Article

A simple mathematical model of mosquito's dynamics at stationary environmental conditions

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Abstract
Mathematical model of population dynamics with two types of individuals (mosquitoes which are malaria transmission vectors, and mosquitoes which are not transmission vectors) is under consideration. Some of basic properties of model were determined. Numerical analysis allowed obtaining typical dynamic regime.

Keywords malaria model; equations with time lag.

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1 Introduction

In literature it is possible to find a lot of various publications which are devoted to problem of modeling and analysis of spread of malaria in human population (see, for example, Ross, 1915; Kermack and McKendrick, 1922; Mandal et al., 2011; Macdonald, 1957, and others). Diversity of publications is determined by goals of investigations, used mathematical approaches, and methods. One group of models can be characterized by small number of variables and based on systems of ordinary differential equations (Ross, 1915, 1916a, b; Ross and Hudson, 1916; Anderson and May, 1991; Macdonald, 1957). Some other models contain big number of variables which describe dynamics of local human population and mosquito's population (with different specification of life cycle of mosquitoes; Koella and Antia, 2003; Depinay, 2004). Such models have a lot of various problems: first of all, this is a problem of determination of model parameters, and, the second, necessity to use huge volume of empirical information. The third problem is following: we have to be sure that there are no possibilities to give sufficient fitting of time series with models which contain smaller number of variables (Nedorezov, 2012; Nedorezov and Utyupin, 2011). Anyway, if we have model which gives good fitting of time series, we can try to decrease number of variables, unknown parameters etc.

Some of considered models cannot be used for the explanation of observed time series and can be used for constructing of forecasts (see, for example, Midekisa, 2012). Note, that autoregressive models (asymmetric filters) are rather popular in analysis and modeling of population dynamics (Isaev, 1984, 2001). It is important to note that such kind of models can be effectively applied to analysis of population if and only if we have no information about population ecology, life cycle, interaction with other components of ecosystem and so on.

In current publication we consider simplest mathematical model which describes mosquito's population dynamics. It is assumed that population contains of two different groups of mosquitoes: mosquitoes which are vectors of malaria, belong to first group; other mosquitoes belong to second group. It is also assumed that human population which can be attacked by mosquitoes contains of individuals which have immunity against malaria, and disabled individuals only (population size is equal to constant and number of disabled individuals is greater than zero). These assumptions allow assuming some parameters of model equal to constant, and concentrate view onto population dynamics.

2 Description of Model

Let's assume that there is a sufficient small pool (it will allow excluding the spatial structure of population, and analyze the model of local dynamics) where the mosquito's larvae have normal conditions for development. Let $x(t)$ be a number of non-vector mosquitoes in a pool at time t, and $y(t)$ be a number of vector mosquitoes. As it was pointed out above human population contains of individuals which have immunity against malaria, and disabled individuals only (population size is equal to constant and number of disabled individuals is greater than zero; it is obvious that if number of disabled individuals is equal to zero then asymptotically $y(t) \to 0$).

Within the framework of model we'll assume that the following processes determine the changing of $x(t)$ in time: appearance of new individuals with the rate $\alpha_1 x + \alpha_2^0 y$ where α_1 and α_2^0 are Malthusian parameters for non-malaria mosquitoes and vectors respectively. In other words, it is assumed that both types of mosquitoes can produce non-malaria mosquitoes only, and vectors can appear in population in a result of interaction between normal mosquitoes and sick people only. Coefficient α_1 is equal to the difference between intensity of birth rate and intensity of death rate plus intensity of migration of individuals out of a pool. Additional death rate (in a result of influence of intra-population self-regulative mechanisms) is equal to $\beta_1 x(x+y)$. Immigration of insects which had interactions with healthy persons only, has the rate $p_1^0 x(t-\tau)$, where $p_1^0 = const > 0$ is a positive coefficient; positive parameter τ is equal to average time of absence of mosquitoes out of the pool, $\tau = const > 0$.

