Analysis of complex wetland ecological system: Effect of harvesting

Nilesh Kumar Thakur, Rashi Gupta
Department of Mathematics, National Institute of Technology, Raipur (C.G.), India
Email: nkthakur.maths@nitrr.ac.in, rashigupta1234@gmail.com

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Abstract
In this paper, we have studied interaction among diffusive phytoplankton, zooplankton and fish population with Beddington-DeAngelis type functional response for the zooplankton and Holling type III for fish. The stability analysis of the model system with diffusion and without diffusion has been analyzed. The conditions for Maximum sustainable yield and Optimal harvesting policy for non-spatial model have been discussed. Our study may be helpful to improve and manage ecosystem services provided by wetlands on an agricultural landscapes include fisheries, water conservation, climate change and many more.

Keywords nonlinear harvesting; optimal harvesting; Turing instability.

1 Introduction
Wetlands are one of the world’s most important environmental assets, existing in all continents and latitudes. Wetlands are home of a large biota diversity and yield significant economic, social and cultural benefits related to timber, fisheries, hunting, recreational and tourist activities, etc. In general they provide a wide array of useful and appreciated ecosystem services related to water quality preservation, erosion shore protection from wave action, nurseries for fish and other freshwater and marine animals.

Modelling of harvesting is an important notion and an interesting topic of mathematical bio-economics in aquatic system. Harvesting is basically of three types: (i) constant harvesting reveals that the constant number of individuals is harvested per unit time. (ii) Linear harvesting \( h(X) = qEX \) shows that the number of individuals harvested per unit time is proportional to the current population. (iii) Non-linear harvesting \( h(X) = qEX / (m_1E + m_2X) \), where \( q \) denotes the catch ability coefficient and \( E \) denotes the attempt applied to harvest the individuals and \( m_1, m_2 \) are suitable positive constants. Optimal control theory is to obtain the increased application in both theoretical and applied ecology. The optimal harvesting policy in their model has been discussed by many authors (Kar and Chaudhuri, 2004; Kar et al. 2009; Dubey and Hussain, 2000). The optimal harvesting policy in the model of fishery resource with reserve area has been discussed by Dubey et al.
(2003). Recently, Dubey et al. (2014) proposed a model to investigate the dynamical interaction among phytoplankton-zooplankton-fish population. A bio-economic model of non-selective harvesting of two competing fish species has been proposed by Purohit and Chaudhuri (2004) and studies the optimal harvesting policy. In this model, they examine non-linear harvesting in both competing species.

The wave phenomenon, non-linear and non-equilibrium pattern formation in a phytoplankton-zooplankton system with Holling type IV functional response has been investigated by Upadhyay et al. (2008). Upadhyay et al. (2011) proposed a reaction-diffusion model to study the dynamics of phytoplankton-zooplankton system in which the consumption of phytoplankton species follows the Beddington-DeAngelis functional response. A population model with diffusion, strong Allee effect and constant yield of harvesting has been investigated by Ali et al. (2009). Spatio-temporal complexity of a three-species ratio-dependent food chain model has been proposed by Rao (2014). Chang and Wei (2012) explore the dynamics of a diffusive delayed predator-prey system with Holling type II functional response and discussed the optimal control problem. Recently, complex dynamics of spatial predator-prey system under non-linear harvesting has been studied by Upadhyay et al. (2015) and interpret how Turing patterns develop gradually under non-linear harvesting. DeAngelis et al. (2010) investigated a model of reaction-diffusion to interpret the fish population dynamics in a seasonally varying wetland of Florida Everglades, USA.

In this paper, we deal with complex dynamical behaviour of phytoplankton-zooplankton model with non-linear harvesting. The paper is organized as follows. In section 2, we present a non-spatial model and discuss the stability analysis, maximum sustainable yield and optimal harvesting policy. In section 3, we propose the spatial model and in section 4, we have discussed the conclusions of present study.

