Article

Analysis of complex wetland ecological system: Effect of harvesting

Nilesh Kumar Thakur, **Rashi Gupta**
Department of Mathematics, National Institute of Technology, Raipur (C.G.), India
Email: nkthakur.maths@nitrr.ac.in, rashigupta1234@gmail.com

Received 12 July 2017; *Accepted 20 August 2017; Published 1 December 2017*

Abstract
In this paper, we have studied interaction among diffusive phytoplankton, zooplankton and fish population with Beddington-DeAngelis type functional response for the zooplankton and Holling type III for fish. The stability analysis of the model system with diffusion and without diffusion has been analyzed. The conditions for Maximum sustainable yield and Optimal harvesting policy for non-spatial model have been discussed. Our study may be helpful to improve and manage ecosystem services provided by wetlands on an agricultural landscapes include fisheries, water conservation, climate change and many more.

Keywords nonlinear harvesting; optimal harvesting; Turing instability.

Computational Ecology and Software
ISSN 2220-721X
URL: http://www.iaees.org/publications/journals/ces/online-version.asp
RSS: http://www.iaees.org/publications/journals/ces/rss.xml
E-mail: ces@iaees.org
Editor-in-Chief: WenJun Zhang
Publisher: International Academy of Ecology and Environmental Sciences

1 Introduction

Wetlands are one of the world's most important environmental assets, existing in all continents and latitudes. Wetlands are home of a large biota diversity and yield significant economic, social and cultural benefits related to timber, fisheries, hunting, recreational and tourist activities, etc. In general they provide a wide array of useful and appreciated ecosystem services related to water quality preservation, erosion shore protection from wave action, nurseries for fish and other freshwater and marine animals.

Modelling of harvesting is an important notion and an interesting topic of mathematical bio-economics in aquatic system. Harvesting is basically of three types: (i) constant harvesting reveals that the constant number of individuals is harvested per unit time. (ii) Linear harvesting $h(X) = qEX$ shows that the number of individuals harvested per unit time is proportional to the current population. (iii) Non-linear harvesting $h(X) = qEX / m_1E + m_2X$, where q denotes the catch ability coefficient and E denotes the attempt applied to harvest the individuals and m_1, m_2 are suitable positive constants. Optimal control theory is to obtain the increased application in both theoretical and applied ecology. The optimal harvesting policy in their model has been discussed by many authors (Kar and Chaudhuri, 2004; Kar et al. 2009; Dubey and Hussain, 2000). The optimal harvesting policy in the model of fishery resource with reserve area has been discussed by Dubey et al.

(2003). Recently, Dubey et al. (2014) proposed a model to investigate the dynamical interaction among phytoplankton-zooplankton-fish population. A bio-economic model of non-selective harvesting of two competing fish species has been proposed by Purohit and Chaudhuri (2004) and studies the optimal harvesting policy. In this model, they examine non-linear harvesting in both competing species.

The wave phenomenon, non-linear and non-equilibrium pattern formation in a phytoplankton-zooplankton system with Holling type IV functional response has been investigated by Upadhyay et al. (2008). Upadhyay et al. (2011) proposed a reaction-diffusion model to study the dynamics of phytoplankton-zooplankton system in which the consumption of phytoplankton species follows the Beddington-DeAngelis functional response. A population model with diffusion, strong Allee effect and constant yield of harvesting has been investigated by Ali et al. (2009). Spatio-temporal complexity of a three-species ratio-dependent food chain model has been proposed by Rao (2014). Chang and Wei (2012) explore the dynamics of a diffusive delayed predator-prey system with Holling type II functional response and discussed the optimal control problem. Recently, complex dynamics of spatial predator-prey system under non-linear harvesting has been studied by Upadhyay et al. (2015) and interpret how Turing patterns develop gradually under non-linear harvesting. DeAngelis et al. (2010) investigated a model of reaction-diffusion to interpret the fish population dynamics in a seasonally varying wetland of Florida Everglades, USA.

In this paper, we deal with complex dynamical behaviour of phytoplankton-zooplankton model with non-linear harvesting. The paper is organized as follows. In secton 2, we present a non-spatial model and discuss the stability analysis, maximum sustainable yield and optimal harvesting policy. In section 3, we propose the spatial model and in section 4, we have discussed the conclusions of present study.

2 The Proposed Model System

In this section, the system of differential equations in which interaction among phytoplankton, zooplankton and fish population is shown as follows:

$$\begin{aligned}
\frac{du}{dt} &= ru(1-u) - \frac{\theta uv}{1+\xi v+\eta u} = ug_1(u,v,w) = f_1(u,v,w),\\
\frac{dv}{dt} &= \frac{\theta_1 uv}{1+\xi v+\eta u} - cv - \frac{wv^2}{v^2+\gamma_1^2} = vg_2(u,v,w) = f_2(u,v,w), \qquad (1)\\
\frac{dw}{dt} &= r_1 w\left(1-\frac{w}{K}\right) + \frac{\theta' wv^2}{v^2+\gamma_1^2} - \frac{qew}{m_1 e + m_2 w} = wg_3(u,v,w) = f_3(u,v,w),
\end{aligned}$$

