Article

The exact solutions of nonlinear problems by Homotopy Analysis Method (HAM)

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Abstract
The present paper presents the comparison of analytical techniques. We establish the existence of the phenomena of the *noise terms* in the perturbation series solution and find the exact solution of the nonlinear problems. If the *noise terms* exist, the *Homotopy Analysis method* gives the same series solution as in *Adomian Decomposition Method* as well as homotopy Perturbation Method (Wahab et al, 2015) and we get the exact solution using the initial guess in Homotopy Analysis Method using the results obtained by *Adomian Decomposition Method*.

Keywords homotopy analysis method; nonlinear problems; perturbation methods.

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1 Introduction

In the recent years, there is a great research in dealing with the nonlinear problems with perturbation techniques. *Adomian Decomposition Method* is proven to be one of those effective and reliable techniques for solving the nonlinear problems. A. M. Wazwaz (2006) is the first one who modified the standard *Adomian decomposition method* to get the exact solution of the differential equations by separating the first component of the iterative solution into two parts. He did not provided any idea to deal with this separation as according to Wazwaz (2006), the separation of the first component of the recursive scheme depends only on the trial basis. Here, we shall critically comment on some idea about the separation of the first component into two parts. We shall see that if we use *standard Adomian method* instead of the *modified technique* developed by Wazwaz, we get the best results if some specific criteria is justified.

This work is originally generated from the M. Phil. Thesis (Wahab, 2006). The other important results of this dissertation have been published in (Wahab et al, 2015).

1.1 The Homotopy Analysis Method (HAM)

Finding the solutions of linear problems has become easy by the appearance of the supercomputers, we face still great difficulty for non-linear problems, mostly when we apply an analytical method. Since the mathematicians and engineers do not still satisfy completely even if the nonlinear analytical methods are fast developing and improving.

The traditional perturbation methods, Like other nonlinear analytical techniques, are restricted by their own limitations. Since all the perturbation methods are mainly based on the existence of a small or a large parameter, but not all nonlinear problems have such an existence. On the other hand, if the existence of such a parameter is ensured, then the results produced by a traditional perturbation technique are mostly valid only for the small parameter. So these restrictions made the applications of perturbation techniques limited. Hence it becomes necessary to overcome this limitation by developing a new nonlinear analytical technique which does not based on the assumption of a small parameter. Liao has developed a nonlinear analytical technique which avoids the requirements of the existence of small parameters and it can be applied for the solution of nonlinear problems. This technique is developed using an interesting property of Homotopy, by which a nonlinear problems can transformed into an infinite number of linear problems. For example see [Shijun Liao, application of process analysis method of solution of two dimensional nonlinear progressive gravity waves, J. Ship Research 36 (1192) 30-37], the developedd method is applied for the solution of two dimensional nonlinear progressive gravity waves and the results obtained are better analytical approximations at the forth order than those given by other perturbation techniques. We will show that the proposed method gives relatively much better approximations than the traditional perturbation methods.

1.2 Basic ideas of homotopy method

For the details of *Homotopy Analysis Method*, we refer the readers to study (Liao, 2003). Here, we simply introduce the basic ideas of Homotopy as an important part of differential topology. The results produced by *Homotopy Analysis Method* are valid uniformly for small as well as large parameters. This is due to the reason that ``*any kth-order deformation equations are linear about the kth-order deformation derivatives*'', which is a simple property of Homotopy in topology, which is the main difference between the traditional perturbation techniques and the proposed method by Liao. The initial approximations can now be chosen with freedom and without the assumption of the existence of the small parameter. Although the examples given in this work are non linear differential equations, the *Homotopy Analysis Method* can also be applied for the solution of nonlinear algebraic equations and other partial differential equations, such as Navier-Stokes equations, equations of gravity waves, the KdV equation, the Boussinesq's equation, the Sine-Gorden equation and so on. However, the *Homotopy Analysis Method* needs certainly many improvements, whether it has proven its effectiveness and reasonableness in many examples (Liao, 2003), because the theoretical research is needed as the examples do not constitute a mathematical proof. The *Homotopy Analysis Method* overcomes most of the limitations of traditional perturbation techniques. However, as a new perturbation technique, this method is not perfect and also has some limitations, even then, has been applied successfully in many nonlinear problems to give satisfactory results. Although, the proposed method seem to be reliable and promising yet more applications, especially the theoretical research are rquired to improve it.

