Article

Period-doubling bifurcation and chaos control in a discrete-time mosquito model

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Abstract
This article deals with the study of some qualitative properties of a discrete-time mosquito Model. It is shown that there exists period-doubling bifurcation for wide range of bifurcation parameter for the unique positive steady-state of given system. In order to control the bifurcation we introduced a feedback strategy. For further confirmation of complexity and chaotic behavior largest Lyapunov exponents are plotted.

Keywords mosquito model; period-doubling bifurcation; hybrid control.

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1 Introduction

Mosquito vector-borne diseases, such as West Nil, dengue fever and malaria spread between humans is a cause of blood-feeding mosquitoes, have been big fears for the public health in the world. The world’s most important tropical parasitic disease is the Malaria. In the whole world it is the fifth leading cause of death due to infectious diseases worldwide (after respiratory infections, HIV/AIDS, diarrheal diseases, and tuberculosis), and in Africa it is the 2nd leading cause of death after HIV/AIDS. At least, 109 countries suffering today by cause of such infectious diseases and nearby half of the world at risk of malaria. Particularly, a brief data of victims of Mosquito vector-borne diseases can be found in (CDC, 2010; WHO, 2010). To prevent and control malaria, an effective weapon provides the transgenic or genetically-altered mosquitoes (Liu and Xiao, 2007; Ito et al., 2002; Lycett and Kafatos, 2002). However, in the laboratories transgenic mosquitoes produces that resistant and can be ultimately introduced into the field, there still exist significant hurdles, including multiple mosquito subspecies, determination mechanisms, transposon stability and, that essential to be overawed (Riehle and Jacobs-Lorena, 2005; Riehle at el., 2007). No vaccines are available for these mosquito-borne diseases. An actual way to prevent diseases spread due to mosquito is to control mosquitoes. We need a better understanding about the dynamics of mosquito population to obtain a suitable optimal strategy, hence to formulate a mathematical model for mosquito dynamic. There are several modeling mechanism on mosquitoes population have been seen in the field of mathematics (Anderson and May, 1991; Li, 2008; MacDonald, 1957).

The dispersal of mosquito's species, for searching the resources of their survival and to reproduce their new generation is an interesting and impartment bio-mathematical process in a malarial endemic region. The growth of such species depend on several factors, such as homogeneous environment, constant re-productivity and carrying capacity. In order to discuss the behavior of mosquitos population, the general equation for mosquitos population is given by:

$$x_{n+1} = f(x_n)g(x_n)x_n, \qquad (1)$$

where the number of wild mosquitoes at n is x_n, the birth function for per-capita offspring is given by f and the fraction of offspring that survive is denoted by g (survival probability). Li (2008) assumed the Ricker-type nonlinearity $g(x_n) = e^{-d-kx_n}$, and discuss the qualitative analysis of the following mosquitos population model:

$$\begin{aligned} x_{n+1} &= ry_n e^{-k_1 x_n}, \\ y_{n+1} &= \beta x_n e^{-k_2 x_n}, \end{aligned} \qquad (2)$$

where the per capita birth rate of adults is assumed to be $r > 0$, and $k_1 > 0$, $k_2 > 0$ denote the carrying capacity of larva and adult, respectively, moreover, the death rate from larva to adult stage is taken as the constant $\beta > 0$. Grove et al., (2000), proposed a mosquitos population model given by the following second-order exponential difference equation:

$$x_{n+1} = (ax_n + bx_{n-1}e^{-x_{n-1}})e^{-x_n}, \; n = 0,1,2,\cdots, \qquad (3)$$

where $a \in (0,1)$ and $b \in [0,\infty)$. Taking $x_{n-1} = y_n$, model (3) can be converted into the following equivalent two-dimensional form:

$$\begin{aligned} x_{n+1} &= (ax_n + by_n e^{-y_n})e^{-x_n}, \\ y_{n+1} &= x_n, \end{aligned} \qquad (4)$$

The model (3) describes the growth of mosquito population at time n. Mosquito lays eggs and hatch them, due to un-favorable satiation some of the eggs remain dormant for one or more years. In this model the dormancy period is assumed to be one year at most. The global attraction of positive steady-state is discussed in Grove et al. (2000). Our aim in this paper is to explore the chaotic nature of system under consideration. Moreover, it is proved that model undergoes period-doubling bifurcation for unique positive steady-state for large range of bifurcation parameter and exhibits complex chaotic nature. In particular, we study not only the period-doubling bifurcation but we alsointroduced a feedback strategy in order to control the bifurcation.

2 Existence of Period-Doubling Bifurcation

In this section, we investigate the existence of equilibrium and the period doubling bifurcation. It is trivial to obtain the coupled algebraic equations from (4):

$$\begin{aligned} x_* &= (ax_* + by_* e^{-y_*})e^{-x_*}, \\ y_* &= x_*, \end{aligned}$$

One can easily obtain the following positive equilibrium point by simply neglecting the trivial fixed point (0,0).

$$(x_*, y_*) = \left(\ln\left(\frac{a+\sqrt{a^2+4\mathrm{b}}}{2}\right), \ln\left(\frac{a+\sqrt{a^2+4b}}{2}\right) \right).$$

Let

$$F_J(x_*, y_*) = \begin{bmatrix} a\mathrm{e}^{-x_*} - x_* & (1-x_*)b\mathrm{e}^{-2\,x_*} \\ 1 & 0 \end{bmatrix}$$

be the Jacobian matrix evaluated at (x_*, y_*). In order to investigate the stability we have the following theorem;

Theorem 2.1 Let $\mathbb{F}(\lambda) = \lambda^2 - \mathrm{A}_1\lambda + \mathrm{A}_2$, and $\mathbb{F}(1) > 0$ withλ_1, λ_2 are roots of $\mathbb{F}(\lambda) = 0$, then:

(i) $|\lambda_1| < 1$ and $|\lambda_2| < 1$if and only if $\mathbb{F}(-1) > 0$ and$\mathbb{F}(0) < 1$;

(ii) $|\lambda_1| < 1$ and $|\lambda_2| > 1$ *or (*$|\lambda_1| > 1$and $|\lambda_2 < |1$*)* if and only if $\mathbb{F}(-1) < 0$;

(iii) $|\lambda_1| > 1$ and $|\lambda_2| > 1$if and only if $\mathbb{F}(-1) > 0$ and $\mathbb{F}(0) > 1$;

(iv) $\lambda_1 = -1$ and $\lambda_2 \neq 1$ if and only if $\mathbb{F}(-1) = 0$ and $\mathbb{F}(0) \neq \pm 1$;

(*v*) λ_1 and λ_2 are complex and $|\lambda_1| = 1$ and $|\lambda_2| = 1$ if and only if ${A_1}^2 - 4A_2 < 0$ and $\mathbb{F}(0) = 1$.

The characteristic polynomial of Jacobian matrix evaluated at positive equilibrium point (x_*, y_*) is given by;

$$\mathbb{F}(\lambda) = \lambda^2 - (ae^{x_*} - x_*)\lambda + be^{-2x_*}(x_* - 1). \qquad (5)$$

Take into account, $\mathbb{F}(1) > 0$ holds trivially. Moreover,

$$\mathbb{F}(-1) = 1 + ae^{-x_*} + be^{-2x_*}(x_* - 1) - x_* \text{and } \mathbb{F}(0) = be^{-2x_*}(x_* - 1).$$

Thus by applying Theorem 2.1 we obtained the following dynamic of system (4).
Lemma 2.1 Let $(\mathrm{x}_*, \mathrm{y}_*)$ be the unique positive fixed point of system (4) then;
(i) $(\mathrm{x}_*, \mathrm{y}_*)$ is sink if one of the the following condition hold true:
$1 - a < b < e^2(e^2 - a)$, and $0 < b < \frac{e^{2x_*}}{x_* - 1}$,

(ii) (x_*, y_*) is source if the following condition hold true:

$1 - a < b < e^2(e^2 - a)$ and $b > \frac{e^{2x_*}}{x_* - 1}$.

(iii) (x_*, y_*) is saddle point if the following condition hold true:

$$b > e^2(e^2 - a).$$

(iv) The equilibrium population (x_*, y_*) of (4) is non-hyperbolic if the following conditions hold true.

$b = e^2(e^2 - a), b \neq \frac{e^{2x_*}}{x_* - 1}$, and $b \neq \frac{e^{2x_*}}{1 - x_*}$.

The main objective of study is to discuss the period-doubling bifurcation of the unique positive equilibrium point (x_*, y_*) of system (4) by using the technique of bifurcation theory and taking b as bifurcation parameter. Similar results are found in (Liu and Xiao, 2007; Jing and Yang, 2006). By using Theorem (2.1) we have the following results.

Assume that

$$T_S = \{(a, b): b = e^2(e^2 - a),\ 0 < a < 1,\ b > 0\ \ and\ A_2 \neq \pm 1\}. \qquad (6)$$

The unique positive equilibrium point of system (4) undergoes period doubling bifurcation when parameters vary in a small neighborhood of T_{NS}. Choosing the parameters (a, b_1) in an arbitrary fashion from the set T_{NS}, we consider system (4) with parameters (a, b), which is described by the following map:

$$\binom{x}{y} \to \binom{(ax + bye^{-y})e^{-x}}{x}. \qquad (7)$$

It is easy to see that map (7) has a unique positive fixed point (x_*, y_*) such that (x_*, y_*) is unique positive equilibrium point of system (4). Since $(a, c) \in T_N S$ and $b_1 << 1$. Taking b_1 as bifurcation parameter and considering the perturbation of (7) as follows:

$$\binom{x}{y} \to \binom{\left(ax + (b_1 + \tilde{b})ye^{-y}\right)e^{-x}}{x}. \qquad (8)$$

Next, we assume the transformations $N = x - x_*,\ P = y - y_*$ so that map (8) is transferred into the following form:

$$\binom{N}{P} \to \begin{pmatrix} a_{11} & a_{12} \\ 1 & 0 \end{pmatrix}\binom{N}{P} + \binom{f(N, P, \tilde{b})}{0}, \qquad (9)$$

where

$$f(N, P, \tilde{b}) = a_{13}N^2 + a_{14}NP + a_{15}P^2 + c_1N^3 + c_2N^2P + c_3P^2N + c_4P^3 + e_1\tilde{b}N + e_2\tilde{b}P$$

$$+e_3\tilde{b}N^2 + e_4\tilde{b}NP + e_5\tilde{b}P^2 + e_6\tilde{b}^2N + e_7\tilde{b}^2P + e_8\tilde{b}^2 + e_9\tilde{b}^3 + O\left((|N| + |P| + |\tilde{b}|)^4\right),$$

$$a_{11} = -\frac{a\ln(K)K - aK + b_1\ln(K)}{K^2},\ a_{12} = -\frac{b_1(\ln(K) - 1)}{K^2} a_{13} = \frac{b_1\left(-2 + \ln(K)\right)}{2K^2}, \qquad (10)$$

$$a_{14} = \frac{b_1(\ln(K) - 1)}{K^2}, a_{15} = \frac{b_1\big(-2 + \ln(K)\big)}{2K^2},$$

$$c_1 = -\frac{a\ln(K)K - 3\,aK + b_1\ln(K)}{6K^2}, c_2 = -\frac{b_1(\ln(K) - 1)}{2K^2},$$