with

$$u(0)>0, v(0)>0, w(0)>0. \qquad (2)$$

In the proposed model (1), r is the intrinsic rate of growth of phytoplankton in the absence of predation. θ is the rate at which phytoplankton is grazed with Holling type II functional response. ξ is the intensity of interference between individuals of zooplankton. η determines how fast the per capita feeding rate approaches to its saturation value. θ_1 is the conversion coefficient from individuals of phytoplankton into individuals of zooplankton. c is the rate of mortality of zooplankton. γ_1 is zooplankton density at which the specific growth rate becomes half of its saturated value. r_1 and K are rate of growth and carrying capacity of fish population respectively. θ' is the conversion coefficient of zooplankton. q is the catch ability coefficient and e is catch per unit effort. m_1 and m_2 are the positive constants.

2.1 Temporal model system

Now, the equilibrium analysis of the model system (1) has been discussed. The system (1) has six non-negative equilibrium points:

$L_0(0,0,0)$, $L_1(1,0,0)$, $L_2(0,0,\hat{w})$, $L_3(\bar{u},\bar{v},0)$, $L_4(1,0,\hat{w})$, $L_5(u^*,v^*,w^*)$,

In the equilibrium point $L_2(0,0,\hat{w})$, the equation whose root is $\hat{w}$ is as follows:

$$r_1 m_2 w^2 + (r_1 m_1 e - r_1 m_2 K)w + (q - m_1 r_1)eK = 0. \tag{3}$$

The equation (3) contain a positive real root w if $q < m_1 r_1$.

The equilibrium point $L_3(\bar{u},\bar{v},0)$ exists if the following equations has the positive solutions $\bar{u}$ and $\bar{v}$:

$$r(1-u) - \frac{\theta v}{1+\xi v+\eta u} = 0, \tag{4}$$

$$\frac{\theta_1 u}{1+\xi v+\eta u} - c = 0. \tag{5}$$

From Eq. (5), we obtain

$$u = \frac{c(1+\xi v)}{\theta_1 - c\eta}. \tag{6}$$

Clearly, $u > 0$ if $\theta_1 > c\eta$.

From Eq. (4), we obtain

$$v = \frac{r(1-u)(1+\eta u)}{\theta - \xi r(1-u)}. \tag{7}$$

The existence of interior equilibrium point $L_5(u^*,v^*,w^*)$. In this case, u^*, v^* and w^* are the positive solutions of the following equations.

$$r - ru - \frac{\theta v}{1+\xi v+\eta u} = 0, \tag{8}$$

$$\frac{\theta_1 u}{1+\xi v+\eta u} - c - \frac{wv}{v^2+\gamma_1^2} = 0, \tag{9}$$

$$r_1\left(1-\frac{w}{K}\right) + \frac{\theta' v^2}{v^2+\gamma_1^2} - \frac{qe}{m_1 e + m_2 w} = 0. \tag{10}$$

From Eq. (8), we get

$$v = \frac{r(1-u)(1+\eta u)}{\theta - \xi r(1-u)}. \tag{11}$$

Substituting this value of v in Eqs. (9) and (10), we obtain

$$G_1(u,w) = \frac{\theta u[\theta - \xi r(1-u)]}{\theta + \eta u\theta} - c - \frac{wr(1-u)(1+\eta u)[\theta - \xi r(1-u)]}{[r(1-u)(1+\eta u)]^2 + \gamma_1^2[\theta - \xi r(1-u)]^2} = 0, \tag{12}$$

$$G_2(u,w) = r_1\left(1-\frac{w}{K}\right) + \frac{\theta' r^2(1-u)^2(1+\eta u)^2}{r^2(1-u)^2(1+\eta u)^2 + \gamma_1^2[\theta - \xi r(1-u)]^2} - \frac{qe}{m_1 e + m_2 w} = 0, \tag{13}$$

From Eq. (12) when $v = 0$, then $G_1(u,0) = 0$ contains a real root u_x, which is given by $u_x = \dfrac{c(1+\xi v)}{\theta - c\eta}$.

We note that,

$$u_x > 0 \text{ if } \theta > c\eta . \tag{14}$$

Now, substituting $u = 0$ in Eq.(12), then $G_1(0, v) = 0$ contains a real root given by

$$w_x = -\frac{c[r^2 + \gamma_1^2 (\theta - \xi r)^2]}{r(\theta - \xi r)} < 0 . \tag{15}$$

Now, we have $\dfrac{du}{dw} = -\dfrac{\frac{\partial G_1}{\partial w}}{\frac{\partial G_2}{\partial u}}$.

Here, $\dfrac{du}{dw} > 0$ if $\dfrac{\partial G_2}{\partial u} > 0$. (16)

From Eq.(13) when $w = 0$, then from $G_2(u, 0) = 0$, we have,

$$\pm \mu = r\eta u^2 + (r - r\eta)u - r , \tag{17}$$

Where $\mu = \sqrt{\dfrac{\gamma_1^2 (\theta - \xi r(1-u))^2 (q - r_1 m_1)}{r_1 m_1 - q + \theta' m_1}}$, $q - r_1 m_1 > 0$ and $r_1 m_1 - q + \theta' m_1 > 0$.