1.2.1 Zero order deformation equation

Mostly, a phenomena of nonlinear problem is described by the set of governing equations prescribed by initial and /or boundary data. We consider here a nonlinear equation in its general form for our convince.

$$N[u(r,t)] = 0, \tag{1}$$

where N is a non linear operator, $u(r,t)$ is an unknown function, r denote spatial variable and t is the temporal independent variable. Let $u_0(r,t),$ is chosen as an initial guess of the exact solution defined by

$u(r,t)$, $h \neq 0,$ an auxiliary parameter, $H(r,t) \neq 0,$ refers to an auxiliary function, and $L,$ is an auxiliary linear operator having the property defined by, $L[f(r,t)] = 0,$ when $f(r,t) = 0.$

Then an embedding parameter $p \in [0,1]$ can be used to construct such a homotopy:

$$\begin{aligned} &H[v(r,t;p);u_0(r,t),H(r,t),h,p] = \\ &\quad (1-p)\{L[v(r,t;p)-u_0(r,t)]\} - p\ h\ H(r,t)N[v(r,t;p)]. \end{aligned} \tag{2}$$

According to Lio, the nonzero auxiliary parameter $\hbar,$ and auxiliary function $H(r,t),$ in the above constructed homotopy are introduced for the first time in this way to construct a Homotopy which makes the constructed Homotopy as more general than traditional ones. The important thing to note is that we are given a great freedom to choose the initial guess $u_0(r,t)$, the linear operator L , the auxiliary parameter $\hbar$, and the auxiliary function H . Let $p \in [0,1]$ be the embedding parameter and taking the Homotopy equal to zero,

i.e., $$H[v(r,t;p);u_0(r,t),H(r,t),h,p] = 0.$$

we shall have the zero deformation equation defined by:

$$(1-p)\{L[v(r,t;p)-u_0(r,t)]\} = p\ h\ H(r,t)N[v(r,t;p)], \tag{3}$$

where $v(r,t;p)$ is the solution which depends upon the following:

- The Initial Guess
- The Auxiliary Linear Operator
- The Auxiliary Parameter
- The Embedding Parameter

The value of the embedding parameter $p = 0,$ leads the zero order deformation equation (3) to,

$$L[v(r,t;p)-u_0(r,t) = 0 \tag{4}$$

which gives $v(r,t;0) = u_0(r,t).$ While he value of the embedding parameter $p = 1,$ with $h \neq 0,$ and $H \neq 0,$ leads the zero order deformation equation (3) to $N[v(r,t;1)] = 0,$ which is the same as the original equation (1), provided

$$v(r,t;1) = u(r,t). \tag{5}$$

Then the embedding parameter p increases from 0 to 1 so that $H,$ continuously defroms the initial guess $u_0,$ to the exact solution $u,$ of the original problem (1). The nth-order deformation derivatives are defined as:

$$u^{[n]}(r,t) = \left.\frac{\partial^n[v(r,t;p)]}{\partial p^n}\right|_{p=0}, \tag{6}$$

where $v(\vec{r},t;p)$ is expanded in power series of q as:

$$v(r,t;p) = v(r,t;0) + \sum_{n=1}^{+\infty} \frac{u_0^{[n]}(r,t)}{n!} p^n. \tag{7}$$

Writing $u_n(r,t) = \frac{u_0^{[n]}(r,t)}{n!} = \frac{1}{n!}\left[\frac{\partial^n v(r,t;p)}{\partial p^n}\right]\Bigg|_{p=0}$, the power series (7) of $v(r,t;p)$ becomes

$$v(r,t;p) = u_0(r,t) + \sum_{m=1}^{+\infty} u_m(r,t) p^m. \tag{8}$$

We assume that the solution $v(\vec{r},t;p)$ of the zero order deformation equation exists for all $p \in [0,1]$, the deformation derivative $u_0^{[n]}(r,t)$ exists for n = 1, 2, 3 ,$+\infty$, and the power series (8) of $v(r,t;p)$ converges when $p = 1$. Then we have the solution series

$$u(r,t) = u_0(r,t) + \sum_{n=1}^{+\infty} u_n(r,t). \tag{9}$$

This expression defined above provides a relation between the exact solution and the initial guess $u_0(r,t)$, of the problem by means of the terms $u_n(r,t)$, which are produces by the high order deformation equations.

1.2.2 High order deformation equation

We define a vector, $\overline{u_n} = \{u_0(r,t), u_1(r,t), u_2(r,t), ..., u_n(r,t)\}$,

and then the governing equations for $u_n(r,t)$, can be obtained from zero order deformation equation (3). We now can define nth-order deformation equation as:

$$L[u_n(r,t) - \chi_n u_{n-1}(r,t)] = \hbar H(x,t) R_n(\overline{u_{n-1}}, r, t), \tag{10}$$

where

$$\begin{cases} \chi_m = & 0 & m \le 1, \\ \quad\;\, = & 1 & Otherwise, \end{cases}$$

and

$$R_n(\overline{u_{n-1}}) = \frac{1}{(n-1)!} \frac{\partial^{n-1}[N[v(r,t;p)]]}{\partial p^{n-1}}\Bigg|_{p=0}.$$

Or finally we have

$$R_n[\overline{u_{n-1}}, r, t] = \left[\frac{1}{n-1)!} \frac{\partial^n}{\partial p^n} N\left[\sum_{n=0}^{\infty} u_n(r,t) p^n\right]\right]\Bigg|_{p=0} = 0 \tag{11}$$

