Article

An approximate solution for a generalized Hirota-Satsom coupled (Kdv) equation

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Abstract

In this paper the Homotopy Analysis Method (HAM), is applied to find the approximate solution of Hirota-Satsuma coupled (KdV) equations, which don't need a small parameter for solution. The results obtained by HAM is compared with exact solution, the results divulge that the Homotopy Analysis Method are most accurate, closed and suitable to exact solution of the equation, as compare to Homotopy Perturbation Method. It is predicated that the HAM can be found usually.

Keywords Homotopy Analysis Method (HAM); non-linear systems of KdV equations.

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1 Introduction

It is more complicated to solve the non-linear equation as compare to others equation, the Homotopy Analysis Method is the best logical method for solving non-linear equations. The Homotopy Analysis Method represents the basic idea of Homotopy to convergence series solution are construct for a non-linear equation from topology (Taiwo, 2012). Homotopy Analysis Method is the best logical method for solving non-linear equations. First time in 1992 Liao Shijun presents the Homotopy Analysis Method. Further Liao modified in 1997 a non-zero auxiliary parameter, to generate a Homotopy on a differential formation in genral form. In different methodology the Homotopy Analysis Method (HAM) was applied by different authors (Ayub et. al., 2003; Hassan et. al., 2013; Kazemian et. al., 2013), to find the solution of differential nonlinear equations in the area of science and mathematics. For Adomian decomposition method the Homotopy Analysis technique is a united method, artificial small parameter and Homotopy Perturbation Method (Sajid et. al., 2008), HAM provide a simple way to generate the convergence of series solution. The Homotopy Perturbation Method are frequently non applicable in case of strong non linearity, but HAM is applicable in case of non-linearity (Liao, 2003).

The HAM used the Homotopy parameter on a theoretical level to show that a non-linear deformation may be break into countless set of linear system which is solve analytically. The Homotopy Analysis Method was useful to solve an increasing number of non-linear differential equation in finance, engineering and science. i.e several steady state thriving wave in trivial and limited water depth.

The main purpose of this paper is to find the approximate solution of non-linear Coupled (KdV) equation by the Homotopy Analysis Method.

2 Basic Idea of Homotopy Analysis Method

The Homotopy Analysis Method is a basic and straightforward technique. HAM was accessible by mean of Homotopy (Hilton, 1953; Liao, 2003)., an elementary idea of topology. Furthermore for necessary idea of HAM, assume the following differential equation.

$$\aleph\left[\mu\left(x,t\right)\right]=0 \tag{1}$$

In equation (1), $\aleph$ is a non-linear operator, independent variables are x and t and the unknown function is $\mu\left(x,t\right)$. The generalization of the conventional Homotopy Method (Liao, 2003) construct a zero-order deformation equation the new type of Homotopy, which is the following,

$$\left(1-\rho\right)L\left[\Upsilon\left(x,t;\rho\right)-\mu_0\left(x,t\right)\right]=\rho\hbar H\left(x,t\right)\aleph\left[\Upsilon\left(x,t;\rho\right)\right]. \tag{2}$$

The Homotopy Analysis Method is the base of continuous mapping $\mu\left(x,t\right)\rightarrow\Upsilon\left(x,t;\rho\right)$, $\Upsilon\left(x,t;\rho\right)$ be a function, that is unknown, $\mu_0\left(x,t\right)$ be the initial approximation for the particular unknown function $\mu\left(x,t\right)$, the embedding parameter is ρ, $H\left(x,t\right)\neq 0$ and $\hbar\neq 0$ are the auxiliary function and auxiliary parameter respectively, L be the operator said linear auxiliary operator. It is important that in HAM we easily chose the auxiliary parameter (Hemeda, 2012), obviously, for embedding parameter $\rho=0$ and ρ=1 the equation (2), becomes:

$$\begin{cases}\Upsilon\left(x,t;0\right)=\mu_0\left(x,t\right),\\ \Upsilon\left(x,t;1\right)=\mu\left(x,t\right).\end{cases} \tag{3}$$

Since the changing of embedding parameter ρ change the solution $\Upsilon\left(x,t;\rho\right)$ from initial gasses to exact solution, this variation is said deformation in method of topology (Liao, 2003).