Finally, we have the following equation for the description of non-malaria mosquito's dynamics in a pool:

$$\frac{dx}{dt} = \alpha_1 x + \alpha_2^0 y - \beta_1 x(x+y) + p_1^0 x(t-\tau). \tag{1}$$

Note, that Malthusian parameters α_1 and α_2^0 in (1) are different, $\alpha_1 \neq \alpha_2^0$. Parameter α_2^0 is an intensity of reproduction rate of malaria mosquitoes; the value of parameter α_1 depends on the intensities of three various processes: it is intensity of birth rate, minus intensity of death rate of individuals, and minus intensity of emigration of mosquitoes out of the pool. It is obvious, if intensities of productivities for malaria and non-malaria mosquitoes are equal, then we have the following inequality for these parameters: $\alpha_1 < \alpha_2^0$.

Let's assume that following processes have a strong influence onto the dynamics of malaria mosquitoes in a pool: immigration with intensity p_1^1 (in a result of interaction between non-malaria mosquitoes with sick persons); natural death with intensity γ_0 and emigration of mosquitoes out of the pool with intensity γ_1; additional death rate in a result of influence of intra-population self-regulative mechanisms with the rate $\beta_2 y(x+y)$ (the assumption that there is the equality between self-regulation coefficients for both types of mosquitoes, $\beta_1 = \beta_2$, can't be excluded), and immigration with intensity p_2 (returning back of malaria mosquitoes). Finally, we have the following equation for the description of malaria mosquito's dynamics in a pool:

$$\frac{dy}{dt} = p_1^1 x(t-\tau) - \alpha_2^1 y - \beta_2 y(x+y) + p_2 y(t-\tau)\,, \qquad (2)$$

where parameter $\alpha_2^1 = \gamma_0 + \gamma_1$.

Taking into account that the number of emigrated non-malaria mosquitoes can't be less than the sum of malaria and non-malaria mosquitoes returning back, we have the following relation between coefficients of model (1)-(2):

$$\alpha_1 = \alpha_1^0 - \alpha_1^1,\ p_1^0 + p_1^1 < \alpha_1^1, \qquad (3)$$

where α_1^0 is Malthusian parameter for non-malaria mosquitoes and α_1^1 is an intensity of emigration process (for non-malaria mosquitoes too). Like for coefficients of self-regulation, we can't exclude the assumption that coefficients α_1^1 and γ_1 are equal, $\gamma_1 = \alpha_1^1$.

Also there is the following inequality for coefficients in equation (2):

$$p_2 < \gamma_1, \qquad (4)$$

because the number of emigrated insects (malaria mosquitoes) can't be bigger than the number of immigrated individuals (during this time interval part of mosquitoes die).

Combining the equations (1) and (2) we obtain the required model. For complete the model we have to point out the initial non-negative functions of mosquitoes dynamics of both types on the time interval $[-\tau,0]$: $x(t) = x_0(t) \ge 0$, $y(t) = y_0(t) \ge 0$.

3 Properties of Model (1)-(2)

1. If at any time moment t_1 the number of non-malaria mosquitoes is equal to zero, $x(t_1) = 0$, for all time moments $t < t_1$ we have $x(t) > 0$, and the number of malaria mosquitoes is greater or equal zero, $y(t_1) \ge 0$, then the following relation is realized:

$$\left.\frac{dx}{dt}\right|_{t=t_1} = \alpha_2^0 y + p_1^0 x(t-\tau) > 0\,.$$

It means that for every sufficient small $\Delta t > 0$ we have $x(t_1 + \Delta t) > 0$.

If at any time moment t_1 the number of malaria mosquitoes is equal to zero, $y(t_1) = 0$, for all time moments $t < t_1$ we have $y(t) > 0$, and the number of non-malaria mosquitoes is greater or equal zero, $x(t_1) \ge 0$, then the following relation is realized:

$$\left.\frac{dy}{dt}\right|_{t=t_1} = p_1^1 x(t-\tau) + p_2 y(t-\tau) > 0\,.$$

Realizations of these inequalities mean that for all non-negative initial functions solutions of model (1)-(2) are non-negative too.