It is interesting to note that high order deformation equation (10) is subject to the same linear operator L, and $R_n[\overrightarrow{u_{n-1}}, r, t],$ may be expressed for any nonlinear operator N. The dependance of the right hand side of equation (10) upon $\overrightarrow{u_{m-1}}$ gives us $u_0(r,t), u_1(r,t), u_2(r,t), ..., u_n(r,t),$ by the solution of the linear high order deformation equation (10). Then the nth-order approximation of $u(r,t),$ is defined as:

$$u(r,t) \approx \sum_{k=0}^{m} u_k(r,t). \tag{12}$$

2 Examples

2.1 Nonlinear partial differential equations

Example 1

$$u_{tt} + u_x^2 + u - u^2 = te^{-x}, \tag{13}$$

with initial conditions $u(x,0) = 0,$ and $u_t(x,0) = e^{-x}.$ Let $u_0(x,t),$ be the initial guess of $u(x,t),$ which satisfies the initial conditions. Since we are free to choose initial guess, so we choose

$$u_0(x,t) = te^{-x}, \tag{14}$$

then clearly $u_0(x,0) = 0,$ and $u_{0t}(x,0) = e^{-x}.$ This chosen guess was the exact solution given by Wazwaz in (Wazwaz. 2006) by choosing it as u_0, and in *Standard Adomian decomposition method* it is a part of $f(x)$ and using the phenomena of the ``*Noise Terms*'', it is a non canceled term of the first component u_0. However, here, if we are ignorant of the exact or approximate solution, then using the rule of solution expression given in (Liao, 2004), the best initial guess is (14). However, we could choose here

$$u_0(x,t) = e^{-x} \sin t,$$

$$u_0(x,t) = te^{-x} \cos t.$$

Instead of (14). These also satisfy the given initial conditions. But using our prior knowledge about the solution of this equation we choose (14). Let $p \in [0,1]$ being an *embedding parameter*. The *Homotopy Analysis Method* being based upon such kind of continuous mapping that $u(x,t) \to v(x,t;p),$ as p goes from 0 to 1, $v(x,t;p),$ varies from the initial guess $u_0(x,t)$ to the exact solution $u(x,t)$.

We define the nonlinear term as

$$N[v(x,t;p)] = v_{tt}(x,t;p) + v_{xx}^2(x,t;p) + v(x,t;p) - v^2(x,t;p) - te^{-x}.$$

Zero order deformation equation

Let $\hbar \neq 0,$ and $H(x,t) \neq 0,$ be the *auxiliary parameter* and the *auxiliary function* respectively. We construct a *homotopy* $H[v(x,t;p),u_0,H,\hbar;p)],$ as

$$v(x,t;p):\cup\Omega j\times[0,1]\to\Re,$$

or
$$v(x,t;p):\Omega\ \times\ [0,1]\to\Re,$$

where $t\in\Omega_1,$ $x\in\Omega_2,$ and $\cup\Omega_j=\Omega.$ This satisfies

$$H[v(x,t;p);u_0,H,\hbar;p]=0\ ,H[v\ ;u_0,H,\hbar;p]=(1-p)L[v(x,t;p)-u_0(x,t)]-\hbar H(x,t)p\ N[v(x,t;p)]=0, \tag{15}$$

$$(1-p)L[v(x,t;p)-u_0(x,t)]=\hbar H(x,t)p\ N[v(x,t;p)],$$

where $v(x,t;p)=u_0(x,t)+\sum_{m=1}^{\infty}u_m(x,t)p^m,$ subject to the $v(x,0;p)=0,$, we have

$$v_t(x,0;p)=e^{-x}. \tag{16}$$

When p=0 : The equation (15) becomes

$$L[v(x,t;0)-u_0(x,t)]=0,$$

subject to the I. C's $v(x,0;0)=0,$ and $v_t(x,0;0)=e^{-x}.$ Then the solution of equation (13) becomes

$$v(x,t;0)=u_0(x,t). \tag{17}$$

When p=1: The equation (15) becomes $\hbar H(x,t)N[v(x,t,1)]=0.$ Since $\hbar\neq 0$ *and* $H\neq 0$, then $N[v(x,t;1)]=0,$ subject to $v(x,0;1)=0,$ and $v_t(x,0;1)=e^{-x}.$ Equations (15) and (16) are called the *Zero Order Deformation Equation*

We now define nth*-order deformation derivative* as

$$u_0^{(n)}(x,t)=\frac{\partial^n}{\partial p^n}v(x,t,p)\bigg|_{p=0}. \tag{18}$$

We expand $v(x,t,p)$ in a power series of p by Tailor's theorem as

$$v(x,t;p)=v(x,t;0)+\sum_{n=1}^{\infty}\frac{1}{n!}\frac{\partial^n}{\partial p^n}[v(x,t;p)]p^n\bigg|_{p=0}. \tag{19}$$

Using (16), we have $v(x,t;p)=v(x,t;0)+\sum_{m=1}^{\infty}\frac{1}{m!}u_o^{(m)}(x,t)p^m\bigg|_{p=0}.$ Define $u_n(x,t)=\frac{u_0}{n!}=\frac{1}{n!}\frac{\partial^n}{\partial p^n}[v(x,t;p)]\bigg|_{p=0}.$ We may write equation (19) as