In the base of AHM, expending Υ, with respect to ρ in the form of power series such as,

$$\Upsilon\left(x,t;\rho\right)=\ \mu_0+\rho\mu_1+\rho^2\mu_2+... \tag{4}$$

$$\left(x,t;\rho\right)=\mu_0\left(x,t\right)+\sum_{m=1}^{\infty}\mu_m\rho^m. \tag{5}$$

where,

$$\mu_m\left(x,t\right)=\frac{1}{m!}\frac{\partial^m \aleph\left(x,t;\rho\right)}{\partial\rho^m}$$

Choosing the parameter $\hbar$, the auxiliary function $H\left(x,t\right)$, linear operator L and initial approximation $\mu_0\left(x,t\right)$ are properly, then at $\rho=1$ series (5) are converges such that,

$$\left(x,t;1\right)=\mu_0\left(x,t\right)+\sum_{m=1}^{\infty}\mu_m. \tag{6}$$

Which is necessity one of the solutions of original non-linear equation, by Liao, 2009.
Now the mth-order deformation equation of Homotopy is the following,

$$L\left[\mu_m-\chi_m\mu_{m-1}\right]=\hbar H\left(x,t\right)\mathrm{K}_m\left(\overrightarrow{\mu_{m-1}}\right). \tag{7}$$

In equation (7),

$$\chi_m=\begin{cases}0 & m\le 1,\\ 1 & \text{otherwise.}\end{cases}\quad \text{and,}\qquad \mathrm{K}_m\left(\overrightarrow{\mu_{m-1}}\right)=\frac{1}{\left(m-1\right)!}\left.\frac{\partial^{m-1}\left[\aleph\left[\mu\left(x,t;\rho\right)\right]\right]}{\partial\rho^{m-1}}\right|_{\rho=0} \tag{8}$$

The generalized Homotopy not depend only on ρ, it moreover dependent on the auxiliary parameter $\hbar$ and auxiliary function $H\left(x,t\right)$. Thus the family of approximation series which given by generalized Homotopy whose convergence regain is depend on $H\left(x,t\right)$ and $\hbar$. For controlling the rate of approximation and region of convergence the straightforward method provide by generalized Homotopy (Liao, 2003).

3 Approximate Solutions of Non-Linear Coupled (kdv) Equation by HAM

Consider the following nonlinear system of KdV equations (Ganji, 2006),

$$u_t=\frac{1}{2}\frac{\partial^3 u}{\partial x^3}-3u\frac{\partial u}{\partial x}+3\frac{\partial}{\partial x}\left(wv\right), \tag{9}$$

$$v_t=-\frac{\partial^3 v}{\partial x^3}+3u\frac{\partial v}{\partial x}, \tag{10}$$

$$w_t=-\frac{\partial^3 w}{\partial x^3}+3u\frac{\partial w}{\partial x}. \tag{11}$$

To solve the above system of equations by HAM, we describe the operator $L=\frac{\partial}{\partial x}$ and $L^{-1}=\int_0^t()dt.$ Now the deformation equation for the above non-linear system of equations as,

$$(1-\rho)L\left[\Upsilon_1(x,t;\rho)-u_0\right]=\rho\hbar H(x,t)\aleph\left[\Upsilon_1(x,t;\rho)\right], \tag{12}$$

$$(1-\rho)L\left[\Upsilon_2(x,t;\rho)-v_0\right]=\rho\hbar H(x,t)\aleph\left[\Upsilon_2(x,t;\rho)\right], \tag{13}$$

$$(1-\rho)L\left[\Upsilon_3(x,t;\rho)-w_0\right]=\rho\hbar H(x,t)\aleph\left[\Upsilon_3(x,t;\rho)\right], \tag{14}$$

