to *F* (i.e., $r_1$ ) respectively.




(a) (b)

**Fig. 6** Temporal evolution ($t$ vs. $F$) for the model system (40) without diffusion with variation of parameters (a) $\gamma$, (b) $r_1$, and other parameters are given in Eq. (53).

The effect of different control parameters on the dynamical behaviour of the conservation model is presented in Table 1. After varying one of the control parameter in its range, while keeping all others constant, we monitor the changes in the dynamical behaviour of the model system, thereby fixing the regimes in which the system exhibit either stable focus or stable limit cycle solution. We have varied the control parameters in the following ranges:

$$15 \le r_0 \le 53,\ \ 26 \le r_{10} \le 49,\ \ \ 57 \le K_0 \le 149, 0.01 \le h \le 0.34, 3.6 \le \mu_1 \le 12.5.$$

From Table 1, it is found that for the parameters $r_0$ in the ranges [22.0, 53.0], $r_{10}$ in the range [26.0, 27.0], $K_0$ in the range [57.0, 92.0], $h$ in the range [0.01, 0.28] and $\mu_1$ in the range [3.6, 8.5], the system dynamics converging to the stable equilibrium and for other ranges it exhibits limit cycle solution. For the lower values of all the control parameters except for $r_0$, the intrinsic growth rate of population, the dynamics settled on equilibrium position and for higher values it shows the periodic nature.

## 9 Discussions and Conclusions

The proposed nonlinear model is analysed to study the effect of pollution on a population, which is living in an environment polluted by its own activities. The model has been studied with and without diffusion. In the case of no diffusion, it has been shown that population density settles down to its equilibrium level, the magnitude of which depends upon the equilibrium levels of emission and washout rates of pollutant as well as on the rate of precursor formation and its depletion. It has been noted that the rate of precursor formation is critical in effecting the population. It has further been noted that if the concentration of pollutant increases unabatedly, the survival of the population would be threatened.

In case of a model with diffusion, it has been shown that the uniform steady state of the system is globally asymptotically stable if the corresponding steady state is globally asymptotically stable in case of without diffusion. It has further been noted that if the positive equilibrium of the system with no diffusion is unstable, then the corresponding uniform steady state of the system with diffusion can be made stable by increasing diffusion coefficients appropriately. Thus, it has been concluded that the global stability is more plausible in the case of diffusion than the case of no diffusion. It is found that the uptake concentration of pollutant by population increases if we increase the growth rate coefficient of *Q* due to *P*, and it decreases, if we increase the values of natural depletion rate coefficient of *Q*, *T* and *U* respectively.

**Table 1** Simulation experiment of model (40) without diffusion with parameter values $K_1 = 40,\ \ \gamma = 1,$ $\gamma_0 = 0.01,\ r_1 = 0.3,\ h_0 = 0.2,\ \theta_1 = 0.02,\ \theta_0 = 0.01,\ \delta_1 = 7.0,\ \alpha = 0.06,\ \upsilon_1 = 0.4,$ and $Q_c = 0.14$ with initial condition $(P_0, Q_0, T_0, U_0, F_0) = (5.0, 5.0, 5.0, 5.0, 5.0)$ and SF: Stable Focus; SLC: Stable Limit Cycle.

| Parameter varied | Range in which parameter varied | Dynamical outcome | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | (*P*,*Q*) | (*P*,*T*) | (*P*,*U*) | (*P*,*F*) | (*Q*,*T*) | (*Q*,*U*) | (*Q*,*F*) | (*T*,*U*) | (*T*,*F*) | (*U*,*F*) |
| $r_0$ | 15-21 | SLC | SLC | SLC | SLC | SLC | SLC | SLC | SLC | SLC | SLC |
| $15 \le r_0 \le 53$ | 22-53 | SF | SF | SF | SF | SF | SF | SF | SF | SF | SF |
| $r_{10}$ | 26-27 | SF | SF | SF | SF | SF | SF | SF | SF | SF | SF |
| $26 \le r_{10} \le 49$ | 28-49 | SLC | SLC | SLC | SLC | SLC | SLC | SLC | SLC | SLC | SLC |
| $K_0$ | 57-92 | SF | SF | SF | SF | SF | SF | SF | SF | SF | SF |
| $57 \le K_0 \le 149$ | 93-149 | SLC | SLC | SLC | SLC | SLC | SLC | SLC | SLC | SLC | SLC |
| $h$ | 0.01-0.28 | SF | SF | SF | SF | SF | SF | SF | SF | SF | SF |
| $0.01 \le h \le 0.34$ | 0.29-0.34 | SLC | SLC | SLC | SLC | SLC | SLC | SLC | SLC | SLC | SLC |
| $\mu_1$ | 3.6-8.5 | SF | SF | SF | SF | SF | SF | SF | SF | SF | SF |
| $3.6 \le \mu_1 \le 12.5$ | 9.0-12.5 | SLC | SLC | SLC | SLC | SLC | SLC | SLC | SLC | SLC | SLC |

In case of conservation model, it has been shown that if the rate of formation of the precursor pollutant is controlled by some external means, its effect on the population can be minimised. All the above results in the absence of diffusion are well supported by computer simulations as explained in Section 8. It is also found that the system dynamics converging to the stable equilibrium for lower values of all the control parameters except for the intrinsic growth rate parameter $r_0$ of the population and for the higher values it exhibit the limit cycle solution.

From this study, it can be concluded that the uncontrolled human activities that polluting the environment may be harmful to itself. Therefore some kind of efforts must be adopted to control the further deterioration of the environment. This study also gives some idea about how to prevent the biological species living in an environment polluted by its own activity and to develop the model related to socio-ecological problems and about its solution.

**Acknowledgements**

The first author (BD) gratefully acknowledges the support received by UGC, New Delhi, Grant No. F.510/2/DRS/2010 (SAP-I).

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