Case I. Taking $-\mu$ in Eq. (17), if

$$(r - r\eta)^2 < 4r\eta(\mu - r), \mu > r, \tag{18}$$

then Eq. (16) contains no real root.

Case II. Taking $+\mu$ in Eq. (17), then Eq. (17) contains a unique positive real root say, P_y .

Substituting $u = 0$ in Eq. (13), then from $G_2(0, w) = 0$, we have

$$r_1 m_2 w^2 + (r_1 m_1 e - r_1 m_2 K - \mu_1 m_2 K)w + (q - m_1 r_1 - \mu_1 m_1)eK = 0 , \tag{19}$$

where $\mu_1 = \dfrac{\theta' r^2}{r^2 + \gamma_1^2 (\theta - \xi r)^2}$.

Eq. (18) contains a real positive root say F_y if

$$q - m_1 r_1 - \mu_1 m_1 < 0 . \tag{20}$$

Now, we have $\dfrac{du}{dw} = -\dfrac{\partial G_2}{\partial w} \Big/ \dfrac{\partial G_2}{\partial u}$.

Hence, $\dfrac{du}{dw} < 0$, if

either (i) $\frac{\partial G_2}{\partial u} > 0$ and $\frac{\partial G_2}{\partial w} > 0$,

(21)

or (ii) $\frac{\partial G_2}{\partial u} < 0$ and $\frac{\partial G_2}{\partial w} < 0$,

holds.

From the above analysis, we consider that the isoclines (12) and (13) inter-sect at a unique point (u^*, w^*) if in addition to condition (12), (16), (18), (20), (21), the condition $P_y > P_x$ holds.

This completes the existence of $L_5(u^*, v^*, w^*)$.

2.2 Stability of non-spatial model system

To study the local stability of the proposed model system, first we find the variational matrices corresponding to each equilibrium point. Then we obtain the following results by using eigen value method and Routh-Hurwitz criterion:

(i) The equilibrium point $L_0(0,0,0)$ is a saddle point with eigen values $r, -c$ and $r_1 - \frac{q}{m_1}$ always exists.

(ii) The equilibrium point $L_1(u^*,0,0)$ is a saddle point with eigen values $-r, \frac{\theta_1}{1+\eta} - c, r_1 - \frac{q}{m_1}$.

(iii) The equilibrium point $L_2(0,0,w^*)$ is a saddle point with eigen values $r, -c, r_1 - \frac{2r_1 w}{K} + \frac{\theta' v^2}{v^2 + \gamma_1{}^2} - \frac{m_1 q e^2}{(m_1 e + m_2 w)^2}$.

(iv) Zooplankton –free equilibrium point $L_3(u^*, v^*, 0)$ is stable or unstable depending on whether $\left(r_1 + \frac{\theta' v^2}{v^2 + \gamma_1{}^2} - \frac{q}{m_1} \right)$ is negative or positive respectively, provided $(r - 2ru)(1 + \xi v + \eta u)^2 < (\theta_1 v + \theta_1 \xi v^2), \frac{\theta_1 u + \theta_1 \eta u^2}{(1 + \xi v + \eta u)^2} - c < \frac{2wv\gamma_1{}^2}{(v^2 + \gamma_1{}^2)^2}$.

(v) TPP-free equilibrium point $L_4(\overline{u}, 0, \overline{w})$ is stable or unstable in the positive direction orthogonal to $uw-$ plane i.e. $v-$direction depending on whether $\rho_2 = \frac{\theta_1 u + \theta_1 \eta u^2}{(1+\eta u)^2} - c$ is negative or positive respectively, if $u > 0$.

(vi) The variational matrix about the equilibrium point $L_5(u^*, v^*, w^*)$ is given by

$$v_5 = \begin{bmatrix} \frac{\theta\eta u^* v^*}{(1+\xi v^* + \eta u^*)^2} - ru^* & -\frac{(\theta u^* + \theta\eta u^{*2})}{(1+\xi v^* + \eta u^*)^2} & 0 \\ \frac{\theta_1 v^* + \theta_1 \xi v^{*2}}{(1+\xi v^* + \eta u^*)^2} & \frac{\theta_1 u^* + \theta_1 \eta u^{*2}}{(1+\xi v^* + \eta u^*)^2} - c - \frac{2wv^*\gamma_1^2}{(v^{*2} + \gamma_1^2)^2} & -\frac{v^{*2}}{v^{*2} + \gamma_1^2} \\ 0 & \frac{2\theta' v^* w^* \gamma_1^2}{(v^{*2} + \gamma_1^2)^2} & r_1 - \frac{2r_1 w^*}{K} + \frac{\theta' v^{*2}}{(v^{*2} + \gamma_1^2)} - \frac{m_1 q e^2}{(m_1 e + m_2 w^*)^2} \end{bmatrix}$$

$$= \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

The characteristic equation of matrix V_5 is given by,

$$\lambda^3 + A_1\lambda^2 + A_2\lambda + A_3 = 0, \tag{22}$$

where

$$A_1 = -(a_{11} + a_{22} + a_{33}),$$

$$A_2 = a_{22}a_{33} - a_{23}a_{32} + a_{11}a_{33} + a_{11}a_{22} - a_{12}a_{21},$$

$$A_3 = -a_{11}a_{22}a_{33} + a_{11}a_{23}a_{32} + a_{12}a_{21}a_{33}.$$

Theorem 1: The unique non-trivial positive equilibrium point $L_5(u^*, v^*, w^*)$ is locally asymptotically stable if and only if the following inequalities hold.