$v(x,t;p) = v(x,t;0) + \sum_{m=1}^{\infty} u_m(x,t)p^m.$

Since by (14) $v(x,t;0) = u_0(x,t),$ then $v(x,t;p) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t)p^m.$ When $p \to 1$ then $v(x,t;p) \to u(x,t),$ and $u(x,t) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t),$

$$\Rightarrow \qquad u(x,t) = \sum_{m=0}^{\infty} u_m(x,t). \tag{20}$$

We make the following assumptions

- Let the solution of *zero order deformation equation* exists for all $\overrightarrow{u_n} = \{u_0(x,t), u_1(x,t), ..., u_n(x,t)\}$ and $p \in [0,1],$
- The deformation derivative $u_0^{(m)}(x,t),$ exists for $m = 1, 2, ...,$ The power series of $v(x,t;p),$ converges at $p = 1.$

Under the assumptions, we have solution series (20).

Higher order deformation equation

Defining a vector $\overrightarrow{u_n} = \{u_0(x,t), u_1(x,t), ..., u_n(x,t)\},$ we have nth *order deformation equation.*

$$L[u_m(x,t) - \chi_m u_{m-1}(x,t)] = \hbar H(x,t) - R_m(\overrightarrow{u_{n-1}}), \tag{21}$$

and

$$R_n(\overrightarrow{u_{n-1}}) = \frac{1}{(n-1)!} \frac{\partial^{n-1}[N[v(x,t;p)]]}{\partial p^{n-1}}\bigg|_{p=0}.$$

and $v(x,t;p) = \sum_{n=0}^{\infty} u_n(x,t)p^n,$ then $R_n(\overrightarrow{u_{n-1}}) = \frac{1}{(m-1)!} \frac{\partial^{n-1}[N[v(x,t;p)]]}{\partial p^{n-1}}\bigg|_{p=0},$ or

$$R_n[\overrightarrow{u_{n-1}}] = \left[\frac{1}{(n-1)!} \frac{\partial^n}{\partial p^n} N\left[\sum_{n=0}^{\infty} u_n(x,t)p^n\right]\right]\Bigg|_{p=0} = 0.$$

The expression is the same expression as ``*Adomian Polynomials*''

$$A_n[\overrightarrow{u_{n-1}}] = \frac{1}{n!} \frac{d^n}{d\lambda^n}\left[N(\sum_{i=0}^{\infty} \lambda^i u_i)\right]\Bigg|_{\lambda=0},$$

in *Adomian decomposition* series solution [1]. So for this problem we have

$$R_n(\overrightarrow{u_{n-1}}) = u_{n-1} + (u_{n-1})_{tt} + \sum_{j=0}^{n-1} u_{jx} u_{n-1-j} - \sum_{j=0}^{n-1} u_j u_{n-1-j} - (1-\chi_n)te^{-x}. \tag{22}$$

From equation (21), we have

$$u_n(x,t) = \chi_n \ u_{n-1}(x,t) + h\ H(x,t) L^{-1}[R_n(\overrightarrow{u_{n-1}})]. \tag{23}$$

Now *first order deformation equation* is

$$u_1(x,t) = \chi_1 \ u_0(x,t) + h\ H(x,t)L^{-1}[R_m(\overline{u_0})],$$

or $u_1(x,t) = h\ H(x,t)L^{-1}[R_n(\vec{u}_0)],$ and from (22), we have

$$R_n(\overline{u_0}) = u_0 + (u_0)_{tt} + (u_0^{'})^2 - u_0^2 - te^{-x},$$

$$R_n(\overline{u_0}) = te^{-x} + 0 + t^2e^{-2x} - t^2e^{-2x} - te^{-x}.$$

Thus $R_n(\overline{u_0}) = 0.$ and $u_1(x,t) = h\ H(x,t)L^{-1}[0],$ and $u_1(x,t) = 0.$ From (23) $u_2(x,t) = \chi_2 \ u_1(x,t) + h\ H(x,t)L^{-1}[R_n(\overline{u_1})],$

$$R_n(\overline{u_0}) = u_1 + (u_1)_{tt} + \sum_{j=0}^{1} u_{jx}u_{n-1-j} - \sum_{j=0}^{1} u_j u_{n-1-j} - (1-\chi_2)te^{-x},$$

$$= 0 + (0)_{tt} + (u_{0x}u_{1x} + u_{1x}u_{0x}) - (u_0u_1 + u_1u_0), = 2u_{0x}u_{1x} - 2u_0u_1.$$

Since $u_1 = 0,$ so $R_n(\overline{u_1}) = 0.$ Hence $R_n(\overline{u_{n-1}}) = 0, \ \ \forall \ \ n \geq 1.$ So from (20) we have $u(x,t) = te^{-x},$ is the exact solution.