$$c_3 = -\frac{b_1\big(-2+\ln(K)\big)}{2K^2},\ c_4 = -\frac{b_1\big(-3+\ln(K)\big)}{6K^2},\ e_1 = -\frac{\ln(K)\sqrt{B}K - a\ln(K)K + 2\,aK - 2\,b_1\ln(K) + b_1}{\sqrt{B}K^3}$$

$$e_2 = -\frac{\ln(K)\sqrt{B}K - \sqrt{B}K - 2\,b_1\ln(K) + 3\,b_1}{\sqrt{B}K^3}, e_3 = \frac{\ln(K)\sqrt{B}K - a\ln(K)K + 3\,aK - 2\,b_1\ln(K) + b_1}{2\sqrt{B}K^3},$$

$$e_4 = \frac{\ln(K)\sqrt{B}K - \sqrt{B}K - 2\,b_1\ln(K) + 3\,b_1}{\sqrt{B}K^3}, e_5 = \frac{\ln(K)\sqrt{B}K - 2\,\sqrt{B}K - 2\,b_1\ln(K) + 5\,b_1}{2\sqrt{B}K^3},$$

$$e_6$$
$$= \frac{-2\,K^2\ln(K)a - 2\,a\ln(K)K\sqrt{B} + 4\,BK\ln(K) + 4\,K^2a - 4\,K\ln(K)b_1 + 5\,aK\sqrt{B}}{2B^{3/2}K^4}\frac{-6\,b_1\ln(K)\sqrt{B} - 2\,BK + 2\,Kb + 5\,b_1}{2B^{3/2}K^4}$$
$$= \frac{4\,BK\ln(K) - 4\,K\ln(K)b_1 - 6\,\ln(K)b_1\sqrt{B} - 6\,BK + 6\,Kb_1 + 11\,b\sqrt{B}}{2B^{3/2}K^4},$$

$$e_8 = -\frac{-2\,K^2\ln(K)a - 2\,a\ln(K)K\sqrt{B} + 4\,BK\ln(K) + 2\,K^2a - 4\,K\ln(K)b_1 + 3\,aK\sqrt{B}}{2B^{3/2}K^4}$$
$$- \frac{-6\,b_1\ln(K)\sqrt{B} - 2\,BK + 2\,Kb + 5\,b_1\sqrt{B}}{2B^{3/2}K^4}, e_9$$
$$= \frac{-12\,K^3\ln(K)a - 12\,K^2\ln(K)a\sqrt{B} + 12\,BK^2\ln(K) - 6\,BK\ln(K)a + 18\,B^{3/2}K\ln(K)}{6B^{5/2}K^5}$$

$$+ \frac{12\,K^3a - 24\,K^2\ln(K)b_1 + 18\,K^2a\sqrt{B} - 36\,K\ln(K)b_1\sqrt{B} - 6\,BK^2 + 11\,BKa}{6B^{5/2}K^5}$$

$$\frac{15\,B^{3/2}K - 24\,B\ln(K)b_1 + 12\,K^2b_1 + 30\,Kb_1\sqrt{B} + 26\,Bb_1}{6B^{5/2}K^5},$$

where

$$K = \frac{a + \sqrt{a^2 + 4b_1}}{2}, \qquad andB = a^2 + 4b_1.$$

In order to obtain the canonical form of (8) at $\tilde{b} = 0$, consider the following mapping:

$$\binom{N}{P} = \begin{pmatrix} a_{12} & a_{12} \\ -1 - a_{11} & \lambda_2 - a_{11} \end{pmatrix}\binom{u}{v}. \qquad (11)$$

Under mapping (11), the canonical form of (8) can be expressed as:

$$\binom{u}{v} \to \begin{pmatrix} -1 & 0 \\ 0 & \lambda_2 \end{pmatrix}\binom{u}{v} + \begin{pmatrix} \tilde{f}(u, v, \tilde{b}) \\ \tilde{g}(u, v, \tilde{b}) \end{pmatrix}, \qquad (12)$$

where

$$\tilde{f}(u,u,\tilde{b}) = -\frac{(-\lambda_2+a_{11})b_1N^3}{a_{12}(\lambda_2+1)} - \frac{(-\lambda_2+a_{11})b_2PN^2}{a_{12}(\lambda_2+1)} - \frac{a_{13}(-\lambda_2+a_{11})N^2}{a_{12}(\lambda_2+1)}$$

$$-\frac{(-\lambda_2+a_{11})b_3P^2N}{a_{12}(\lambda_2+1)} - \frac{a_{14}(-\lambda_2+a_{11})PN}{a_{12}(\lambda_2+1)} - \frac{(-\lambda_2+a_{11})b_4P^3}{a_{12}(\lambda_2+1)}$$

$$-\frac{a_{15}(-\lambda_2+a_{11})P^2}{a_{12}(\lambda_2+1)} - \frac{(-\lambda_2+a_{11})e_1\tilde{b}N}{a_{12}(\lambda_2+1)} - \frac{(-\lambda_2+a_{11})e_2\tilde{b}P}{a_{12}(\lambda_2+1)}$$

$$-\frac{(-\lambda_2+a_{11})\tilde{b}e_3N^2}{a_{12}(\lambda_2+1)} - \frac{(-\lambda_2+a_{11})\tilde{b}e_4PN}{a_{12}(\lambda_2+1)} - \frac{(-\lambda_2+a_{11})\tilde{b}e_5P^2}{a_{12}(\lambda_2+1)}$$