2. It is naturally to assume that there exist two various values r_0^x and r_0^y and for all values $t \in [-\tau,0]$ the following inequalities are realized:

$$x(t) \le r_0^x,\ y(t) \le r_0^y\,.$$

For next time interval $[0,\tau]$ we have the following relation:

$$\frac{dy}{dt} = p_1^1 x(t-\tau) - \alpha_2^1 y - \beta_2 y(x+y) + p_2 y(t-\tau) < p_1^1 r_0^x + p_2 r_0^y - \alpha_2^1 y - \beta_2 y^2\,.$$

It means that if y is greater than

$$r_1^y = \frac{-\alpha_2^1 + \sqrt{(\alpha_2^1)^2 + 4\beta_2 (p_1^1 r_0^x + p_2 r_0^y)}}{2\beta_2},$$

then the rate of changing of number of malaria mosquitoes becomes negative. Denote as r_k^x and r_k^y the upper limits for $x(t)$ and $y(t)$ respectively on the time interval $\delta_k = [(k-1)\tau, k\tau]$. Then the last relation can be presented in the form:

$$r_{k+1}^y = \frac{-\alpha_2^1 + \sqrt{(\alpha_2^1)^2 + 4\beta_2(p_1^1 r_k^x + p_2 r_k^y)}}{2\beta_2}.$$

Let's consider the situation when

$$x \ge \frac{\alpha_2^0}{\beta_1}.$$

If the last inequality is realized then we have the following inequality:

$$\frac{dx}{dt} = \alpha_1 x + \alpha_2^0 y - \beta_1 x(x+y) + p_1^0 x(t-\tau) \le \alpha_1 x - \beta_1 x^2 + p_1^0 r_0^x.$$

Consequently, if variable x is greater than the value

$$r_1^x = \frac{\alpha_1 + \sqrt{(\alpha_1)^2 + 4\beta_1 p_1^0 r_0^x}}{2\beta_1},$$

then the rate of changing of number of malaria mosquitoes becomes negative. Taking into account all notifications pointed out above, we can write

$$r_{k+1}^x = \frac{\alpha_1 + \sqrt{(\alpha_1)^2 + 4\beta_1 p_1^0 r_k^x}}{2\beta_1}.$$

We can't assume that $r_{k+1}^x < r_k^x$ because the extreme of solution value can be at boundary (right) point of time interval δ_k (if $x(t) \ge r_{k+1}^x$ on the interval δ_{k+1} the rate of $x(t)$ changing is negative and its amount decreases monotonously). Consequently, we have to construct a new sequence of values which organize the upper limits for solution:

$$\tilde{r}_{k+1}^x = \max(r_k^x, r_{k+1}^x).$$

Note, that if the following inequality is realized

$$r_k^x > \frac{p_1^0 + \alpha_1}{\beta_1},$$

then $r_{k+1}^x < r_k^x$. This property of sequence $\{r_k^x\}$ allows us to proof that solutions of model are bounded. We can choose the first value r_0^x:

$$r_0^x = \max\left\{\frac{p_1^0 + \alpha_1}{\beta_1}, \max_{t \in \delta_0} x(t)\right\}.$$

If so, the sequence $\tilde{r}_k^x \equiv r_0^x$ and, respectively, for all $t > 0$ we have $x(t) \le r_0^x$.

The similar procedure can be realized for second variable. Let denote as u^* the solution (the biggest positive root) of the following equation (with respect to r_k^y):

$$p_1^1 r_0^x + p_2 r_k^y = \alpha_2^1 r_k^y + \beta_2 (r_k^y)^2.$$

It is obvious, that this solution u^* exists. If we have the inequality $r_k^y > u^*$ then $r_k^y > r_{k+1}^y$. As it was assumed before, on the initial time interval $\delta_0 = [-\tau, 0]$ the following inequality is realized:

$$\max_{t \in \delta_0} y(t) \le r_0^y .$$

In this situation we can choose the following value of r_0^y:

$$r_0^y = \max\left\{u^*, \max_{t \in \delta_0} y(t)\right\}.$$

Finally, we can conclude that all solutions of considering model are bounded and for all $t > 0$ point $(x(t), y(t)) \in [0, r_0^x] \times [0, r_0^y]$.