2 The Proposed Model System

In this section, the system of differential equations in which interaction among phytoplankton, zooplankton and fish population is shown as follows:

\[
\frac{du}{dt} = ru(1-u) - \frac{\theta uv}{1 + \xi v + \eta u} = ug_1(u,v,w) = f_1(u,v,w),
\]

\[
\frac{dv}{dt} = \frac{\theta uv}{1 + \xi v + \eta u} - cv - \frac{wv^2}{v^2 + \gamma_1^2} = vg_2(u,v,w) = f_2(u,v,w),
\]

\[
\frac{dw}{dt} = r_ww\left(1 - \frac{w}{K}\right) + \frac{\theta^* wv^2}{v^2 + \gamma_1^2} - \frac{qew}{m_1e + m_2w} = wg_3(u,v,w) = f_3(u,v,w),
\]

with

\[
 u(0) > 0, v(0) > 0, w(0) > 0.
\]

In the proposed model (1), \( r \) is the intrinsic rate of growth of phytoplankton in the absence of predation. \( \theta \) is the rate at which phytoplankton is grazed with Holling type II functional response. \( \xi \) is the intensity of interference between individuals of zooplankton. \( \eta \) determines how fast the per capita feeding rate approaches to its saturation value. \( \theta_1 \) is the conversion coefficient from individuals of phytoplankton into individuals of zooplankton. \( c \) is the rate of mortality of zooplankton. \( \gamma_1 \) is zooplankton density at which the specific growth rate becomes half of its saturated value. \( \theta^* \) and \( K \) are rate of growth and carrying capacity of fish population respectively. \( \theta^* \) is the conversion coefficient of zooplankton. \( q \) is the catch ability coefficient and \( e \) is catch per unit effort. \( m_1 \) and \( m_2 \) are the positive constants.

2.1 Temporal model system

Now, the equilibrium analysis of the model system (1) has been discussed. The system (1) has six non-negative equilibrium points:
In the equilibrium point \( L_2(0,0,\dot{w}) \), the equation whose root is \( \dot{w} \) is as follows:

\[
r r_m \dot{w} + (r_m e - r_m K)\dot{w} + (q - m_r r)eK = 0. \tag{3}
\]

The equation (3) contain a positive real root \( w \) if \( q < m_r r \).

The equilibrium point \( L_3(\bar{u},\bar{v},0) \) exists if the following equations has the positive solutions \( \bar{u} \) and \( \bar{v} \):

\[
(1-r\theta\varphi)(1-r\theta\varphi) = 0, \tag{4}
\]

\[
(1-r\theta\varphi)\bar{v} = 0. \tag{5}
\]

From Eq. (5), we obtain

\[
u = \frac{c(1+\xi\varphi)}{\theta_1-c\eta}. \tag{6}
\]

Clearly, \( u > 0 \) if \( \theta_1 > c\eta \).

From Eq. (4), we obtain

\[
\nu = \frac{r(1-u)(1+\eta u)}{\theta-\xi r(1-u)}. \tag{7}
\]

The existence of interior equilibrium point \( L_4(u',v',w') \). In this case, \( u' \), \( v' \) and \( w' \) are the positive solutions of the following equations.

\[
r - ru - \frac{\theta v}{1+\xi\varphi+\eta u} = 0, \tag{8}
\]

\[
(\theta_1 c - c) = 0. \tag{9}
\]

\[
r\left(1-\frac{w}{K}\right) + \frac{\theta' v^2}{v^2+\gamma_1^2} - \frac{q e}{m e + m w} = 0. \tag{10}
\]

From Eq. (8), we get

\[
\nu = \frac{r(1-u)(1+\eta u)}{\theta-\xi r(1-u)}. \tag{11}
\]