$$-\frac{(-\lambda_2+a_{11})\tilde{b}^2e_6N}{a_{12}(\lambda_2+1)} - \frac{(-\lambda_2+a_{11})\tilde{b}^2e_7P}{a_{12}(\lambda_2+1)} - \frac{(-\lambda_2+a_{11})\tilde{b}^2e_8}{a_{12}(\lambda_2+1)}$$

$$-\frac{(-\lambda_2+a_{11})\tilde{b}^3e_9}{a_{12}(\lambda_2+1)} + O\big((|u|+|v|+|\tilde{b}|)^4\big),$$

$$\tilde{g}(u,v,\tilde{b}) = \frac{(1+a_{11})b_1N^3}{a_{12}(\lambda_2+1)} + \frac{(1+a_{11})b_2PN^2}{a_{12}(\lambda_2+1)} + \frac{a_{13}(1+a_{11})N^2}{a_{12}(\lambda_2+1)}$$

$$+\frac{(1+a_{11})b_3P^2N}{a_{12}(\lambda_2+1)} + \frac{a_{14}(1+a_{11})PN}{a_{12}(\lambda_2+1)} + \frac{(1+a_{11})b_4P^3}{a_{12}(\lambda_2+1)}$$

$$+\frac{a_{15}(1+a_{11})P^2}{a_{12}(\lambda_2+1)} + \frac{(1+a_{11})e_1\tilde{b}N}{a_{12}(\lambda_2+1)} + \frac{(1+a_{11})e_2cP}{a_{12}(\lambda_2+1)}$$

$$+\frac{(1+a_{11})\tilde{b}e_3N^2}{a_{12}(\lambda_2+1)} + \frac{(1+a_{11})\tilde{b}e_4PN}{a_{12}(\lambda_2+1)} + \frac{(1+a_{11})\tilde{b}e_5P^2}{a_{12}(\lambda_2+1)}$$

$$++\frac{(1+a_{11})\tilde{b}^2e_6N}{a_{12}(\lambda_2+1)} + \frac{(1+a_{11})\tilde{b}^2e_7P}{a_{12}(\lambda_2+1)} + \frac{(1+a_{11})\tilde{b}^2e_8}{a_{12}(\lambda_2+1)}$$

$$+\frac{(1+a_{11})\tilde{b}^3e_9}{a_{12}(\lambda_2+1)} + O\big((|u|+|v|+|\tilde{b}|)^4\big).$$

$N = a_{12}(u+v)$ and $P = (-1-a_{11})N + (\lambda_2 - a_{11})P$. Let $W^c(0,0,0)$ be the center manifold of (12) evaluated at (0,0) in a small neighborhood of $\tilde{b}$ = 0, then $W^c(0,0,0)$ can be approximated as follows:

$$W^c(0,0,0) = \{(u,v,\tilde{b} \in R^3): v = M_3\tilde{b}^2 + M_2\tilde{b}u + M_1u^2 + (O|u|,|\tilde{b}|)^3\},$$

where

$$M_1 = -\frac{(1+a_{11})({a_{11}}^2a_{15} - a_{14}a_{12}a_{11} + {a_{12}}^2a_{13} + 2\,a_{11}a_{15} - a_{14}a_{12} + a_{15})}{a_{12}\left({\lambda_2}^2-1\right)}, M_2 = \frac{(1+a_{11})(e_2a_{11} - e_1a_{12} + e_2)}{a_{12}\left({\lambda_2}^2-1\right)}, M_3 = -\frac{(1+a_{11})e_8}{a_{12}\left({\lambda_2}^2-1\right)}.$$

Hence, the map restricted to the center manifold $W^c(0,0,0)$ is expressed as:

$$F: u \to -u + q_1 u^2 + q_2 u\tilde{b} + q_3 u^2 \tilde{b} + q_4 u\tilde{b}^2 + q_5 u^3 + (O|u|, |\tilde{b}|)^4,$$

where

$$q_1 = -\frac{(-\lambda_2 + a_{11})({a_{11}}^2 a_{15} - a_{14}a_{12}a_{11} + {a_{12}}^2 a_{13} + 2\, a_{11}a_{15} - a_{14}a_{12} + a_{15})}{a_{12}(\lambda_2 + 1)}, q_2 = \frac{(-\lambda_2 + a_{11})(e_2 a_{11} - e_1 a_{12} + e_2)}{a_{12}(\lambda_2 + 1)},$$

$$q_3 = \frac{(-\lambda_2 + a_{11})(2\, a_{14}M_2 a_{11} - 2\, a_{12}a_{13}M_2 - a_{14}M_2\lambda_2 - a_{12}e_3 + a_{14}M_2)}{\lambda_2 + 1}$$
$$- \frac{(-\lambda_2 + a_{11})({a_{11}}^2 e_5 - e_4 a_{12}a_{11} - e_2 M_1 a_{11} + e_2 M_1 \lambda_2 + 2\, a_{11}e_5 - e_4 a_{12} + e_5)}{a_{12}(\lambda_2 + 1)}$$
$$- \frac{(-\lambda_2 + a_{11})(2\, {a_{11}}^2 a_{15}M_2 - 2\, a_{11}a_{15}\lambda_2 M_2 + 2\, a_{11}a_{15}M_2 + e_1 M_1 a_{12} - 2\, a_{15}\lambda_2 M_2)}{a_{12}(\lambda_2 + 1)},$$

$$q_4 = \frac{(-\lambda_2 + a_{11})M_3(2\, a_{14}a_{11} - 2\, a_{12}a_{13} - a_{14}\lambda_2 + a_{14})}{\lambda_2 + 1} + \frac{(-\lambda_2 + a_{11})(e_2 M_2 a_{11} - e_1 M_2 a_{12} - e_2 M_2 \lambda_2 - e_6 a_{12})}{a_{12}(\lambda_2 + 1)}$$
$$- \frac{(-\lambda_2 + a_{11})(1 + a_{11})(2\, a_{11}a_{15}M_3 - 2\, a_{15}\lambda_2 M_3 - e_7)}{a_{12}(\lambda_2 + 1)},$$