Where u_0, v_0 and w_0 be the initial guess (Ganji, 2004), such as,

$$u_0=\frac{1}{3}\left(\beta-8\kappa^2\right)+4\kappa^2\tanh^2(\kappa x), \tag{15}$$

$$v_0=-\frac{4\kappa^2\ 3\kappa^2 a_0-2\beta a_2+4\kappa^2 a_2}{3a_2^2}+\frac{4\kappa^2}{a_2}\tanh^2\ \kappa x\ , \tag{16}$$

$$w_0=a_0+a_2\tanh^2\ \kappa x\ . \tag{17}$$

Since ρ is embedding parameter whose value is changed from 0 to 1, thus for $\rho=0$ and 1 the above deformation equation give us.

$$\Upsilon_1(x,t;0)=u_0,\ \Upsilon_2(x,t;0)=v_0 \text{ and } \Upsilon_3(x,t;0)=w_0. \tag{18}$$

and,

$$\Upsilon_1(x,t;1)=u,\ \Upsilon_2(x,t;1)=v \text{ and } \Upsilon_3(x,t;1)=w. \tag{19}$$

The solution can be expressed of the above system of equation in the power of embedding parameter ρ such that,

$$\aleph_1(x,t;\rho)=u_0+\sum_{m=1}^{+\infty}u_m\rho^m, \tag{20}$$

$$\aleph_2(x,t;\rho)=v_0+\sum_{m=1}^{+\infty}v_m\rho^m, \tag{21}$$

$$\aleph_3(x,t;\rho)=w_0+\sum_{m=1}^{+\infty}w_m\rho^m. \tag{22}$$

In equation (20)-(22),

$u_m(x,t)=\frac{1}{m!}\frac{\partial^m\aleph_1(x,t;\rho)}{\partial\rho^m}$, $\quad v_m(x,t)=\frac{1}{m!}\frac{\partial^m\aleph_2(x,t;\rho)}{\partial\rho^m}$ and

$w_m(x,t)=\frac{1}{m!}\frac{\partial^m\aleph_3(x,t;\rho)}{\partial\rho^m}$. Now for $\rho=1$ the above system is converges which gives the following form,

$$u = u_0 + \sum_{m=1}^{+\infty} u_m \ , \ v = v_0 + \sum_{m=1}^{+\infty} v_m \text{ and } w = w_0 + \sum_{m=1}^{+\infty} w_m \ . \tag{23}$$

According to HAM, the following are the m[th]-order deformation equation for the given system,

$$L\left[u_m - \chi_m u_{m-1}\right] = \hbar H\left(x,t\right) \mathrm{K}_m\left(\overrightarrow{u_{m-1}}\right), \tag{24}$$

$$L\left[v_m - \chi_m v_{m-1}\right] = \hbar H\left(x,t\right) \mathrm{K}_m\left(\overrightarrow{v_{m-1}}\right), \tag{25}$$

$$L\left[w_m - \chi_m w_{m-1}\right] = \hbar H\left(x,t\right) \mathrm{K}_m\left(\overrightarrow{w_{m-1}}\right). \tag{26}$$

Applying $L^{-1} = \int_0^t (\ )\,dt$ on equation (24)-(26) to both sides and setting $H\left(x,t\right) = 1$ we get,

$$u_m = \chi_m u_{m-1} + \hbar \int_0^t \mathrm{K}_m\left(\overrightarrow{u_{m-1}}\right) dt, \tag{27}$$

$$v_m = \chi_m v_{m-1} + \hbar \int_0^t \mathrm{K}_m\left(\overrightarrow{v_{m-1}}\right) dt, \tag{28}$$

$$w_m = \chi_m w_{m-1} + \hbar \int_0^t \mathrm{K}_m\left(\overrightarrow{w_{m-1}}\right) dt. \tag{29}$$