(i) $$ru^* + \frac{2r_1 w^*}{K} + \frac{m_1 q e^2}{(m_1 e + m_2 w^*)^2} + \frac{2wv^*\gamma_1^2 - \theta' v^{*4} - \theta' v^{*2}\gamma_1^2}{(v^{*2} + \gamma_1^2)^2} > r_1 + \frac{\theta\eta u^* v^* - \theta u^* - \theta\eta u^{*2}}{(1+\xi v^* + \eta u^*)^2},$$

(ii) $$\left\{\frac{\theta\eta u^* v^*}{(1+\xi v^* + \eta u^*)^2} - ru^*\right\}\left\{\begin{array}{l}\frac{2\theta' v^{*3} w^* \gamma_1^2}{(v^{*2} + \gamma_1^2)^3} - \left(\frac{\theta_1 u^* + \theta_1\eta u^{*2}}{(1+\xi v^* + \eta u^*)^2} - c - \frac{2wv^*\gamma_1^2}{(v^{*2} + \gamma_1^2)^2}\right) \\ \left(r_1 - \frac{2r_1 w^*}{K} + \frac{\theta' v^{*2}}{v^{*2} + \gamma_1^2} - \frac{m_1 q e^2}{(m_1 e + m_2 w^*)^2}\right)\end{array}\right\}$$
$$> \left(\frac{(\theta_1 u^* + \theta_1\eta u^{*2})(\theta_1 v^* + \theta_1 \xi v^{*2})}{(1+\xi v^* + \eta u^*)^4}\right)\left(r_1 - \frac{2r_1 w^*}{K} + \frac{\theta' v^{*2}}{v^{*2} + \gamma_1^2} - \frac{m_1 q e^2}{(m_1 e + m_2 w^*)^2}\right),$$

(iii)

$$\left(\frac{\theta\eta u^* v^*}{(1+\xi v^*+\eta u^*)^2}-ru^*\right)\left(-\frac{(\theta u^*+\theta\eta u^{*2})(\theta v^*+\theta\xi v^{*2})}{(1+\xi v^*+\eta u^*)^4}\right)-\left(\frac{\theta\eta u^* v^*}{(1+\xi v^*+\eta u^*)^2}-ru^*\right)$$

$$\left(r_1-\frac{2r_1w^*}{K}+\frac{\theta' v^{*2}}{v^{*2}+\gamma_1{}^2}-\frac{m_1qe^2}{(m_1e+m_2w^*)^2}\right)-\left(r_1-\frac{2r_1w^*}{K}+\frac{\theta' v^{*2}}{v^{*2}+\gamma_1{}^2}-\frac{m_1qe^2}{(m_1e+m_2w^*)^2}\right)^2>$$

$$\left\{\frac{\theta_1u^*+\theta_1\eta u^{*2}}{(1+\xi v^*+\eta u^*)^2}-c-\frac{2wv^*\gamma_1{}^2}{(v^{*2}+\gamma_1{}^2)^2}\right\}\left(\frac{\theta\eta u^* v^*}{(1+\xi v^*+\eta u^*)^2}-ru^*\right)^2-\left(-\frac{v^{*2}}{v^{*2}+\gamma_1{}^2}\right)\left(\frac{2\theta' v^* w^*\gamma_1{}^2}{(v^{*2}+\gamma_1{}^2)^2}\right)+$$

$$\left(\frac{\theta\eta u^* v^*}{(1+\xi v^*+\eta u^*)^2}-ru^*\right)\left(\frac{\theta_1u^*+\theta_1\eta u^{*2}}{(1+\xi v^*+\eta u^*)^2}-c-\frac{2wv^*\gamma_1{}^2}{(v^{*2}+\gamma_1{}^2)^2}\right)+\frac{(\theta u^*+\theta\eta u^{*2})(\theta v^*+\theta\xi v^{*2})}{(1+\xi v^*+\eta u^*)^4}+$$

$$2\left(\frac{\theta\eta u^* v^*}{(1+\xi v^*+\eta u^*)}-ru^*\right)\left(r_1-\frac{2r_1w^*}{K}+\frac{\theta' v^{*2}}{v^{*2}+\gamma_1{}^2}-\frac{m_1qe^2}{(m_1e+m_2w^*)^2}\right)+$$

$$\left\{\frac{\theta_1u^*+\theta_1\eta u^{*2}}{(1+\xi v^*+\eta u^*)^2}-c-\frac{2wv^*\gamma_1{}^2}{(v^{*2}+\gamma_1{}^2)^2}\right\}\left\{r_1-\frac{2r_1w^*}{K}+\frac{\theta' v^{*2}}{v^{*2}+\gamma_1{}^2}-\frac{m_1qe^2}{(m_1e+m_2w^*)^2}\right\}+$$

$$\left\{r_1-\frac{2r_1w^*}{K}+\frac{\theta' v^{*2}}{v^{*2}+\gamma_1{}^2}-\frac{m_1qe^2}{(m_1e+m_2w^*)^2}\right\}^2$$

$$+\frac{2v^{*3}\theta' w^* r_1{}^2}{(v^{*2}+\gamma_1{}^2)^3}\left(r_1-\frac{2r_1w^*}{K}+\frac{\theta' v^{*2}}{v^{*2}+\gamma_1{}^2}-\frac{m_1qe^2}{(m_1e+m_2w^*)^2}\right).$$