Example 2 $$u_{xx} + uu_x = x + \ln t, \tag{24}$$

with initial conditions $u(0,t) = \ln t,$ and $u_x(0,t) = 1.$ Let the initial guess of $u(x,t),$ be $u_0(x,t) = x + \ln t.$ This chosen guess is the exact solution given by Wazwaz in (Wazwaz, 2006). However, we could choose here $u_0(x,t) = xe^x + \ln t \cos x,$ or $u_0(x,t) = e^x - \cos x + \ln t.$ We define the nonlinear term as

$$N[v(x,t,p)] = v_{xx}(x,t;p) + v_x(x,t;p)v(x,t;p) - (x + \ln t). \tag{25}$$

Zero Order Deformation Equation

We construct a *Homotopy* which satisfies

$$(1-p)L[v(x,t;p) - u_0(x,t)] = \hbar H(x,t)\, p\, N[v(x,t;p)], \tag{26}$$

Subject to the $v(0,t;p) = \ln t,$ and $$v_x(0,t;p) = 1. \tag{27}$$

When p=0: The equation (26) becomes $L[v(x,t;0) - u_0(x,t)] = 0,$ subject to $v(0,t;0) = \ln t,$ and $v_t(0,t;0) = 1.$ Then the solution of equation 26 becomes $v(x,t;0) = u_0(x,t).$

When p=1: The equation (26) becomes $\hbar H(x,t)N[v(x,t,1)] = 0,$ or $N[v(x,t;1)] = 0,$ subject to the

$v(0,t;1)=\ln t,$ and $v_t(0,t;1)=1.$ Equations (26) and (27) are called the *Zero Order Deformation Equation*

Higher order deformation equation

For this problem we have $$R_m(\overrightarrow{u_{m-1}})=(u_{m-1})_{xx}+\sum_{j=0}^{m-1}u_j(u_{m-1-j})_x-(1-\chi_m)(x+\ln t). \qquad (28)$$

Now *first order deformation equation* is $u_1(x,t)=\chi_1\ u_0(x,t)+h\,H(x,t)L^{-1}[R_m(\overrightarrow{u_0})],$

or $u_1(x,t)=h\,H(x,t)L^{-1}[R_m(\overrightarrow{u_0})].$ And from (28), we have

$$R_m(\overrightarrow{u_0})=(u_0)_{xx}+u_0u_{0x}-(1-\chi_1)(x+\ln t),$$

$$R_m(\overrightarrow{u_0})=(x+\ln t)_{xx}+(x+\ln t)(x+\ln t)_x-(1-0)(x+\ln t),$$

$$R_m(\overrightarrow{u_0})=(x+\ln t)-(x+\ln t).$$

Therefore $R_m(\overrightarrow{u_0})=0.$ Now, $u_1(x,t)=h\,H(x,t)L^{-1}[0],\quad \Rightarrow \quad u_1(x,t)=0.$ And

$$u_2(x,t)=\chi_2\ u_1(x,t)+h\,H(x,t)L^{-1}[R_m(\overrightarrow{u_1})],$$

with $R_m(\overrightarrow{u_1})=(u_1)_{xx}+\sum_{j=0}^{1}u_j(u_{1-j})_x-(1-\chi_2)(x+\ln t),$

$$=(0)_{xx}+(u_0u_{1x}+u_1u_{0x})-(1-1)(x+\ln t).$$

So $R_m(\overrightarrow{u_0})=0,$ Hence $R_m(\overrightarrow{u_{m-1}})=0,\quad \forall\quad m\geq 1.$ So from (20) we have $u(x,t)=x+\ln t,$ is the exact solution.

2.2 Nonlinear Klein Gordon Equations

Example 1 $$u_{tt}-u_{xx}+u^2=6xt(x^2-t^2)+x^6t^6, \qquad (29)$$

with $u(x,0)=0,$ and $u_t(x,0)=0.$ Let initial guess of $u(x,t)$ be $u_0(x,t)=x^3t^3.$ This chosen guess was the exact solution given by Wazwaz in [1]. However, we could choose here $u_0(x,t)=x^nt^n,\quad \forall n\geq 2,$ or $u_0(x,t)=xt^n,\quad \forall n\geq 2,$ or $u_0(x,t)=t^n,\quad \forall n\geq 2,$ or $u_0(x,t)=t^n\sin t,\quad \forall n\geq 1.$ We define the nonlinear term as

$$N[v(x,t,p)]=v_{tt}(x,t;p)-v_{xx}(x,t;p)-v^2(x,t;p)-[6xt(x^2-t^2)+x^6t^6]. \qquad (30)$$

We construct a *Homotopy* which satisfies

$$(1-p)L[v(x,t;p)-u_0(x,t)]=\hbar H(x,t)\,p\,N[v(x,t;p)], \qquad (31)$$

subject to the initial conditions $v(x,0;p)=0,$ and

$$v_t(x,0;p)=0. \tag{32}$$

Equations (31) and (32) are called the *Zero Order Deformation Equation*.Then we define

$$R_n(\overrightarrow{u_{n-1}})=(u_{n-1})_{tt}-(u_{m-1})_{xx}+\sum_{j=0}^{n-1}u_j u_{n-1-j}-(1-\chi_n)[6xt(x^2-t^2)+x^6t^6],$$