For $m = 1$ the equation (27)-(29), becomes,

$$u_1 = \hbar \int_0^t \mathrm{K}_1(\overrightarrow{u_0}) dt, \ v_1 = \hbar \int_0^t \mathrm{K}_2(\overrightarrow{v_0}) dt \text{ and } w_1 = \hbar \int_0^t \mathrm{K}_3(\overrightarrow{w_0}) dt. \tag{30}$$

Now using the initial value of (15)-(17), in equation (30) we get,

$$\begin{aligned} u_1 = h\Bigg[& 32\kappa^5 \left(1 - \left(\tanh\left(\kappa x\right)\right)^2\right)^2 \tanh\left(\kappa x\right) \mathrm{t} - 16\kappa^5 \left(\tanh\left(\kappa x\right)\right)^3 \left(1 - \left(\tanh\left(\kappa x\right)\right)^2\right) \mathrm{t} \\ & + 24\left(-\frac{8}{3}\kappa^2 + \frac{1}{3}\beta + 4\kappa^2 \left(\tanh\left(\kappa x\right)\right)^2\right) \kappa^3 \tanh\left(\kappa x\right) \left(1 - \left(\tanh\left(\kappa x\right)\right)^2\right) \mathrm{t} \\ & - 6\left(-\frac{4}{3}\frac{\kappa^2\left(4a_2\kappa^2 + 3a_o\kappa^2 - 2\beta a_2\right)}{a_2} + 4\frac{\kappa^2\left(\tanh\left(\kappa x\right)\right)^2}{a_2}\right) a_2 \tanh\left(\kappa x\right) \\ & \left(1 - \left(\tanh\left(\kappa x\right)\right)^2\right) \kappa \mathrm{t} - 24\frac{\left(a_o + a_2\left(\tanh\left(\kappa x\right)\right)^2\right) \kappa^3 \tanh\left(\kappa x\right) \left(1 - \left(\tanh\left(\kappa x\right)\right)^2\right) \mathrm{t}}{a_2} \Bigg]. \end{aligned} \tag{31}$$

$$\begin{aligned} v_1 = h\Bigg[& \frac{24}{a_2}\left(-\frac{8}{3}\kappa^2 + \frac{1}{3}\beta + 4\kappa^2 \tanh\left(\kappa x\right)^2\right) \kappa^3 \tanh\left(\kappa x\right) \left(1 - \tanh\left(\kappa x\right)^2\right) \mathrm{t} \\ & + \frac{64}{a_2}\left(1 - \tanh\left(\kappa x\right)^2\right)^2 \tanh\left(\kappa x\right) \kappa^5 t - \frac{32}{a_2}\kappa^5 \tanh\left(\kappa x\right)^3 \left(1 - \tanh\left(\kappa x\right)^2\right) \mathrm{t} \Bigg]. \end{aligned} \tag{32}$$

$$w_1 = h\left[-6\left(-\frac{8}{3}\kappa^2 + \frac{1}{3}\beta + 4\kappa^2 \tanh(\kappa x)^2\right) a_2 \tanh(\kappa x)\left(1 - \tanh(\kappa x)^2\right)\kappa t \right.$$
$$\left. -16a_2\left(1 - \tanh(\kappa x)^2\right)^2 \kappa^3 \tanh(\kappa x) t + 8a_2 \tanh(\kappa x)^3 \left(1 - \tanh(\kappa x)^2\right)\kappa^3 t\right]. \quad (33)$$

Similarly we can find $u_2,\ v_2,\ w_2$ and so on. using $m = 2,3,...$ in equation (27)-(29),

to obtaining the approximate solution by Homotopy Analysis Method for the above system of KdV equations such as,

$$u(x,t) = u_0(x,t) + u_1(x,t) + ... \quad (34)$$

$$v(x,t) = v_0(x,t) + v_1(x,t) + ... \quad (35)$$

$$w(x,t) = w_0(x,t) + w_1(x,t) + ... \quad (36)$$

Now through the approximation $u_1,\ v_1,\ w_1$ we also fined the 2^{nd} component as, $u_2,\ v_2,\ w_2$ which is obtained by Maple, from equation (34)-(36), find the solution with HAM.