3. Coordinates of stationary states of model are determined by the following system of algebraic equations:

$$\alpha_1 x + \alpha_2^0 y - \beta_1 x(x+y) + p_1^0 x = 0,$$
$$p_1^1 x - \alpha_2^1 y - \beta_2 y(x+y) + p_2 y = 0.$$

Obviously, point $(0,0)$ satisfies to this system. And there are no other stationary states, which belong to coordinate lines. From the second equation of this system we have the following relation:

$$x = \frac{Ay + \beta_2 y^2}{p_1^1 - \beta_2 y},$$

where $A = \alpha_2^1 - p_2 > 0$ is a positive parameter. It is obvious, that this curve comes through origin $(0,0)$. From basic conditions for relations between model parameters we have that this function is monotonous increasing function and the second derivative is positive. Also there exists asymptotic value, which is determined by the following expression: $y = p_1^1 / \beta_2$,

$$\frac{dx}{dy} = \frac{p_1^1 (A + 2\beta_2 y) - \beta_2^2 y^2}{(p_1^1 - \beta_2 y)^2} > 0, \quad \frac{d^2 x}{dy^2} = \frac{(p_1^1)^2 + Ap_1^1}{(p_1^1 - \beta_2 y)^3} > 0.$$

Note that first derivative has extreme points in “non-biological part” of phase plane.

From the first equation we have the following function:

$$y = \frac{\beta_1 x^2 - x(\alpha_1 + p_1^0)}{\alpha_2^0 - \beta_1 x}.$$

It is obvious, that this curve comes through origin $(0,0)$. Let

$$x^* = \frac{\alpha_2^0}{\beta_1}, \quad x^{**} = \frac{\alpha_1 + p_1^0}{\beta_1}.$$

If we have the following inequality: $x^{**} > x^*$ or

$$\alpha_1 + p_1^0 > \alpha_2^0,$$

function $y = y(x)$ has negative values only. It means that there are no stationary states in non-negative part of plane (in “biological part” of phase plane). Also it means that population eliminates for all possible initial values of sizes of mosquitoes. On the other hand, realization of last inequality means that transformation of non-malaria mosquito into malaria mosquito leads to catastrophic decrease of insect productivity. Consequently, realization of this inequality seems unreal for natural conditions and later we'll assume that the inverse inequality is realized for considering situation, $x^{**} < x^{*}$.

The first derivative of this function is presented in the form:

$$\frac{dy}{dx} = \frac{\alpha_2^0(2\beta_1 x - \alpha_1 - p_1^0) - \beta_1^2 x^2}{(\alpha_2^0 - \beta_1 x)^2}.$$

The upper part of this fraction has two various roots:

$$x_{1,2} = \frac{\alpha_2^0 \pm \sqrt{(\alpha_2^0)^2 - \alpha_2^0(\alpha_1 + p_1^0)}}{\beta_1},$$

that can be presented in the following form:

$$x_{1,2} = x^* \pm \sqrt{x^*(x^* - x^{**})}.$$

The biggest root of last expression is greater than x^* where function $y = y(x)$ is negative. And lowest root belongs to domain $x < x^{**}$ where function $y = y(x)$ is also negative. Finally, on the interval of argument changing $[x^{**}, x^*]$ function $y = y(x)$ increases monotonously (up to plus infinity).

The second derivative of function $y = y(x)$ has the form:

$$\frac{d^2 y}{dx^2} = \frac{2\beta_1\alpha_2^0(\alpha_2^0 - \alpha_1 - p_1^0)}{(\alpha_2^0 - \beta_1 x)^3}.$$

Taking into account our assumption we made above about the relation between model parameters, the second derivative is greater than zero on the interval $[x^{**}, x^*]$. Finally, in considering model we have two various situations: if the inequality $\alpha_1 + p_1^0 > \alpha_2^0$ is realized in phase space there is a unique stationary state $(0,0)$; at inverse inequality there are two stationary states on phase plane (origin and non-zero equilibrium).

4. Numerical analysis of model (1)-(2) allowed obtaining some of dynamic regimes which were observed for initial conditions $\forall t \in [-\tau, 0)$: $x(t) \equiv 0$, $y(t) \equiv 0$, $x(0) = x_0 > 0$, $y(0) = y_0 > 0$. Typical dynamic regime was observed for following values of parameters (Fig. 1): $x_0 = 0.05$, $y_0 = 0.01$, $a_1 = 120$, $a_2^0 = 7 \cdot 10^{-12}$, $a_2^1 = 3$, $p_1^0 = 7 \cdot 10^{-12}$, $p_1^1 = 3.5$, $p_2 = 0.15$, $\beta_1 = 13$, $\beta_2 = 3.5 \cdot 10^{-10}$, $\tau = 1.5$. As we can see, in asymptotic we have periodic fluctuations of both variables near stationary level.