2.3 Maximum sustainable yield

In the above model (1), the fish population depends on zooplankton. We have,

$$h=\frac{qew^*}{m_1e+m_2w^*}$$
$$=r_1w^*\left(1-\frac{w^*}{K}\right)+\frac{\theta' w^* v^2}{v^2+\gamma_1{}^2}.$$

Hence $\frac{\partial h}{\partial w^*}=0$ yields $w^*=\frac{K}{2}\left(1+\frac{\theta' v^2}{r_1(v^2+\gamma_1{}^2)}\right)$ and $\frac{\partial^2 h}{\partial w^{*2}}<0$.

Thus, we have

$$h_{MSY}=\frac{r_1K}{4}\left\{1+\frac{\theta' v^2}{r_1(v^2+\gamma_1{}^2)}\right\}^2.$$

If $h>h_{MSY}$, then it indicate the over exploitation of fish population. If $h<h_{MSY}$, then the fish population is under exploitation.

2.4 Optimal harvesting policy

The present value J of a continuous time-stream of revenues is given by

$$J=\int_0^\infty e^{-\delta t}\left(\frac{pqw}{m_1e+m_2w}-c\right)E(t)dt,$$

where δ is the instantaneous rate of annual discount, c is the utilized cost per unit effort and p is the price per unit harvested fish. Thus, our objective is to max J , subject to conditions (8)-(10) and control constraints $0<E<E_{\max}$.

Now to find the optimal level of equilibrium, we use Pontryagins Maximum Principle (21). The associated Hamiltonian function is given by

$$H=e^{-\delta t}\left(\frac{pqw}{m_1e+m_2w}-c\right)E(t)+\lambda_1(t)\left(ru(1-u)-\frac{\theta uv}{1+\xi v+\eta u}\right)+\lambda_2(t)\left(\frac{\theta_1 uv}{1+\xi v+\eta u}-cv-\frac{wv^2}{v^2+\gamma_1^{\ 2}}\right)+$$
$$\lambda_3(t)\left(r_1w\left(1-\frac{w}{K}\right)+\frac{\theta' wv^2}{v^2+\gamma_1^{\ 2}}-\frac{qew}{m_1e+m_2w}\right), \tag{23}$$

where $\lambda_1,\lambda_2,\lambda_3$ are adjoint variables and

$$\sigma(t)=e^{-\delta t}\left(\frac{pqm_2w^2}{(m_1e+m_2w)^2}-c\right)-\lambda_3\left(\frac{qm_2w^2}{(m_1e+m_2w)^2}\right)$$

is switching function.

The optimal control $E(t)$ which maximizes H must satisfy

$$E(t)=\begin{cases}E_{\max},\sigma(t)>0 i.e.\lambda_3e^{\delta t}<\left(p-\dfrac{c(m_1e+m_2w)^2}{qm_2w^2}\right),\\ 0,\sigma(t)<0 i.e.\lambda_3e^{\delta t}>\left(p-\dfrac{c(m_1e+m_2w)^2}{qm_2w^2}\right).\end{cases}$$

Now, the usual shadow price is $\lambda_3e^{\delta t}$ and the net revenue on a unit harvest is $\left(p-\dfrac{c(m_1e+m_2w)^2}{qm_2w^2}\right)$ Thus, if the shadow price is less than the net economic revenue on a unit harvest, then $E=E_{\max}$, if the shadow price is greater than the net economic revenue on a unit harvest then $E=0$ and when shadow price equals the net economic revenue on a unit harvest, i.e $\sigma(t)=0$, then the Hamiltonian becomes independent of the control variable $E(t)$ i.e. $\dfrac{\partial H}{\partial E}=0$. This is the necessary condition for singular control $E^*(t)$ to be optimal over control set $0<E^*<E_{\max}$.