4 Comparison of Ham Result with Exact Solutions

The accuracy of Homotopy analysis Method for the given Hirota-Satsuma coupled KdV is convenient, and the absolute errors are extremely small for choosing the different values for $x,\ t$. The following Table 1; show that the HAM achieves the minimum accuracy of six and maximum accuracy of eleven significant figures (Figs 1-6) for equations (9)-(11), for the first three approximations.

Table 1 The HAM result for $u(x,t),\ v(x,t)$ and $w(x,t)$, comparing with the exact solution when, $\kappa = 0.1,\ \beta=1.5,\ a_2=1.5,\ a_0=1.5$ and $h = -1$ for solitary wave solution with initial gausses (15)-(17) of equations (9)-(11) respectively.

(x,t)	$\lvert u_{exact} - u_{HAM}\rvert$	$\lvert v_{exact} - v_{HAM}\rvert$	$\lvert w_{exact} - w_{HAM}\rvert$
(0.1,0.1)	2.37543×10^{-6}	1.37041×10^{-10}	7.18763×10^{-9}
(0.1,0.2)	4.3205×10^{-5}	1.62076×10^{-8}	1.02353×10^{-7}
(0.1,0.3)	7.21061×10^{-6}	6.46572×10^{-9}	3.56727×10^{-6}
(0.2,0.1)	4.76252×10^{-7}	2.33742×10^{-10}	1.34572×10^{-8}
(0.2,0.2)	8.78542×10^{-6}	2.14321×10^{-8}	1.31370×10^{-7}
(0.2,0.3)	1.43514×10^{-5}	1.00210×10^{-8}	5.42523×10^{-6}
(0.3,0.1)	7.23120×10^{-6}	3.35673×10^{-10}	2.00432×10^{-8}
(0.3,0.2)	1.24387×10^{-6}	3.02341×10^{-8}	1.43256×10^{-7}
(0.3,0.3)	2.2503×10^{-6}	1.16353×10^{-8}	6.45723×10^{-6}

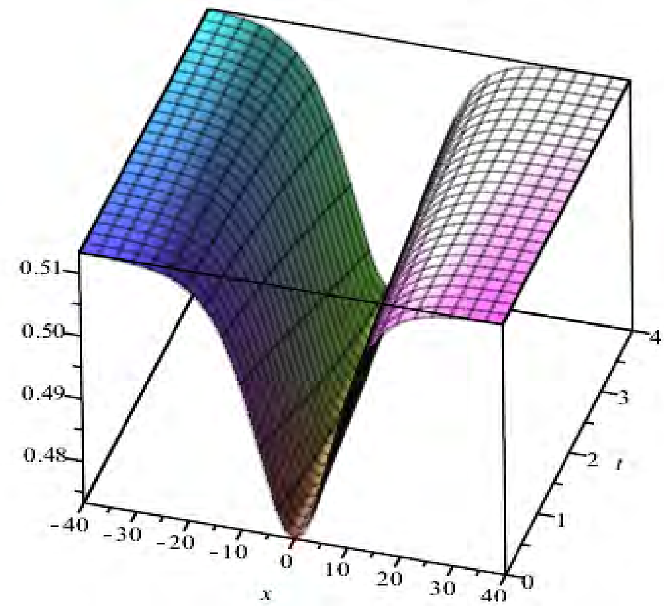

Fig. 1 The approximate result of HAM for $u(x,t)$ when $\kappa = 0.1$, β=1.5, a_2=1.5, a_0=1.5 and $h = -1$.