Then *first order deformation equation* is $u_1(x,t)=\chi_1\ u_0(x,t)+h\ H(x,t)L^{-1}[R_n(\overrightarrow{u_0})].$

or $u_1(x,t)=h\ H(x,t)L^{-1}[R_n(\overrightarrow{u_0})].$ we get

$$R_n(\overrightarrow{u_0})=(u_0)_{tt}-(u_0)_{xx}+u_0u_0-(1-\chi_1)[6xt(x^2-t^2)+x^6t^6=6x^3t-6xt^3+x^6t^6-(1)[6x^3t-6xt^3+x^6t^6],$$

Hence $R_m(\overrightarrow{u_0})=0.$ So $R_m(\overrightarrow{u_{n-1}})=0,\quad \forall\quad n\geq 1,$ and $u(x,t)=x^3t^3,$ is the exact solution

Example 2 $$u_{tt}-u_{xx}+u^2=-x\cos t+x^2cos^2t, \tag{33}$$

with $u(x,0)=x,$ and $u_t(x,0)=0.$ Let $u_0(x,t)=x\cos t,$ be the initial guess of $u(x,t),$ which satisfies the initial conditions. However, we could choose here $u_0(x,t)=x(1-t^n\sin t),\quad \forall n\geq 1,$ or $u_0(x,t)=x(1-t^n\cos t),\ ;\forall n\geq 2,$ or $u_0(x,t)=x+t^ne^x,\ ;\forall n\geq 2.$ We define the nonlinear term as

$$N[v(x,t,p)]=v_{tt}(x,t;p)-v_{xx}(x,t;p)-v^2(x,t;p)-[-x\cos t+x^2\cos^2 t],$$

and construct a *Homotopy* as

$$(1-p)L[v(x,t;p)-u_0(x,t)]=\hbar H(x,t)p\ N[v(x,t;p)], \tag{34}$$

subject to $v(x,0;p)=x,$ and $v_t(x,0;p)=0.$ For this problem we have

$$R_n(\overrightarrow{u_{m-1}})=(u_{m-1})_{tt}-(u_{m-1})_{xx}+\sum_{j=0}^{m-1}u_j u_{m-1-j}-(1-\chi_m)[-x\cos t+x^2cos^2t].$$

which gives

$$R_n(\overrightarrow{u_0})=(u_0)_{tt}-(u_0)_{xx}+u_0u_0-(1-\chi_1)[-x\cos t+x^2\cos^2 t]=-x\cos t-0+x^2\cos^2+x\cos t-x^2\cos^2 t=0.$$

and $R_n(\vec{u}_{n-1})=0,\quad \forall\quad n\geq 1,$ implies that $u(x,t)=x\cos t,$ is the exact solution.

2.3 Lane-Emden equations

Example 1 $$u_{xx}+\frac{2}{x}u_x+u^3=6+x^6, \tag{35}$$

with $u(0)=0,$ and $u_x(0)=0.$ Let $u_0(x)=x^2$ be the initial guess of $u(x)$ which satisfies the initial conditions. However, we could choose here $u_0(x,t)=x^n,\quad \forall n\geq 2,$ or $u_0(x,t)=x^ne^x,\quad \forall n\geq 2,$

or $u_0(x,t) = x^n \sin x, \quad \forall n \geq 1,$ or $u_0(x,t) = x^n \cos x, \quad \forall n \geq 2.$ Let $p \in [0,1],$ being the *embedding parameter*. The *Homotopy analysis Method* being based on the continuous mapping $u(x) \to v(x;p),$ such that as p goes from 0 to 1, $v(x;p),$ varies from the initial guess $u_0(x),$ to the exact solution $u(x).$ We define the nonlinear term as

$$N[v(x;p)] = v_{xx}(x;p) + \frac{2}{x} v_x(x;p) + v^3(x;p) - (6 + x^6). \tag{36}$$

We construct a *homotopy,* $H[v(x;p), u_0, H, \hbar; p)],$ as $v(x;p) : \Omega \times [0,1] \to IR,$ where $x \in \Omega.$ This satisfies

$$H[v\,; u_0, H, \hbar; p] = (1-p)L[v(x;p) - u_0(x)] - \hbar H(x)p\, N[v(x;p)] = 0,$$

subject to the initial conditions $v(0;p) = 0,$ and $v_x(0;p) = 0.$

When p=0: we have $L[v(x;0) - u_0(x)] = 0,$ subject to $v(0;0) = 0,$ and $v_x(0;0) = 0.$ Then the solution of equation becomes $v(x;0) = u_0(x).$