Hence, the optimal harvesting policy is

$$E(t)=\begin{cases} E_{\max}, \sigma(t)>0, \\ 0, \sigma(t)<0, \\ E^{*}(t), \sigma(t)=0. \end{cases} \tag{24}$$

For the singular control to be optimal, we must have $\frac{\partial H}{\partial e}=\sigma(t)=0$. This yields

$$\lambda_3=\left(p-\frac{c(m_1e+m_2w)^2}{qm_2w^2}\right)e^{-\delta t}. \tag{25}$$

According to the Pontryagin's Maximum Principle, the adjoint equations are

$$\frac{d\lambda_1}{dt}=-\frac{\partial H}{\partial P}, \frac{d\lambda_2}{dt}=-\frac{\partial H}{\partial Z}, \frac{d\lambda_3}{dt}=-\frac{\partial H}{\partial F}. \tag{26}$$

From the first adjoint equation, we have

$$\frac{d\lambda_1}{dt}=-\lambda_1(t)\left[r-2ru-\frac{\theta v+\theta\xi v^2}{(1+\xi v+\eta u)^2}\right]-\lambda_2(t)\left[\frac{\theta_1 v+\theta_1\xi v^2}{(1+\xi v+\eta u)^2}\right].$$

Using Eq. (8), this equation becomes

$$\frac{d\lambda_1}{dt}=-\lambda_1\left[-ru+\frac{\theta\eta uv}{(1+\xi v+\eta u)^2}\right]-\lambda_2\left[\frac{\theta_1 v+\theta_1\xi v^2}{(1+\xi v+\eta u)^2}\right]. \tag{27}$$

The second adjoint equation becomes

$$\frac{d\lambda_2}{dt}=\lambda_1\frac{\theta u+\theta\eta u^2}{(1+\xi v+\eta u)^2}-\lambda_2\left[\frac{\theta_1 u+\eta\theta_1 u^2}{(1+\xi v+\eta u)^2}-c-\frac{((v^2+\gamma_1^{\ 2})^2 2wv-2wv^3)}{(v^2+\gamma_1^{\ 2})}\right]-\lambda_3\frac{2\theta' wv\gamma_1^{\ 2}}{(v^2+\gamma_1^{\ 2})^2}.$$

Using Eq. (13), this equation becomes

$$\frac{d\lambda_2}{dt}=\lambda_1\frac{\theta u+\theta\eta u^2}{(1+\xi v+\eta u)^2}-\lambda_2\left[\frac{-\theta_1 u\xi v}{(1+\xi v+\eta u)^2}+\frac{3wv^3+wv\gamma_1^{\ 2}-2wv^5-2wv\gamma_1^{\ 4}-4wv^3\gamma_1^{\ 2}}{(v^2+\gamma_1^{\ 2})^2}\right]$$
$$-\lambda_3\frac{2\theta' wv\gamma_1^{\ 2}}{(v^2+\gamma_1^{\ 2})^2}. \tag{28}$$

The third adjoint equation becomes

$$\frac{d\lambda_3}{dt}=-\left[e^{-\delta t}\left(\frac{pqm_1e}{(m_1e+m_2w)^2}\right)\right]+\lambda_2\left(\frac{v^2}{v^2+\gamma_1^{\ 2}}\right)-\lambda_3\left[r_1-\frac{2r_1w}{K}+\frac{\theta' v^2}{v^2+\gamma_1^{\ 2}}-\frac{m_1qe^2}{(m_1e+m_2w)^2}\right].$$

Using Eq.(10), this equation becomes

$$\frac{d\lambda_3}{dt}=-\left[e^{-\delta t}\left(\frac{pqm_1e}{(m_1e+m_2w)^2}\right)\right]+\lambda_2\left(\frac{v^2}{v^2+\gamma_1^{\ 2}}\right)-\lambda_3\left[-\frac{r_1w}{K}+\frac{m_2qew}{(m_1e+m_2w)^2}\right]. \tag{29}$$

Differentiating Eq. (26), with respect to t, we have

$$\frac{d\lambda_3}{dt}=-\delta\lambda_3. \tag{30}$$

Using Eq. (28) in Eq. (27), we obtain

$$\lambda_2 = B_1 e^{-\delta t}, \tag{31}$$

where

$$B_1 = -\frac{\gamma_1^{\ 2} + v^2}{v^2}\left[\left(\delta + \frac{r_1 w}{K} - \frac{q m_2 e w}{(m_1 e + m_2 w)^2}\right)\left(p - \frac{c(m_1 e + m_2 w)^2}{q m_2 w^2}\right) - \frac{p q m_1 e}{(m_1 e + m_2 w)^2}\right] e^{-\delta t}.$$

Putting the value of λ_2 from Eq. (31) in Eq. (28), we find

$$\frac{d\lambda_1}{dt} - B_2 \lambda_1 = -B_3 e^{-\delta t}, \tag{32}$$

where $B_3 = \left[\dfrac{B_1(\theta_1 v + \theta_1 \xi v^2)}{(1+\xi v + \eta u)^2}\right], B_2 = -\left[-ru + \dfrac{\theta \eta u v}{(1+\xi v + \eta u)^2}\right].$

solving Eq. (31), we obtain

$$\lambda_1 = \frac{B_3}{B_2 + \delta} e^{-\delta t} + k_0 e^{B_2 t}. \tag{33}$$

When $t \to \infty$, then shadow price $\lambda_1 e^{\delta t}$ is bounded iff $k_0 = 0$. Hence, we have $\lambda_1 = \dfrac{B_3}{B_2 + \delta} e^{-\delta t}$.