$$q_5 = -\frac{(-\lambda_2 + a_{11})({a_{11}}^2 b_3 - a_{12}b_2 a_{11} + {a_{12}}^2 b_1 + 2\, a_{12}a_{13}M_1 + 2\, a_{11}b_3 - a_{12}b_2 + b_3)}{\lambda_2 + 1}$$
$$+ \frac{(-\lambda_2 + a_{11})({a_{11}}^3 b_4 + 2\, a_{14}M_1 a_{12}a_{11} - a_{14}M_1 a_{12}\lambda_2 + 3\, {a_{11}}^2 b_4 + a_{14}M_1 a_{12} + 3\, a_{11}b_4 + b_4)}{a_{12}(\lambda_2 + 1)} + 2\, \frac{(-\lambda_2 + a_{11})^2 a_{15}(1 + a_{11})M_1}{a_{12}(\lambda_2 + 1)}.$$

Next, we define the following two nonzero real numbers:

$$s_1 = \left(\frac{\partial^2 \tilde{f}}{\partial \mathrm{u}\, \partial \tilde{b}} + \frac{1}{2}\frac{\partial F}{\partial \tilde{b}}\frac{\partial^2 F}{\partial u^2}\right)_{(0,0)} = -\frac{(-\lambda_2 + a_{11})e_1}{\lambda_2 + 1} - \frac{(-\lambda_2 + a_{11})e_2(-1 - a_{11})}{a_{12}(\lambda_2 + 1)},$$

and

$$s_2 = \left(\frac{1}{6}\frac{\partial^3 F}{\partial u^3} + \left(\frac{1}{2}\frac{\partial^2 F}{\partial u^2}\right)^2\right)_{(0,0)} = {q_1}^2 + q_5 \neq 0.$$

The aforementioned computation surmise the following result about period-doubling bifurcation of system(4).

Theorem 2.2 If $s_2 \neq 0$ then there exists period-doubling bifurcation for

$$(x_*, y_*) = \left(\ln\left(\frac{a+\sqrt{a^2+4b}}{2}\right), \ln\left(\frac{a+\sqrt{a^2+4b}}{2}\right)\right)$$

of (4) whenever b varies in small neighborhood of b_1. Furthermore, if $s_2 < 0$, $(s_2 > 0)$ then the orbit having period-2 unstable or stable, respectively.

3 Hybrid Control of Period-Doubling Bifurcation

We consider the hybrid control strategy for controlling the period-doubling bifurcation in non-linear dynamical system (4). Such strategies are found in (Luo et al., 2004; Luo et al., 2003; Chen and Yu, 2005; ELabbasy et al., 2007). Consider the following controlled system crossposting to system (4):

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} \to \begin{pmatrix} \beta(ax_n + by_n e^{-y_n})e^{-x_n} + (1-\beta)x_n \\ \beta x_n + (1-\beta)y_n \end{pmatrix}, \qquad (13)$$

where $0 < \beta < 1$. Moreover, by suitable choice of controlled parameter β, the period-doubling bifurcation of the equilibrium point (x_*, y_*) of controlled system (13) can be advanced (delayed) or even completely eliminated. The original system (4) and the crossposting controlled (13) has the same fixed point, the variational matrix at positive fixed point (x_*, y_*) of controlled system can be written as:

$$\begin{bmatrix} \frac{2b + a\sqrt{a^2+4b}\beta - \beta\left(a^2 + 2b + 2b\ln\left[\frac{1}{2}\left(a+\sqrt{a^2+4b}\right)\right]\right)}{2b} & \frac{4b\beta\left(1+\ln[2]-\ln\left[a+\sqrt{a^2+4b}\right]\right)}{\left(a+\sqrt{a^2+4b}\right)^2} \\ \beta & 1-\beta \end{bmatrix}.$$

The following result gives conditions for local asymptotic stability of positive equilibrium (x_*, y_*) of the controlled system (13).

Theorem 3.1 The equilibrium population (x_*, y_*) of control system (13) is locally asymptotically stable iff the following condition hold.

$$\left|\frac{4b + a\sqrt{a^2+4b}\beta - \beta\left(a^2 + 4b + 2b\ln\left[\frac{1}{2}\left(a+\sqrt{a^2+4b}\right)\right]\right)}{2b}\right| < 1 +$$

$$\left(\frac{2\left(-2b + a^2(-1+\beta) + a\sqrt{a^2+4b}(-1+\beta) + 4b\beta\right)\left(-1+\beta\ln\left[\frac{1}{2}\left(a+\sqrt{a^2+4b}\right)\right]\right)}{\left(a+\sqrt{a^2+4b}\right)^2}\right) < 2.$$

4 Numerical Simulation and Discussion

In this section we prove the correctness of above mathematical analysis and visualize the further complex and interesting behavior of system (4). For this we consider the following special cases of (4).

Example 4.1 Assuming the parameters$a = 0.5$, $b \in [1,300]$ and initial conditions $(x_0, y_0) = (2.512424567, 2.512424567)$. Then both the population undergoesperiod-doubling bifurcation (see Fig. 1 and Fig. 2). Furthermore, MLE is shown in Fig. 3 which conform the existence of chaotic behavior.