Substituting this value of \( \nu \) in Eqs. (9) and (10), we obtain

\[
G_1(u,w) = \frac{\theta_1 r(1-u)(1+\eta u)}{\theta+\eta u\theta} - c - \frac{r(1-u)(1+\eta u)[\theta-\xi r(1-u)]}{[r(1-u)(1+\eta u)]^2 + \gamma_1^2[\theta-\xi r(1-u)]^2} = 0, \tag{12}
\]

\[
G_2(u,w) = \frac{\theta r^2(1-u)^2(1+\eta u)^2}{r^2(1-u)^2(1+\eta u)^2 + \gamma_1^2[\theta-\xi r(1-u)]^2} - \frac{q e}{m e + m w} = 0. \tag{13}
\]

From Eq. (12) when \( \nu = 0 \), then \( G_1(u,0) = 0 \) contains a real root \( u_* \), which is given by \( u_* = \frac{c(1+\xi\varphi)}{\theta-\eta \varphi} \).

We note that,
$u_x > 0$ if $\theta > c\eta$. \hfill (14)

Now, substituting $u = 0$ in Eq. (12), then $G_i(0,v) = 0$ contains a real root given by

$$w_x = -\frac{c[r^2 + \gamma_i^2(\theta - \xi r)^2]}{r(\theta - \xi r)} < 0. \hfill (15)$$

Now, we have $\frac{du}{dw} = -\frac{\partial G_i}{\partial w}. \hfill (16)$

Here, $\frac{du}{dw} > 0$ if $\frac{\partial G_x}{\partial u} > 0$. \hfill (16)

From Eq. (13) when $w = 0$, then from $G_z(u,0) = 0$, we have,

$$\pm \mu = r\eta u^2 + (r - \eta)(u - r), \hfill (17)$$

Where $\mu = \sqrt{\gamma_i^2(\theta - \xi r(1-u)^2)(q-rm_i)}$, $q - rm_i > 0$ and $rm_i - q + \theta^m m_i > 0$.

**Case I.** Taking $-\mu$ in Eq. (17), if

$$(r - \eta)^2 < 4r\eta(\mu - r), \mu > r, \hfill (18)$$

then Eq. (16) contains no real root.

**Case II.** Taking $+\mu$ in Eq. (17), then Eq. (17) contains a unique positive real root say, $P_y$.

Substituting $u = 0$ in Eq. (13), then from $G_z(0,w) = 0$, we have

$$rm_zw^2 + (rm_1e - rm_zK - \mu m_zK)w + (q - rm_1 - \mu m_z)eK = 0, \hfill (19)$$

where $\mu_i = \frac{\theta^m r^2}{r^2 + \gamma_i^2(\theta - \xi r)^2}$. 

Eq. (18) contains a real positive root say $F_y$ if

$$q - rm_1 - \mu m_z < 0. \hfill (20)$$

Now, we have $\frac{du}{dw} = \frac{\partial G_z}{\partial w} / \frac{\partial G_z}{\partial u}$. 

Hence, $\frac{du}{dw} < 0$, if
either (i) \[
\frac{\partial G_u}{\partial u} > 0 \quad \text{and} \quad \frac{\partial G_w}{\partial w} > 0,
\]

or (ii) \[
\frac{\partial G_u}{\partial u} < 0 \quad \text{and} \quad \frac{\partial G_w}{\partial w} < 0,
\]
holds.

From the above analysis, we consider that the isoclines (12) and (13) inter-sect at a unique point \((u^*, w^*)\) if in addition to condition (12), (16), (18), (20), (21), the condition \(P_y > P_z\) holds.

This completes the existence of \(L_5(u^*, v^*, w^*)\).

### 2.2 Stability of non-spatial model system

To study the local stability of the proposed model system, first we find the variational matrices corresponding to each equilibrium point. Then we obtain the following results by using eigen value method and Routh-Hurwitz criterion:

(i) The equilibrium point \(L_3(0,0,0)\) is a saddle point with eigen values \(r, -c\) and \(r - \frac{q}{m_i}\) always exists.

(ii) The equilibrium point \(L_4(u^*, 0, 0)\) is a saddle point with eigen values \(-r, -\frac{\theta v^2}{1 + \eta} - c, r - \frac{q}{m_i}\).