Putting the values of λ_1 and λ_3 in Eq. (28), we obtain when $t \to \infty$, then shadow price $\lambda_1 e^{\delta t}$ is bounded if $k_0 = 0$. Hence, we have

$$\lambda_1 = \frac{B_3}{B_2 + \delta} e^{-\delta t}. \tag{34}$$

Substituting the values of λ_1 and λ_3 in Eq. (27), we obtain

$$\frac{d\lambda_2}{dt} - B_4 \lambda_2 = -B_5 e^{-\delta t}, \tag{35}$$

where $B_4 = \left[\dfrac{\theta_1 u \xi v}{(1+\xi v + \eta u)^2} - \dfrac{3wv^3 + wv\gamma_1^{\ 3} - 2wv^5 - 2wv\gamma_1^{\ 4} - 4wv^3\gamma_1^{\ 2}}{(v^2 + \gamma_1^{\ 2})}\right],$

$$B_5 = \left(p - \frac{c(m_1 e + m_2 w)^2}{q m_2 w^2}\right)\frac{2\theta' wv\gamma_1^{\ 2}}{(v^2 + \gamma_1^{\ 2})^2} - \frac{B_3(\theta u + \theta \eta u^2)}{(B_2 + \delta)(1+\xi v + \eta u)^2}.$$

Solving Eq. (35), we obtain

$$\lambda_2 = \frac{B_5 e^{-\delta t}}{B_4 + \delta} + k_1 e^{B_4 t}.$$

When $t \to \infty$, then shadow price $\lambda_2 e^{\delta t}$ is bounded iff $k_1 = 0$. Hence, we have

$$\lambda_2 = \frac{B_5 e^{-\delta t}}{B_4 + \delta}. \tag{36}$$

Hence from Eqs.(32) and (37), we have

$$B_1 = \frac{B_5}{B_4 + \delta}. \tag{37}$$

By solving Eq. (37) we get $E = E_\delta$, where E_δ is the optimal harvesting effort. Hence by solving Eqs. (8)-(10) using Eq. (37), we find the optimal solution $(P_\delta, Z_\delta, F_\delta)$.

3 Spatial Model System

In this section, the model system is given by:

$$\frac{\partial u}{\partial t} = ru(1-u) - \frac{\theta uv}{1+\xi v+\eta u} + d_1\nabla^2 u, \tag{38}$$

$$\frac{\partial v}{\partial t} = \frac{\theta_1 uv}{1+\xi v+\eta u} - cv - \frac{wv^2}{v^2+\gamma_1{}^2} + d_2\nabla^2 v, \tag{39}$$

$$\frac{\partial w}{\partial t} = r_1 w\left(1-\frac{w}{K}\right) + \frac{\theta' wv^2}{v^2+\gamma_1{}^2} - \frac{qew}{m_1 e+m_2 w} + d_1\nabla^2 w, \tag{40}$$

with initial conditions,

$$u(x,y,0)>0, v(x,y,0)>0, w(x,y,0)>0 \qquad \text{for} \qquad (x,y)\in\Omega \tag{41}$$

and boundary condition,

$$\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = \frac{\partial w}{\partial n} = 0 \qquad \text{for} \qquad (x,y)\in\Omega\ ,\ t>0, \tag{42}$$

where, d_1, d_2 and d_3 are diffusion coefficient for phytoplankton, zooplankton and fish populations respectively, n is the outward normal to $\partial\Omega$ and $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$.

To understand the spatial dynamics of the model system (38)-(40), we consider the linearized form of the system about $L_5(u^*, v^*, w^*)$ as follows:

$$\begin{aligned}
\frac{\partial u}{\partial t} &= a_{11}u + a_{12}v + a_{13}w + d_1\nabla^2 u,\\
\frac{\partial v}{\partial t} &= a_{21}u + a_{22}v + a_{23}w + d_2\nabla^2 v,\\
\frac{\partial w}{\partial t} &= a_{31}u + a_{32}v + a_{33}w + d_3\nabla^2 w.
\end{aligned} \tag{43}$$

where we introduce small perturbations $u = u - u^*, v = v - v^*$ and $w = w - w^*$. Let us suppose that the model system (42) has the solution which is of the form

$$\begin{pmatrix} U \\ V \\ W \end{pmatrix} = \begin{pmatrix} l_1 \\ l_2 \\ l_3 \end{pmatrix} \exp(\lambda_k t)\cos(k_x x)\cos\left(k_y y\right),$$

where l_1, l_2 and l_3 are sufficiently small constants. k_x and k_y are the components of wave number along x and y directions respectively and λ_k denotes the wavelength.