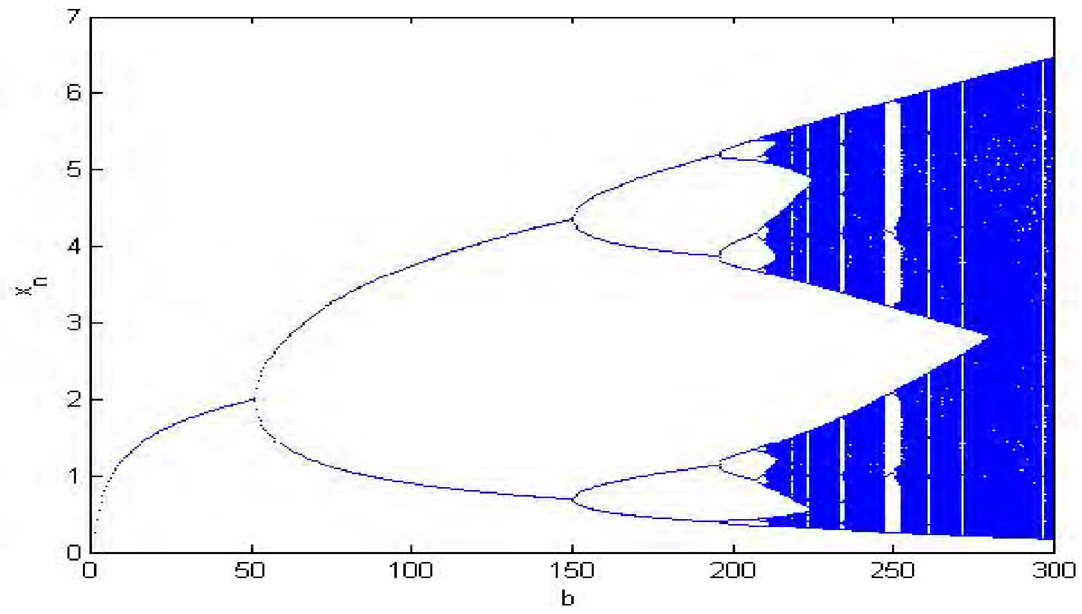

Fig. 1 Bifurcation diagram of x_n for system (4).

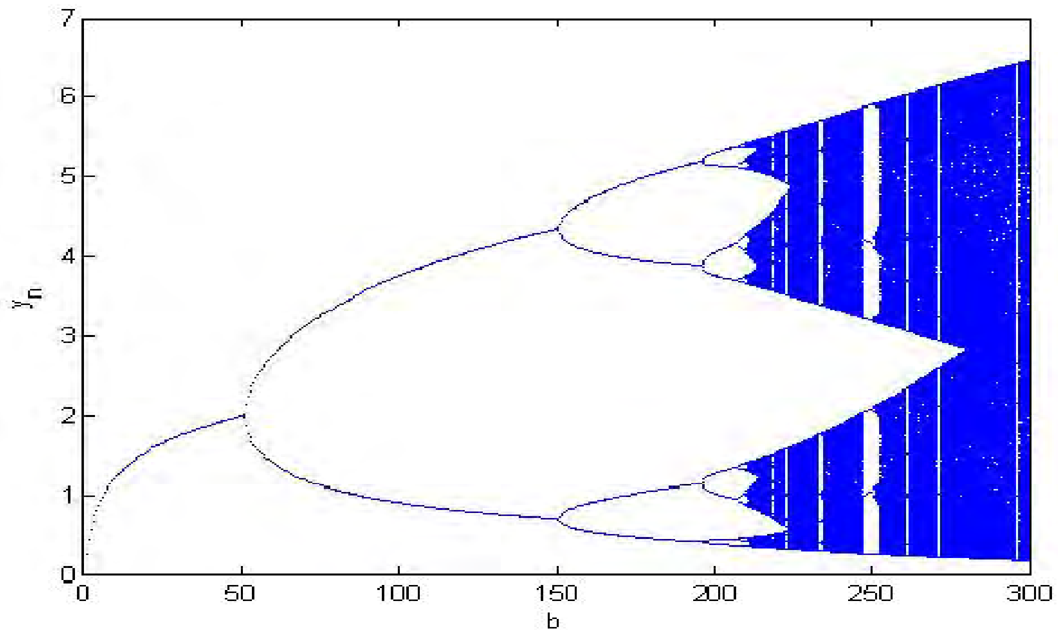

Fig. 2 Bifurcation diagram of y_n for system (4).

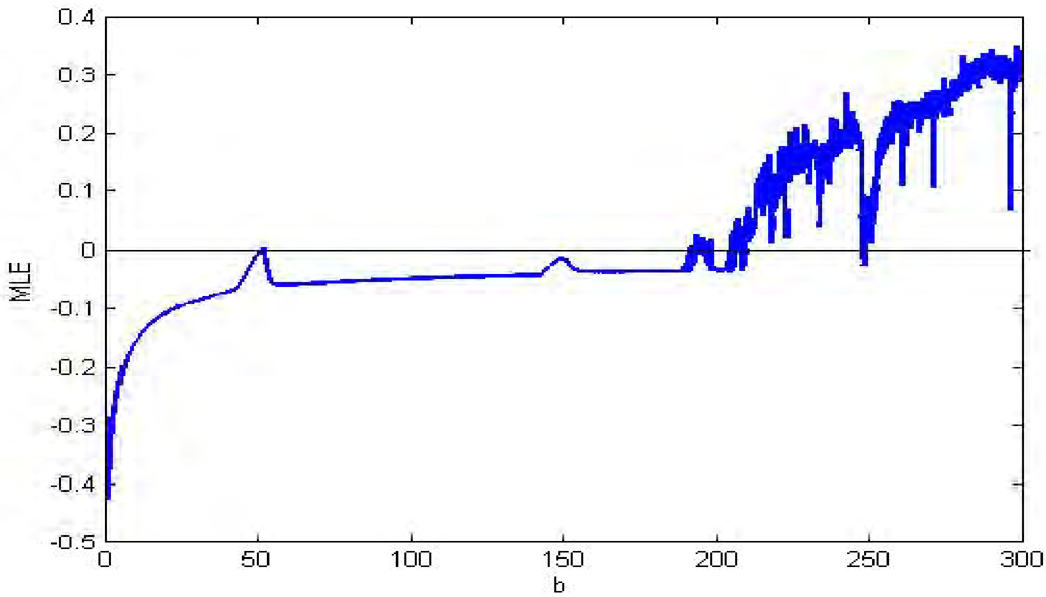

Fig. 3 Maximum Lypunov exponents for system (4).