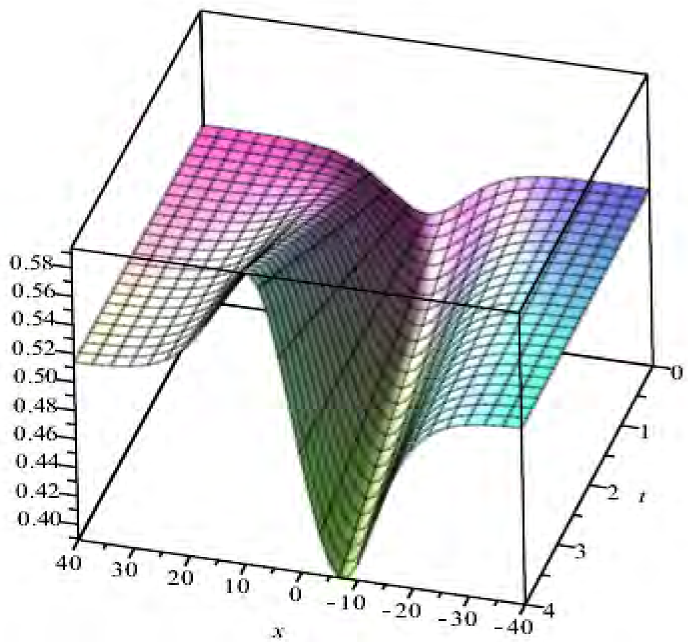

Fig. 2 The exact for $u(x,t)$ when $\kappa = 0.1$, β=1.5, a_2=1.5, a_0=1.5 and $h = -1$.

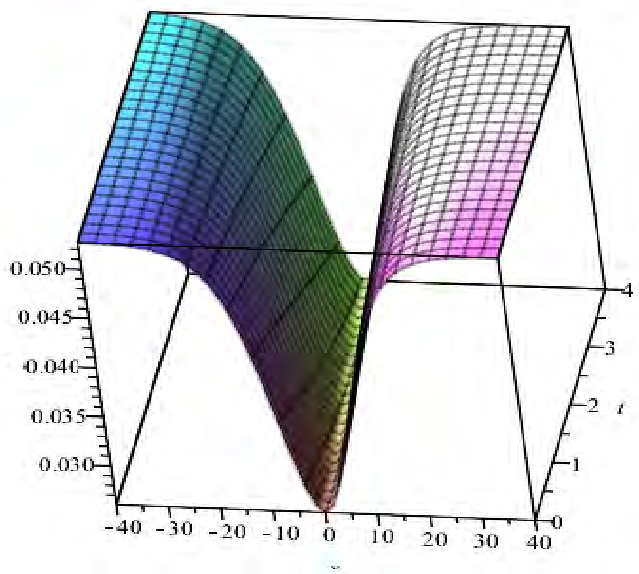

Fig. 3 The exact for $u(x,t)$ when $\kappa = 0.1$, β=1.5, a_2=1.5, a_0=1.5 and $h = -1$.

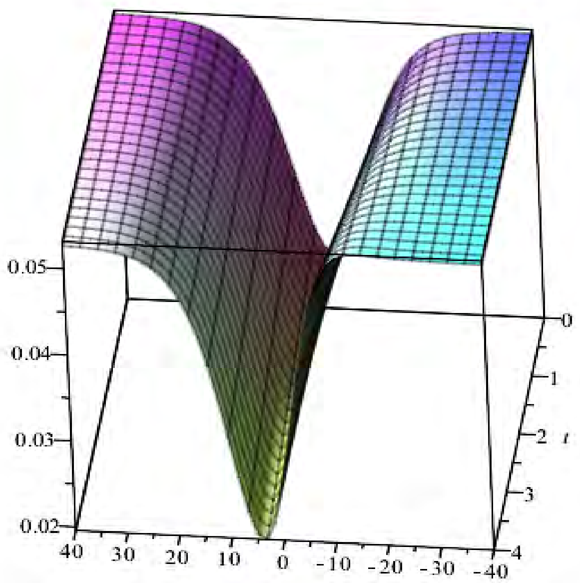

Fig. 4 The approximate for $v(x,t)$ when $\kappa = 0.1$, β=1.5, a_2=1.5, a_0=1.5 and $h = -1$.