(iii) The equilibrium point \(L_2(0,0, w^*)\) is a saddle point with eigen values \(r, -c, r - \frac{2rww}{K} + \frac{\theta v^2}{v^2 + \gamma_i^2} - \frac{m_ie^2}{(m_i + m_e + m_w)v^2}\).

(iv) Zooplankton-free equilibrium point \(L_5(u^*, v^*, 0)\) is stable or unstable depending on whether

\[
\left( r - c, r - \frac{\theta v^2}{v^2 + \gamma_i^2} - \frac{q}{m_i} \right)
\]

is negative or positive respectively, provided

\[
(r - 2ru)(1 + \xi v + \eta u)^2 < (\theta v + \xi v^2)\left(\frac{\theta u + \theta \eta u^2}{1 + \xi v + \eta u}\right) - c < \frac{2wv\gamma_i^2}{(v^2 + \gamma_i^2)^2}.
\]

(v) TPP-free equilibrium point \(L_4(\bar{u}, 0, \bar{w})\) is stable or unstable in the positive direction orthogonal to \(uw\) plane i.e. \(v\) direction depending on whether \(\rho_z = \frac{\theta u + \theta \eta u^2}{(1 + \eta u)^2} - c\) is negative or positive respectively, if \(u > 0\).

(vi) The variational matrix about the equilibrium point \(L_4(u^*, v^*, w^*)\) is given by
The characteristic equation of matrix $V_5$ is given by,

$$\lambda^3 + A_4\lambda^2 + A_2\lambda + A_3 = 0,$$

where

$$A_4 = -(a_{11} + a_{22} + a_{33}),$$
$$A_2 = a_{23}a_{33} + a_{23}a_{32} + a_{11}a_{33} + a_{11}a_{22} - a_{12}a_{21},$$
$$A_3 = -a_{11}a_{22}a_{33} + a_{11}a_{23}a_{32} + a_{12}a_{21}a_{33}.$$

**Theorem 1:** The unique non-trivial positive equilibrium point $L_*$ is locally asymptotically stable if and only if the following inequalities hold.

(i) $$ru^* + \frac{2r_iw^*}{K} + m,\sigma^2 + \frac{2wv^*\theta^*v^* - \theta^*v^* - \theta^*v^* - \theta^*v^*}{(v^* + \gamma_i^2)^2} > r_i + \frac{\theta^*v^* - \theta^*v^* - \theta^*v^*}{(1 + \xi v^* + \eta v^*)^2}.$$

(ii) $$\left\{\frac{\theta^*v^* - \theta^*v^*}{(1 + \xi v^* + \eta v^*)^2} - ru^*\right\} \left\{\frac{2\theta^*v^*w^*\gamma_i^2}{(v^* + \gamma_i^2)^2} + \frac{\theta^*u^* + \theta^*\eta u^*}{(1 + \xi v^* + \eta v^*)^2} - \frac{2wv^*\gamma_i^2}{(v^* + \gamma_i^2)^2}\right\}$$

$$\left\{\frac{r_i - \frac{2r_iw^*}{K} + \frac{\theta^*v^*}{v^* + \gamma_i^2} - \frac{m,\sigma^2}{(m,\sigma + m,\sigma)^2}}{r_i - \frac{2r_iw^*}{K} + \frac{\theta^*v^*}{v^* + \gamma_i^2} - \frac{m,\sigma^2}{(m,\sigma + m,\sigma)^2}}\right\}.$$
2.3 Maximum sustainable yield

In the above model (1), the fish population depends on zooplankton. We have,

\[
h = \frac{q_{ew}^*}{m.e + m_w^*}
\]

\[= r_{1w^*} \left(1 - \frac{w^*}{K}\right) + \frac{\theta^* w^* v^2}{v^2 + \gamma_1^2}.
\]

Hence \(\frac{\partial h}{\partial w^*} = 0\) yields \(w^* = \frac{K}{2} \left(1 + \frac{\theta^* v^2}{r_i(v^2 + \gamma_1^2)}\right)\) and \(\frac{\partial^2 h}{\partial w^{2*}} < 0\).