When p=1: we have $N[v(x;1)] = 0,$ subject to $v(0;1) = 0,$ and $v_t(0;1) = 0.$ These equations are equivalent to the original equations provided $v(x;1) = u(x).$ We now define nth*-order deformation derivative* as

$$u_0^{(n)}(x) = \frac{\partial^n}{\partial p^n} v(x;p)\Big|_{p=0}, \tag{37}$$

and expanding $v(x;p)$ in a power series of p by Tailor's theorem to get, using (37), we have

$$v(x;p) = v(x;0) + \sum_{m=1}^{\infty} \frac{1}{m!} u_o^{(m)}(x) p^m \Big|_{p=0}. \tag{38}$$

which can be written as $v(x;p) = v(x;0) + \sum_{m=1}^{\infty} u_m(x) p^m.$ Under the assumptions, we now define a vector $\overrightarrow{u_n} = \{u_0(x), u_1(x), ..., u_n(x)\}.$ and then we have nth *order deformation equation.* defined as:

$$L[u_n(x) - \chi_n u_{n-1}(x)] = \hbar H(x) - R_m(\overrightarrow{u_{n-1}}), \tag{39}$$

where

$$R_n[\overrightarrow{u_{n-1}}] = \left[\frac{1}{(n-1)!} \frac{\partial^n}{\partial p^n} N\left[\sum_{n=0}^{\infty} u_n(x) p^n \right] \right]\Bigg|_{p=0} = 0,$$

For this problem we have

$$R_n(\overrightarrow{u_{n-1}}) = (u_{m-1})_{xx} + \frac{2}{x}(u_{m-1})_x + \sum_{j=0}^{m-1} u_j \sum_{r=0}^{m-1-j} u_r u_{m-1-j-r} - (1-\chi_m)(6+x^6). \quad (40)$$

From equation (39), we have

$$u_n(x) = \chi_n \; u_{n-1}(x) + h\, H(x) L^{-1}[R_m(\overrightarrow{u_{n-1}})]. \quad (41)$$

where

$$R_m(\overrightarrow{u_{m-1}}) = (u_0)_{xx} + \frac{2}{x}(u_0)_x + u_0u_0u_0 - (1-\chi_1)(6+x^6) = (x^2)_{xx} + \frac{2}{x}(x^2)_x + u_0^3 - (1-0)(6+x^6) = 0$$

So $R_m(\overrightarrow{u_{n-1}}) = 0, \quad \forall \quad n \geq 1,$ implies $u(x) = x^2,$ is the exact solution.

Example 2
$$u_{xx} + \frac{4}{x}u_x + u^2 = 4 + 18x + 4x^3 + x^6, \quad (42)$$

with $u(0) = 2,$ and $u_x(0) = 0.$ Let $u_0(x) = 2 + x^3,$ be the initial guess, however, we could choose here $u_0(x,t) = 2 + x^n, \; ; \forall n \geq 2,$ or $u_0(x,t) = 2 + x^n \sin x \; ; \forall n \geq 1,$ or $u_0(x,t) = 2 + x^n \cos x \; ; \forall n \geq 2.$ We define the nonlinear term as

$$N[v(x;p)] = v_{xx}(x;p) + \frac{4}{x}v_x(x;p) + v^2(x;p) - (4 + 18x + 4x^3 + x^6). \quad (43)$$

and construct a *homotopy* which satisfies,

$$(1-p)L[v(x;p) - u_0(x)] = \hbar H(x)\, p\, N[v(x;p)], \quad (44)$$

subject to $v(0;p) = 2,$ and $v_x(0;p) = 0.$ Then we have

$$R_n(\overrightarrow{u_{n-1}}) = (u_{n-1})_{xx} + \frac{4}{x}(u_{n-1})_x + \sum_{j=0}^{n-1} u_j u_{n-1-j} - (1-\chi_n)(4 + 18x + 4x^3 + x^6).$$

Now, *first order deformation equation* is

$$u_1(x) = \chi_1 \; u_0(x) + h\, H(x) L^{-1}[R_n(\overrightarrow{u_{n-1}})],$$

where
$$R_n(\overrightarrow{u_0}) = (u_0)_{xx} + \frac{4}{x}(u_0)_x + u_0u_0 - (1-\chi_1)(4 + 18x + 4x^3 + x^6)$$
$$= (4 + 18x + 4x^3 + x^6) - (4 + 18x + 4x^3 + x^6) = 0.$$

Thus $R_n(\overrightarrow{u_0}) = 0.$ Hence $R_n(\overrightarrow{u_0}) = 0, \quad \forall \quad n \geq 1.$ So, $u(x) = 2 + x^3,$ is the exact solution.