The variational matrix of the linearized model system (42) is given by

$$J = \begin{pmatrix} a_{11} - d_1 k^2 & a_{12} & 0 \\ a_{21} & a_{22} - d_2 k^2 & a_{23} \\ 0 & a_{32} & a_{33} - d_3 k^2 \end{pmatrix} \tag{44}$$

where k is the wave number given by $k^2 = k_x{}^2 + k_y{}^2$. The characteristic equation of J is given by

$$\lambda_k{}^3 + \rho_1 \lambda_k{}^2 + \rho_2 \lambda_k + \rho_3 = 0, \tag{45}$$

where $\rho_1 = (d_1 + d_2 + d_3)k^2 + A_1$,

$$\rho_2 = (d_1 d_2 + d_2 d_3 + d_3 d_1)k^4 - \left[d_1(a_{22} + a_{33}) + d_2(a_{11} + a_{33}) + d_3(a_{11} + a_{22})\right]k^2 + A_2,$$

$$\rho_3 = d_1 d_2 d_3 k^6 - [a_{11} d_2 d_3 + a_{22} d_1 d_3 + a_{33} d_1 d_2]k^4 + [d_1(a_{22}a_{33} - a_{23}a_{32}) + d_2 a_{11} a_{33} + d_3(a_{11}a_{22} - a_{12}a_{21})]k^2 + A_3$$

and A_1, A_2 and A_3 are defined in Eq. (22).

Theorem 2: The equilibrium point $L_5(u^*, v^*, w^*)$ is locally asymptotically stable in the presence of diffusion if the conditions of Theorem 1 hold.

The proof of theorem follows from Routh –Hurwitz criterion, hence omitted.

4 Conclusions

Human activity has been the major threat to wetlands. Agriculture, industrial development, and urban and suburban sprawl have caused the greatest losses of freshwater wetlands. The biggest current source of loss for fresh-water coastal wetlands is from urban sprawl. Subsidence causes the land surface to drop, which can then become flooded if the surface is already very near to sea level. Wetland management generally involves activities that can be conducted with, in, and around wetlands, both natural and man-made. Two major facets of managing wetlands for protection include buffering wetlands from direct human pressures, and maintaining natural processes in surrounding lands that affect wetlands and that may be disrupted by human activities. Our study is based on the study of system dynamics and stability of the wetland system. In this paper, we have analytically investigated a diffusive three species phytoplankton-zooplankton-fish model. The non-trivial equilibrium where all the species coexist is locally asymptotically stable under a fixed region of attraction when certain conditions are satisfied. Next, the optimal harvesting policy is discussed. The result of the model system (1) shows that a proper management of the wetland is required to control the growth of the wild grasses. So there should be taken some proper management of the wetland system by means of the comfortable survival of the species of the system and for the good economic values.

Acknowledgement

This work is supported by CCOST (C.G., India) under grant no. 2238/CCOST/MRP/2015 to the author (Nilesh Kumar Thakur).

References

Ali J, Shivaji R, Wampler K. 2009. Population models with diffusion, strong Allee effect and constant yield harvesting. Journal of Mathematical Analysis and Applications, 352: 907-913

Baek H, Jung DI, Wang ZW. 2013. Pattern formation in a semi-ratio-dependent predator prey system with diffusion. Discrete Dynamics in Nature and Society, 1-14

Chang, Wei. 2012. Hopf bifurcation and optimal control in a diffusive predator prey system with time delay and prey harvesting. Nonlinear Analysis: Modelling and Control, 17: 379-409

DeAngelis DL, Trexler JC, Cosner C, Obaza A, Jopp F. 2010. Fish population dynamics in a seasonally varying wetland. Ecological Modelling, 221: 1131-1137

Dubey B, Hussain, J. 2000. Modelling the interaction of two bio-logical species in the polluted environment. Journal of Mathematical Analysis and Applications, 246: 58-79

Dubey B, Patra A. 2013. A mathematical model for optimal management and utilization of a renewable resource by population. Journal of Mathematics, 2013: 1-9

Dubey B, Chandra P, Sinha P. 2003. A model for fishery resource with reserve area. Nonlinear Analysis Real World Applications, 4: 625-637

Dubey B, Patra A, Upadhyay RK. 2014. Dynamics of phyto-plankton, zooplankton and fishery resource. Applied Mathematics and Computation, 9: 217-245

Jorgensen SE. 2013. Handbook of Environmental Data and Ecological Parameter: Environmental Sciences and Applications. Elsevier.

Kar TK, Chaudhuri KS. 2004. Harvesting in a two-prey one-predator fishery: A bio economic model. ANZIAM Journal, 45: 443-456

Kar TK, Chattopadhyay SK, Pati CK. 2009. A bio-economic model of two-prey one predator system. Journal of Applied Mathematics and Informatics, 27: 1411-1427

Purohit D, Chaudhuri KS. 2004. A bioeconomic model of non-selective harvesting of competing fish species. ANZIAM Journal, 46(2): 299-308

Rao F. 2014. Spatio temporal complexity of a three species ratio-dependent food chain model. Nonlinear Dynamics, 76: 1661-1676

Upadhyay RK, Kumari N, Rai V. 2008. Wave of chaos and pattern formation in spatial predator prey systems with Holling type IV predator response. Mathematical Modelling of Natural Phenomena, 3: 71-95

Upadhyay RK, Thakur NK, Rai V. 2011. Diffusion-driven instabilities and spatio-temporal patterns in an aquatic predator prey system with Beddington-DeAngelis type functional response. International Journal of Bifurcation and Chaos, 21: 663-684

Upadhyay RK, Roy P, Datta J. 2015. Complex dynamics of eco-logical systems under nonlinear harvesting: Hopf bifurcation and Turing instability. Nonlinear Dynamics, 79: 2251-2270