Thus, we have

\[
h_{MSY} = \frac{r_i K}{4} \left(1 + \frac{\theta^* v^2}{r_i(v^2 + \gamma_1^2)}\right)^2.
\]

If \(h > h_{MSY}\), then it indicate the over exploitation of fish population. If \(h < h_{MSY}\), then the fish population is under exploitation.

2.4 Optimal harvesting policy

The present value \(J\) of a continuous time-stream of revenues is given by
\[ J = \int_0^\infty e^{-\delta t} \left( \frac{pqw}{m_1 e + m_2 w} - c \right) E(t) dt, \]

where \( \delta \) is the instantaneous rate of annual discount, \( c \) is the utilized cost per unit effort and \( p \) is the price per unit harvested fish. Thus, our objective is to max \( J \), subject to conditions (8)-(10) and control constraints \( 0 < E < E_{\text{max}} \).

Now to find the optimal level of equilibrium, we use Pontryagins Maximum Principle (21). The associated Hamiltonian function is given by

\[
H = e^{-\delta t} \left( \frac{pqw}{m_1 e + m_2 w} - c \right) E(t) + \dot{\lambda}_1(t) \left( ru(1-u) - \frac{\theta uv}{1 + \xi v + \eta u} \right) + \dot{\lambda}_2(t) \left( \frac{\theta uv}{1 + \xi v + \eta u} - cv - \frac{wv^2}{v^2 + \gamma_1^2} \right) + \\
\dot{\lambda}_3(t) \left( r_w \left( 1 - \frac{w}{K} \right) + \frac{\theta' w v^2}{v^2 + \gamma_1^2} - \frac{qew}{m_1 e + m_2 w} \right),
\]

where \( \dot{\lambda}_1, \dot{\lambda}_2, \dot{\lambda}_3 \) are adjoint variables and

\[
\sigma(t) = e^{-\delta t} \left( \frac{pqm_2 w^2}{(m_1 e + m_2 w)^2} - c \right) - \lambda_3 \left( \frac{qm_2 w^2}{(m_1 e + m_2 w)^2} \right)
\]

is switching function.

The optimal control \( E(t) \) which maximizes \( H \) must satisfy

\[
E(t) = \begin{cases} 
E_{\text{max}}, & \sigma(t) > 0 i.e. \dot{\lambda}_3 e^{\delta t} < \left( p - \frac{c(m_1 e + m_2 w)^2}{qm_2 w^2} \right), \\
0, & \sigma(t) < 0 i.e. \dot{\lambda}_3 e^{\delta t} > \left( p - \frac{c(m_1 e + m_2 w)^2}{qm_2 w^2} \right),
\end{cases}
\]

Now, the usual shadow price is \( \dot{\lambda}_3 e^{\delta t} \) and the net revenue on a unit harvest is \( p - \frac{c(m_1 e + m_2 w)^2}{qm_2 w^2} \). Thus, if the shadow price is less than the net economic revenue on a unit harvest, then \( E = E_{\text{max}} \), if the shadow price is greater than the net economic revenue on a unit harvest then \( E = 0 \) and when shadow price equals the net economic revenue on a unit harvest, i.e. \( \sigma(t) = 0 \), then the Hamiltonian becomes independent of the control variable \( E(t) \) i.e. \( \frac{\partial H}{\partial E} = 0 \). This is the necessary condition for singular control \( E^* (t) \) to be optimal over control set \( 0 < E^* < E_{\text{max}} \).