3 Concluding Remarks

We have made a comparison of the ADM and HPM on the same examples in (wahab, 2015) and observe here the solutions obtained by HAM are more appropriate than that given by *Adomian Decomposition Method* (Adomian. 1994). As pointed out by Liao in (Liao. 2004), that the HAM logically contains *Adomian Decomposition Method*, if we use the same linear operator as in *Adomian Decomposition Method*, the initial guess chosen in HAM is the same as the first component of the series solution of the *Adomian Decomposition Method*. Since we have seen that in our work, we have chosen the same initial guess as in *Adomian Decomposition Method* and get the exact solution after the first iteration without any need of the so the auxiliary function and the auxiliary parameter. However, they have their own advantages. Then this leads us to the exact solution of the differential equations. So we can say that before applying the HAM we must either have ``prior knowledge" about the solution of the governing problem or we should use *Adomian Decomposition Method* get the best form of the initial guess, if we are going to use the same linear operator as in *Adomian Decomposition Method*, as pointed out by Liao (2004).

On the other hand, we have made guess to choose the initial guess for the series solution. However, *Homotopy Analysis Method* has some advantages over *Adomian Decomposition Method* by various examples as shown by many authors. We have chosen the same initial guesses for the zero order deformation equation as was made in Modified technique (Wazwaz, 2006). Due to its advantages over the *Standard Adomian Method*, we shall decide that the *Homotopy Analysis Method* is more appropriated than the *Adomian Decomposition Method*. Moreover we could choose different base functions as initial guesses instead of the pre obtained solutions to approximate solution of the give problem. In this way we could obtain different series solutions of the nonlinear problems.

Moreover, the comparison between the two techniques has been established in various works (for example see (Wahab et al, 2013; Wahab et al, 2015, Wahab, 2016), and shown that the homotopy analysis method is a powerful technique than homotopy perturbation method, as we have done in Wahab (2015). The difference is clear just as the Tailor series method is different from the perturbation method. The HAM uses the Homotopy parameter, , to obtain a Tailor series While the HPM applies the Homotopy parameter as the embedding parameter If we stop at this step, the HAM is equivalent to the HPM (He, 2003). However, the later can so successfully apply the knowledge of the various perturbation methods, that the low-order approximate solution leads to high accuracy there require no infinite series as the former does. Liao in Liao (2004) showed that HAM is only the special case of the HAM. Both methods are in principle based on Taylor series with respect to an embedding parameter. Besides, both can give very good approximations by means of a few terms, if the initial guess and the auxiliary linear operator are good enough. The difference is that HPM had to use a good enough initial guess as we have done in our work, that the initial guess is chosen satisfying the initial conditions, but it is not absolutely necessary for the HAM. This is because the HAM contains the auxiliary parameter, , which provides us with a simple way to adjust and control the convergence region and rate of solution series. However in our work, we have not used the advantages of this auxiliary parameter, and the auxiliary function. So the HAM is more general than HPM.

Finally, we say conclude that, since modified *Adomian decomposition method* (Wazwaz, 2006), depends upon the two components of the series solutions of *Adomian Decomposition Method* and there are many features discussed in this work. Also HPM is a special case of HAM and HAM logically contains *Adomian Decomposition Method*. We have found the HAM best analytical technique among these as we have discussed in our work. But before applying the HAM, we must be able to choose the best initial guess, which may be obtained by applying the *Adomian Decomposition Method* to get the first iteration. Therefore, it is reasonable to apply ADM before applying the HAM to obtain the initial guess.

References

Adomian G. 1994. Solving Frontier Problems of Physics: The Decomposition Method. Kluwer Academic Publishers. Boston, USA

He JH. 2003. Homotopy perturbation method; a new nonlinear analytic technique. Applied Mathematics and Computation, 135: 73-79

He JH. 1999. Homotopy perturbation technique. Computer Methods in Applied Mechanics and Engineering, 178: 257-262

Liao SJ. 1995. An approximate solution technique not depending upon small parameters: A special example. 1995. International Journal of Non-linear Mechanics, 30: 371-380

Liao SJ. 2004. Beyond Perturbation: Introduction to The Homotopy Analysis Method. Chapman and Hall CRC Press, Boca Raton, USA

Wazwaz AM. 2006. The decomposition method for analytic treatment of differential equations. Applied Mathematics and Computation, 173: 165-176

Wahab HA, Hussain S, Bhatti S, Naeem M. 2016. Mixed convection flow of Powell-Eyring fluid over a stretching cylinder with Newtonian heating. Kuwait Journal of Science, 43(3) (Accepted)

Wahab HA, 2006. Comparison of Perturbation Techniques for Nonlinear Problems, Hafiz Abdul Wahab, Analytical Techniques for Nonlinear Problems. M. Phil. Thesis, Department of Mathematics, Quaid-i-Azam University, Islamabad, Pakistan

Wahab, HA, Bhatti S, Naeem M. 2015. A comparison of perturbation techniques for nonlinear problems, Journal of Prime Research in Mathematics, 10: 59-79

Wahab HA, Bhatti S, Saifullah K, Naeem M, Hussain S. 2015. A study of the flow of non-Newtonian fluid between heated parallel plates by HAM. Journal of Prime Research in Mathematics, 11: 93-105

Wahab HA, Shakil M, Khan T, Bhatti S, Naeem M. 2013. A comparative study of a system of Lotka-Voltera type of PDEs through perturbation methods. Computational Ecology and Software, 3(4): 110-125