Example 4.2 Assuming the parameters $a = 0.5$, $b = 53$. In this case the unique positive equilibrium $(x_*, y_*) = (2.019479352, 2.019479352)$, which is unstable and undergoes period-doubling bifurcation (see Fig. 1and Fig. 2). In order to control the bifurcation one has the following control system;

$$\begin{aligned} x_{n+1} &= \beta(0.5x_n + 53y_n e^{-y_n})e^{-x_n} + (1-\beta)x_n, \\ y_{n+1} &= \beta x_n + (1-\beta)y_n. \end{aligned} \tag{14}$$

Thus, the controlled system (13) has has a fixed point $(x_*, y_*) = (2.019479352, 2.019479352)$ which is similar to the un-control system (4). Moreover, the variational matrix evaluated at fixed point $(x_*, y_*) = (2.019479352, 2.019479352)$ is given by;

$$\begin{bmatrix} -2.953117078\,\beta + 1 & -0.9518243829\,\beta \\ \beta & 1-\beta \end{bmatrix}.$$

Furthermore,the characteristic equation of variational matrix is given by;

$$\lambda^2 + (3.953117078\,\beta - 2)\lambda + 1 - 3.953117078\,\beta + 3.904941461\,\beta^2 = 0. \tag{15}$$

According to Theorem 3.1 the control system is locally asymptotically stable if $0 < \beta < 0.990354$. Moreover, one can see that the bifurcation control for $0 < \beta < 0.990354$ (see Fig. 4 and Fig. 5).

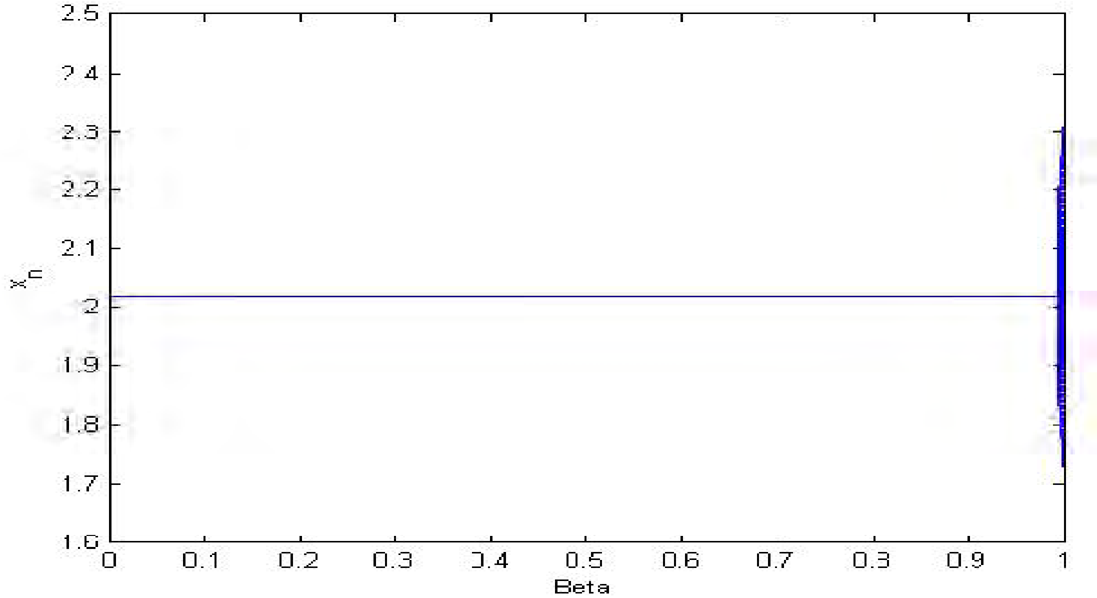

Fig. 4 Bifurcation diagram of x_n for system (14).

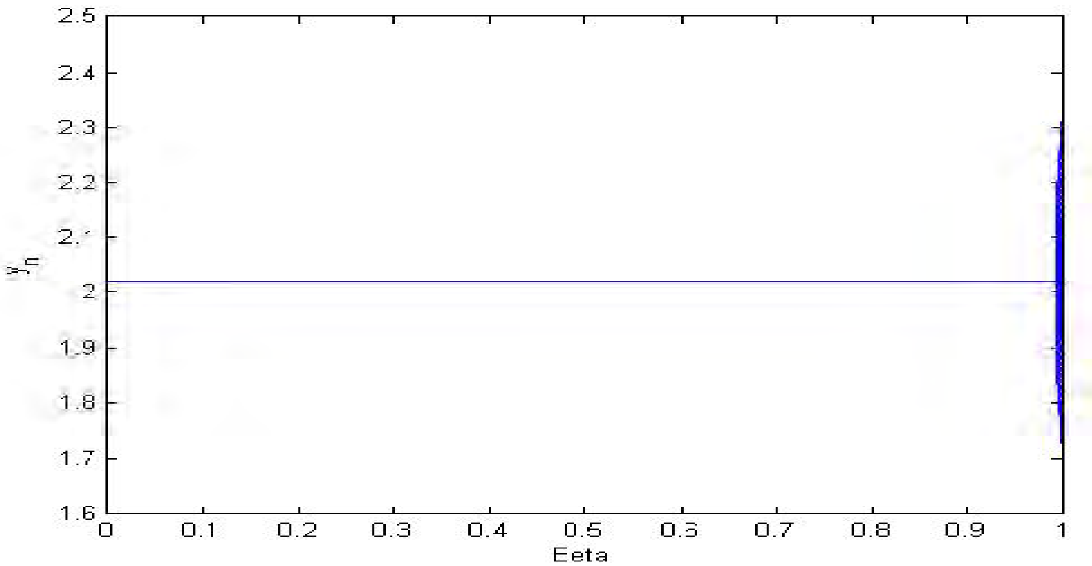

Fig. 5 Bifurcation diagram of y_n for system (14).