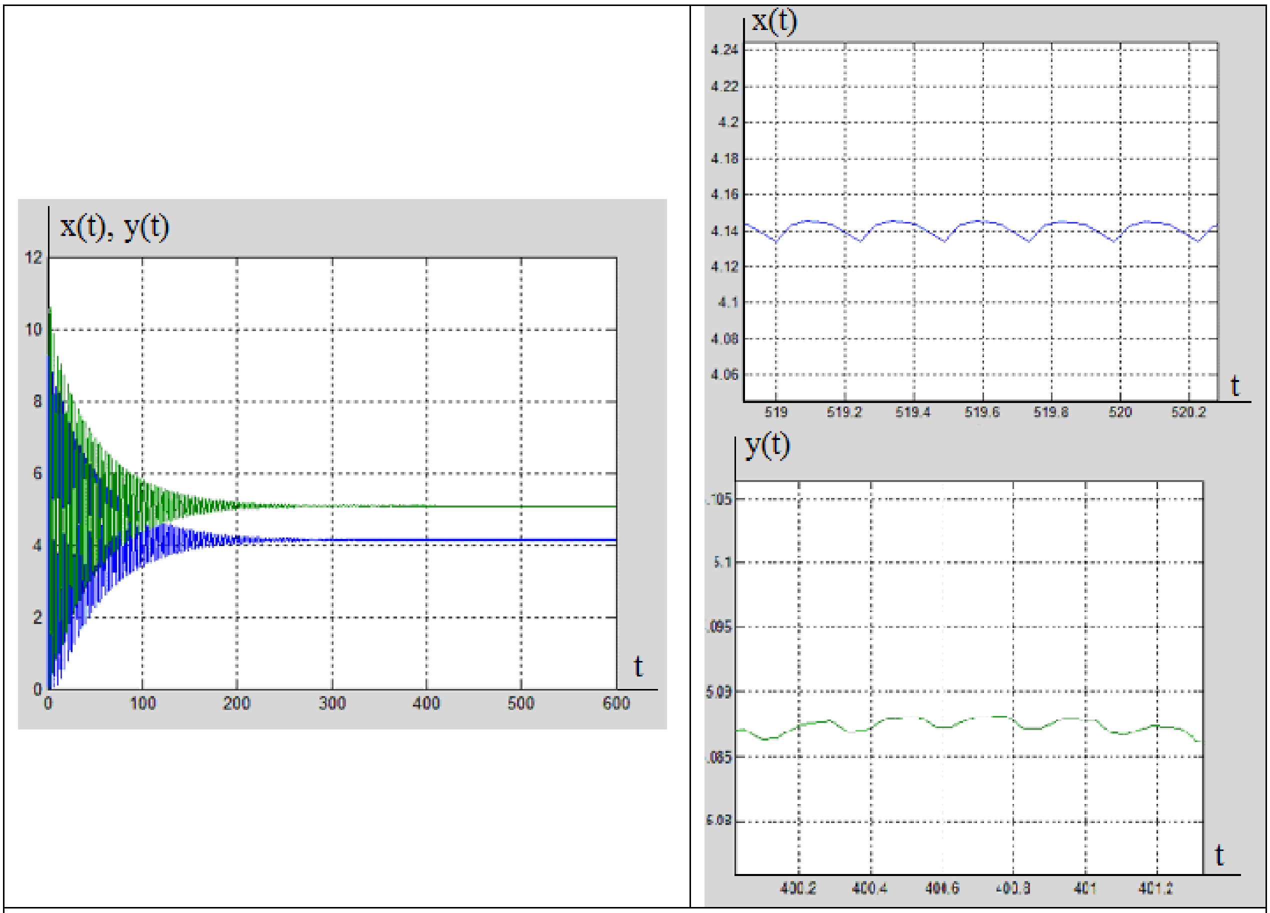

Fig. 1 Typical dynamic regime in model (1)-(2). Initial stage of process is in the left; asymptotic fluctuations of variables are in the right. Blue line corresponds to $x(t)$ dynamics; green line corresponds to $y(t)$ dynamics.

4 Conclusion

Considered model (1)-(2) needs in further modification. It can be provided in several possible ways. First of all, it is important to take into account influence of life cycle of insect onto population dynamics, and existence of several different stages of insect development. Another way must have relation to existence of more complicated structure of local human population. In all situations model (1)-(2) can be used as basic for further model constructions.

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