Hence, the optimal harvesting policy is
For the singular control to be optimal, we must have \( \frac{\partial H}{\partial e} = \sigma(t) = 0 \). This yields

\[
\lambda_3 = \left( p - \frac{c(m_1 e + m_2 w)^2}{qm_2 w^2} \right) e^{-\beta t}.
\]  

(25)

According to the Pontryagin's Maximum Principle, the adjoint equations are

\[
\frac{d\lambda_1}{dt} = -\frac{\partial H}{\partial P}, \quad \frac{d\lambda_2}{dt} = -\frac{\partial H}{\partial Z}, \quad \frac{d\lambda_3}{dt} = -\frac{\partial H}{\partial F}.
\]  

(26)

From the first adjoint equation, we have

\[
\frac{d\lambda_1}{dt} = -\lambda_1(t) \left[ r - 2ru - \frac{\theta v + \theta \xi v^2}{(1 + \xi v + \eta u)^2} \right] - \lambda_2(t) \left[ \frac{\theta v + \theta \xi v^2}{(1 + \xi v + \eta u)^2} \right].
\]

Using Eq. (8), this equation becomes

\[
\frac{d\lambda_1}{dt} = -\lambda_1 \left[ -ru + \frac{\theta \eta u}{(1 + \xi v + \eta u)^2} \right] - \lambda_2 \left[ \frac{\theta v + \theta \xi v^2}{(1 + \xi v + \eta u)^2} \right].
\]  

(27)

The second adjoint equation becomes

\[
\frac{d\lambda_2}{dt} = \frac{\lambda_3}{1 + \xi v + \eta u} - \lambda_2 \left[ \frac{-\theta v \xi v}{(1 + \xi v + \eta u)^2} + \frac{3wv^3 + wv\gamma_1^2 - 2wv^5 - 2wv\gamma_1^3 - 4wv^3\gamma_1^2}{(v^2 + \gamma_1^2)^2} \right]
\]

\[
- \lambda_3 \cdot \frac{2\theta v w\gamma_1^2}{(v^2 + \gamma_1^2)^2}.
\]  

(28)

The third adjoint equation becomes

\[
\frac{d\lambda_3}{dt} = -\left[ e^{-\beta t} \left( \frac{pq m_1 e}{(m_1 e + m_2 w)^2} \right) + \lambda_2 \left( \frac{v^2}{v^2 + \gamma_1^2} \right) \right] - \lambda_3 \left[ \frac{r_1 - 2r_1 w}{K} + \frac{\theta v^2}{v^2 + \gamma_1^2} - \frac{m_1 q e^2}{(m_1 e + m_2 w)^2} \right].
\]

Using Eq. (10), this equation becomes

\[
\frac{d\lambda_3}{dt} = -\left[ e^{-\beta t} \left( \frac{pq m_1 e}{(m_1 e + m_2 w)^2} \right) + \lambda_2 \left( \frac{v^2}{v^2 + \gamma_1^2} \right) \right] - \lambda_3 \left[ \frac{-r w}{K} + \frac{m_2 q e w}{(m_1 e + m_1 w)^2} \right].
\]  

(29)

Differentiating Eq. (26), with respect to \( t \), we have

\[
\frac{d\lambda_3}{dt} = -\delta \lambda_3.
\]  

(30)
Using Eq. (28) in Eq. (27), we obtain

$$\lambda_2 = B_t e^{-\delta t},$$

(31)

where

$$B_t = -\gamma^2_1 + v^2 \left[ \left( \delta + \frac{r_w}{K} \frac{q m_2 w}{(m_e + m_2 w)^2} \right) \left( p - \frac{c(m_e + m_2 w)^2}{q m_2 w^2} \right) - \frac{p q m_2 e}{(m_e + m_2 w)^2} \right] e^{-\delta t}.$$  

Putting the value of $\lambda_2$ from Eq. (31) in Eq. (28), we find

$$\frac{d\lambda_1}{dt} - B_3 \lambda_1 = -B_t e^{-\delta t},$$

(32)

where

$$B_3 = \left[ \frac{B_t (\theta \nu + \theta \xi v)}{(1 + \xi v + \eta u)^2} \right], B_2 = \left[ -ru + \frac{\theta \eta uv}{(1 + \xi v + \eta u)^2} \right].$$