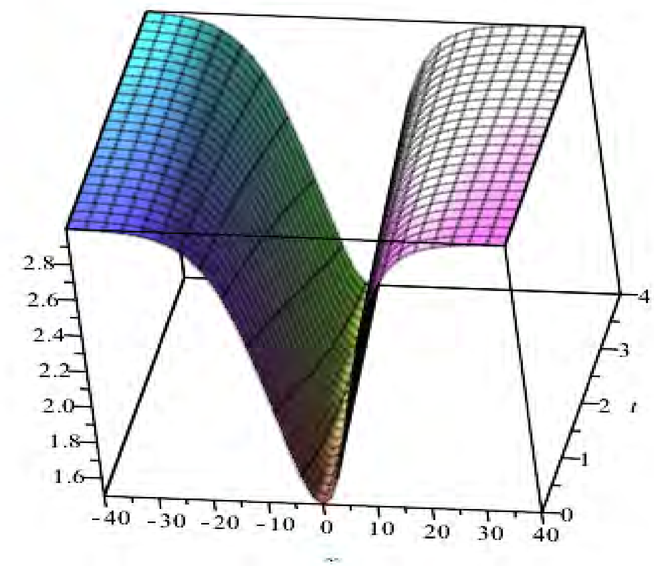

Fig. 5 The approximate for $w(x,t)$ when $\kappa = 0.1$, β=1.5, a_2=1.5, a_0=1.5 and $h = -1$.

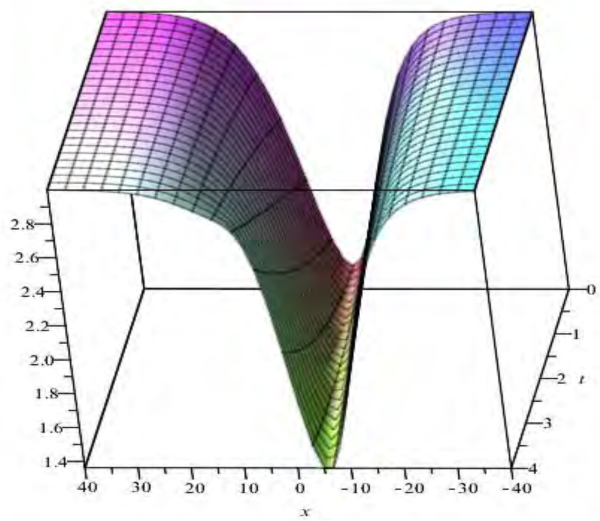

Fig. 6 The approximate for $w(x,t)$ when $\kappa = 0.1$, β=1.5, a_2=1.5, a_0=1.5 and $h = -1$.

5 Conclusions

In this paper, the Homotopy Analysis Method was applied for obtaining the soliton solution of Korteweg-de Vries (KdV) equation with initial guesses. We have concluded that the HAM is a powerful and resourceful technique to find the approximate solutions used for large classes of problems. It is point out the HAM present a fast convergence for the general solutions. The calculated results show that the HAM are outstanding technique with those which obtained by Homotopy Perturbation Method. The HAM has many virtues and more advantage then the Homotopy Perturbation Method. For solving non-linear equations the HAM do not depend upon a small parameter or large parameter, so the Homotopy Analysis Method is very simple and Straight-forward method.

Also the result demonstrates that the HAM is dominant mathematical tool to solve the non-linear system of differential equations have a extensive application in mathematics and engineering.

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