In addition, a local amplification for control system (13) corresponding to Fig. 4 and Fig. 5 is shown in Fig. 6 and Fig. 7 which shows that population bifurcate for least range of control parameter β.

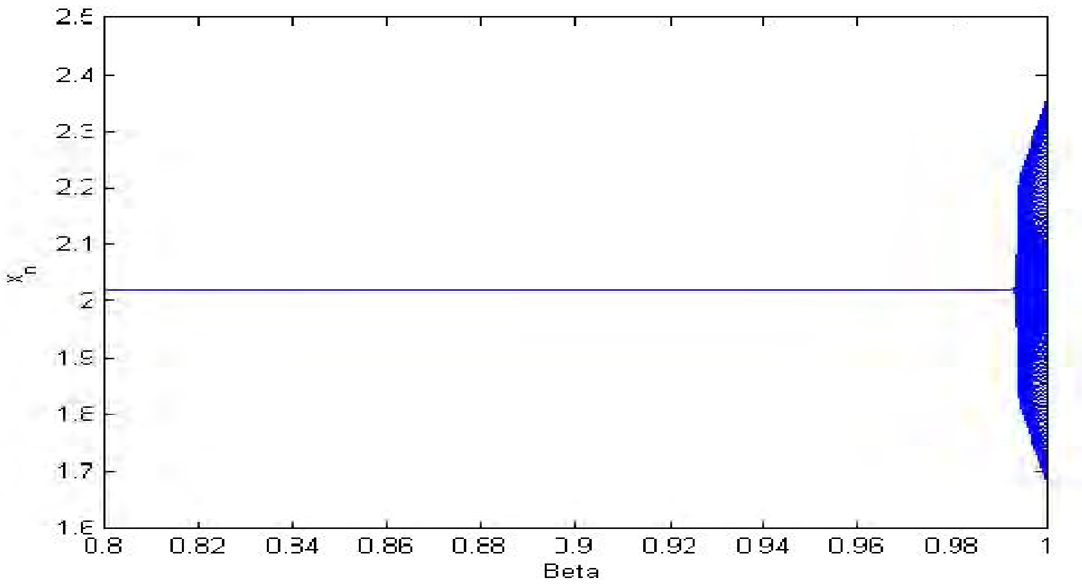

Fig. 6 Local amplification corresponding to Fig. 4.

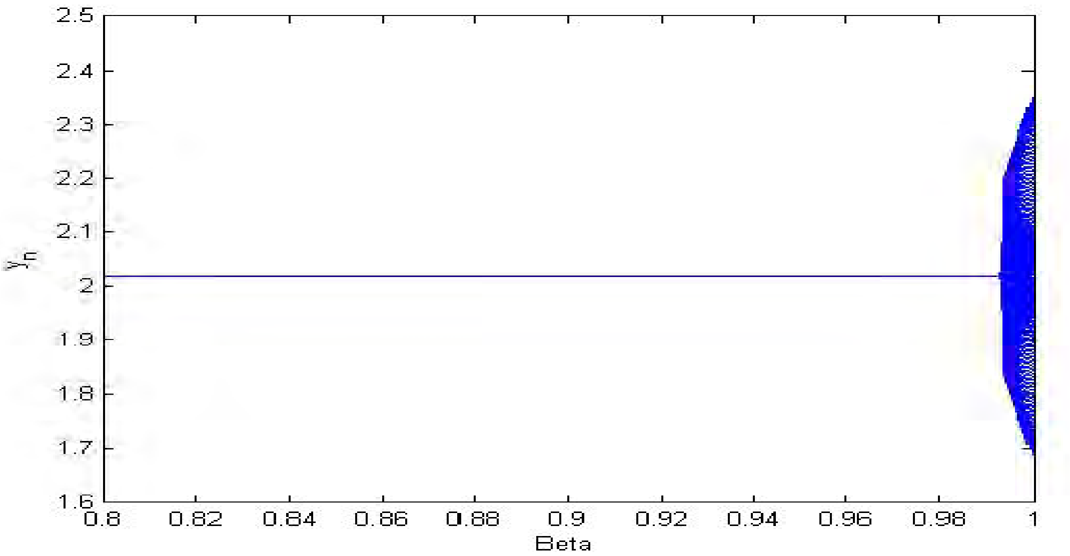

Fig. 7 Local amplification corresponding to Fig. 5.

5 Conclusion

We explore the dynamics of a discrete-time mosquito population model, and discuss the non-hyperbolic case for the unique positive fixed point. We investigate the parametric conditions for the existence and direction of period-doubling at uinque positive steady-state by implementing center manifold theorem and theory of bifurcation. The existence of period-doubling bifurcation is found for wide range of bifurcation parameter b. Moreover, it is investigated that the effects of period-doubling bifurcation can be controlled through implementation of feedback strategy, which is known as hybrid control. For some recent results related to feedback control strategies, we can refer to Din (2017a, b, c, d), Din and Saeed (2017).

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