Solving Eq. (31), we obtain

$$\lambda_1 = \frac{B_3}{B_2 + \delta} e^{-\delta t} + k_0 e^{\beta_2 t}.$$  

(33)

When $t \to \infty$, then shadow price $\lambda_1 e^{\delta t}$ is bounded iff $k_0 = 0$. Hence, we have

$$\lambda_1 = \frac{B_3}{B_2 + \delta} e^{-\delta t}. $$

Putting the values of $\lambda_1$ and $\lambda_3$ in Eq. (28), we obtain when $t \to \infty$, then shadow price $\lambda_4 e^{\delta t}$ is bounded if $k_0 = 0$. Hence, we have

$$\lambda_1 = \frac{B_3}{B_2 + \delta} e^{-\delta t}. $$

(34)

Substituting the values of $\lambda_1$ and $\lambda_3$ in Eq. (27), we obtain

$$\frac{d\lambda_2}{dt} - B_3 \lambda_2 = -B_t e^{-\delta t},$$

(35)

where

$$B_4 = \left[ \frac{\theta \mu \xi v}{(1 + \xi v + \eta u)^2} \right] - \frac{3 w v ^ 3 + w v ^ 3 - 2 w ^ 5 - 2 w v ^ 4 - 4 w v ^ 3 \gamma_1^2}{(v ^ 2 + \gamma_1^2)^2},$$

$$B_5 = \left( p - \frac{c(m_e + m_2 w)^2}{q m_2 w^2} \right) \frac{2 \theta' v w \gamma_1^2}{(v ^ 2 + \gamma_1^2)^2} - \frac{B_3 (\theta \mu + \theta \eta u)}{(B_2 + \delta)(1 + \xi v + \eta u)^2}. $$

Solving Eq. (35), we obtain

$$\lambda_2 = \frac{B_4 e^{-\delta t}}{B_4 + \delta} + k_1 e^{\beta_2 t}. $$

(36)

When $t \to \infty$, then shadow price $\lambda_2 e^{\delta t}$ is bounded iff $k_1 = 0$. Hence, we have
\[
\lambda_2 = \frac{B_s e^{-\delta t}}{B_s + \delta}.
\] (36)

Hence from Eqs.(32) and (37), we have
\[
B_i = \frac{B_s}{B_s + \delta}.
\] (37)

By solving Eq. (37) we get \( E = E_\delta \), where \( E_\delta \) is the optimal harvesting effort. Hence by solving Eqs. (8)-(10) using Eq. (37), we find the optimal solution \((P_\delta, Z_\delta, F_\delta)\).

3 Spatial Model System

In this section, the model system is given by:
\[
\frac{\partial u}{\partial t} = ru(1-u) - \frac{\theta uv}{1 + \xi v + \eta u} + d_1 \nabla^2 u, \tag{38}
\]
\[
\frac{\partial v}{\partial t} = \frac{\theta uv}{1 + \xi v + \eta u} - cv - \frac{wv^2}{v^2 + \gamma_1^2} + d_2 \nabla^2 v, \tag{39}
\]
\[
\frac{\partial w}{\partial t} = ru(1-u) - \frac{\theta' wv^2}{v^2 + \gamma_1^2} - m_1 e + m_2 \frac{qe_1}{K} + d_3 \nabla^2 w. \tag{40}
\]

with initial conditions,
\[
u(x, y, 0) > 0, v(x, y, 0) > 0, w(x, y, 0) > 0 \quad \text{for} \quad (x, y) \in \Omega \tag{41}
\]

and boundary condition,
\[
\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = \frac{\partial w}{\partial n} = 0 \quad \text{for} \quad (x, y) \in \Omega, \quad t > 0, \tag{42}
\]

where, \( d_1, d_2 \) and \( d_3 \) are diffusion coefficient for phytoplankton, zooplankton and fish populations respectively, \( n \) is the outward normal to \( \partial \Omega \) and \( \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \).

To understand the spatial dynamics of the model system (38)-(40), we consider the linearized form of the system about \( L_\delta(u^*, v^*, w^*) \) as follows:
\[
\frac{\partial u}{\partial t} = a_{11} u + a_{12} v + a_{13} w + d_1 \nabla^2 u, \tag{43}
\]
\[
\frac{\partial v}{\partial t} = a_{21} u + a_{22} v + a_{23} w + d_2 \nabla^2 v,
\]
\[
\frac{\partial w}{\partial t} = a_{31} u + a_{32} v + a_{33} w + d_3 \nabla^2 w.
\]
where we introduce small perturbations \( u = u - u^* \), \( v = v - v^* \) and \( w = w - w^* \). Let us suppose that the model system (42) has the solution which is of the form

\[
\begin{bmatrix}
U \\
V \\
W
\end{bmatrix} = 
\begin{bmatrix}
l_1 \\
l_2 \\
l_3
\end{bmatrix} \exp(\lambda_k t) \cos(k_x x) \cos(k_y y),
\]

where \( l_1, l_2 \), and \( l_3 \) are sufficiently small constants. \( k_x \) and \( k_y \) are the components of wave number along \( x \) and \( y \) directions respectively and \( \lambda_k \) denotes the wavelength.

The variational matrix of the linearized model system (42) is given by

\[
J = 
\begin{pmatrix}
      a_{11} - d_1 k^2 & a_{12} & 0 \\
      a_{21} & a_{22} - d_2 k^2 & a_{23} \\
      0 & a_{32} & a_{33} - d_3 k^2
\end{pmatrix}
\]

The characteristic equation of \( J \) is given by

\[
\lambda_k^3 + \rho_1 \lambda_k^2 + \rho_2 \lambda_k + \rho_3 = 0,
\]

where \( \rho_1 = (d_1 + d_2 + d_3)k^2 + A_1 \), \( \rho_2 = (d_1d_2 + d_2d_3 + d_3d_1)k^4 - [d_1(a_{22} + a_{33}) + d_2(a_{11} + a_{33}) + d_3(a_{11} + a_{22})]k^2 + A_2 \), \( \rho_3 = d_1d_2d_3k^6 - [a_{11}d_2d_3 + a_{22}d_1d_3 + a_{33}d_1d_2]k^4 + [d_1(a_{22}a_{33} - a_{23}a_{32}) + d_2a_{11}a_{33} + d_3(a_{11}a_{22} - a_{12}a_{21})]k^2 + A_3 \)

and \( A_1, A_2 \), and \( A_3 \) are defined in Eq. (22).

**Theorem 2:** The equilibrium point \( L^*(u^*, v^*, w^*) \) is locally asymptotically stable in the presence of diffusion if the conditions of Theorem 1 hold.

The proof of theorem follows from Routh–Hurwitz criterion, hence omitted.

4 Conclusions

Human activity has been the major threat to wetlands. Agriculture, industrial development, and urban and suburban sprawl have caused the greatest losses of freshwater wetlands. The biggest current source of loss for fresh-water coastal wetlands is from urban sprawl. Subsidence causes the land surface to drop, which can then become flooded if the surface is already very near to sea level. Wetland management generally involves activities that can be conducted with, in, and around wetlands, both natural and man-made. Two major facets of managing wetlands for protection include buffering wetlands from direct human pressures, and maintaining natural processes in surrounding lands that affect wetlands and that may be disrupted by human activities. Our study is based on the study of system dynamics and stability of the wetland system. In this paper, we have analytically investigated a diffusive three species phytoplankton-zooplankton-fish model. The non-trivial
equilibrium where all the species coexist is locally asymptotically stable under a fixed region of attraction when certain conditions are satisfied. Next, the optimal harvesting policy is discussed. The result of the model system (1) shows that a proper management of the wetland is required to control the growth of the wild grasses. So there should be taken some proper management of the wetland system by means of the comfortable survival of the species of the system and for the good economic values.

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