Article

Crowding effects and depletion mechanisms for population regulation in prey-predator intraspecific competition model

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Abstract

The current investigation centres on the consequences of intra-specific rivalry involving predators in the predator-prey equation. A careful account of the investigation is offered mathematically of the model to offer insights into important outcomes that result from the interplay of deterministic and stochastic process. In particular, the steadiness and bifurcation investigation of this model find mention. Allowance is also made in this model for a stochastic environment impacted by white noise. While for this particular version, the global stability is predicated under conditions bordering on stochasticity close to environmental concerns. Rivalry among the predator population is without a doubt accommodating for a various predator-prey models by keeping population stable at a positive interior equilibrium. Numerical solutions obtained for the models support the assumptions governing the study.

Keywords intraspecific competition; Hopf-bifurcation; stochasticity; Lyapunov function; discrete model; white noise.

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1 Introduction

Mathematical models of population dynamics find expression in terms of difference or differential equations that detail how populations change with time, space or particular stages of development (Zhang, 2015, 2016). Although research in the field of population dynamics, traditionally the preserve of mathematical ecology, can be found in accounts of 18^{th}century, a watershed of its arrival as a scientific discipline and its subsequent respectability may be attributed to the works of Alfred J.Lokta and Vito Volterra.

To begin with, actual instances of individual behaviour accompany the concept of functional response, defined as the rate at which an individual predator consumes prey in terms of density of prey. Plenty of literature is available that has already discussed ecological systems (Arditi et al., 1989; Freedman et al., 1980; Gatto et al., 1991; May et al., 1973), where the model systems are based on prey-dependent model systems. In all the cases the functional response which is prey-dependent is modelled as $ax/\theta + x,\ \ ax^2/(\theta + x)^2,\ \ ax^2/\theta + x + \alpha x^2$ or some equivalent form. The Holling-Tanner model deals with the Michaelis-Menten or Holling type –II functional response of the form $cx/m + x$. Where c is the maximal predator, per capita utilization rate, i.e., the most extreme number of prey that can be devoured by a predator in each time unit and *m* is the half catching immersion consistent i.e., the quantity of prey important to accomplish one-half portion of the greatest rate c. Such functional responses go by the name of “prey-dependent”, named as such by Arditi and Ginzburg, in view of the fact that it depends on prey density only. It was realised early on that the predator density could directly impact functional response. Plenty of such predator-dependent models (Akcakaya et al., 1995; Cosner et al., 1999; Gutierrez et al., 1992; Hanski et al., 1991; Arditi et al., 1992; Gutirrez et al., 1992; Blaine et al., 1997; Poggiale et al., 1998; Cosner et al., 1999] have been in the offing, the most well - known being Hassell and Varley, 1969; Detngeles et al., 1975. Arditi and Ginzburg 1989, 1991; presented a Michaelis-Menten type ratio dependent functional response of the form $cx/my + x$, where x, y stand for densities of prey and predator respectively.

This article intends to propose a model with intraspecific competition between predators with half saturation constant, before this so many authors investigated prey-predator models with intraspecific competition without saturation constant. Intraspecific competition among predators for prey starts the minute the ratio of predators to prey is sufficiently large, leading to individuals among the predator population undergoing reduced fitness from absence of sustenance (Purves et al., 2001). This extreme competition happens in blue crab populations where they express brutal behaviour, leading to bloodied wounds due to scarcity (Clark et al., 1999). More extreme intra specific competition has been known to happen in intra specific predation in a variety of predator populations due to limited availability of alternative food source (Fox, 1975). Finally, it is surmised that competition within the predator population might be advantageous for predator species under specified circumstances in deterministic environment.

However, the deterministic environment rarely occurs in reality since most natural environments display randomness. Most of what shows up in the models proposed and explored in the ecological literature operate is in the framework of an unchanging, deterministic ambience. That is, real environments tend to be uncertain and stochastic.

Frankly, randomness or stochasticity majorly impacts the structure and function of biological systems (May, 1974; Renshaw, 1995; Nisbet and Gurney, 1982; Samanta, 1996). The environmental factors are usually time-dependent, randomly changing and are to be treated as stochastic. Renshaw (1995) maintain that natural phenomena of whatever persuasion defy purely deterministic laws and toggle randomly around some average, so much so as to enable the deterministic equilibrium to suffer the loss of an absolutely fixed state; instead it pans out towards a “fuzzy” value, surrounding the biological system it is concerned with. A primary hurdle in the stochastic modelling of an ecosystem is the lack of reliable mathematical wherewithal at hand to analyze non-linear multi-dimensional stochastic process. Many researchers (Dimentberg, 1988, 2002; Samanta, 1994; Samanta and Maiti, 2003, 2004; Bandyopadhyay and Chakrabarti, 2003; Bandyopadhyay and Chattopadhyay, 2005; Maiti and Samanta, 2005, 2006; Maiti et al., 2007) found out the comparison between deterministic and stochastic models with good accuracy.

We also considered discrete version of the considered continuous population model. Generally, discrete model shows richer dynamics than continuous models. Based on this many researchers (Liu et al., 2007; Lopez-Ruiz et al., 2004; Elsadany et al., 2012) have considered these models to discuss the population dynamics of prey- predator models. Usually discrete models are described with difference equations. Also, we can easily obtain numerical simulations for discrete models.

This paper is organized as follows: Section 3 discusses the boundedness of the system; Section 4 permanence of the system, Section 5 throws equilibrium points, existence, and stability and bifurcation analysis of the model. Section 6 defined discrete version of the mathematical models, Section 7 and 8 throws equilibrium points, their stability and bifurcation analysis, Section 9 throws light on stochastic stability of the system by using Lyapunov function. Section 10 explores numerical simulation of the system which supports analytical results arrived at.

2 Formulation of Deterministic Mathematical Model

In this section we have constructed a mathematical model of Prey-predator with intraspecific competition between predators. The model can be presented by the following set of ordinary differential equations

$$\begin{aligned} \frac{dx}{dt} &= rx\left(1-\frac{x}{k}\right)-\frac{\beta xy}{\theta+x} \\ \frac{dy}{dt} &= \frac{e\beta xy}{\theta+x}-dy-\frac{\gamma y^2}{L+y} \end{aligned} \tag{2.1}$$

with initial conditions $x(0)\geq 0, y(0)\geq 0$, where x(t), y(t) represents prey and predator population biomass, r stands for intrinsic growth rate of prey, k carrying capacity, β conversion rate , θ saturation constant, e another conversion rate, d represents death coefficient of predator, γ rate of predator inter competition, L saturation constant. All the model parameters are positive constants.

This model involves certain assumptions which consist of the followings:

(i) Prey individuals are assumed to have the logistic growth rate with carrying capacity K.
(ii) The Prey- dependent functional response is assumed in the interaction between prey and predator Population.
(iii) We assumed intraspecific competition among the predators.

3 Behaviour of the Solutions of Deterministic Model

Theorem 1. The solutions of the system (1) are invariant under R_+^2.

Proof: we can easily verify that $F_1(0,0)=F_2(0,0)=0$, then from this we can say that $F_1(x,y)$ and $F_2(x,y)$ are continuous on R_+^2 and also Lipschizian on R_+^2. Hence solution of (1) exists and it is unique. These solutions are exist for all $t>0$ and stay non-negative. Hence, the interior of R_+^2 is invariant under model system (1).

Theorem 2. The solutions of the system (1) are bounded.

Proof: Let $W=x+y$

$$\frac{dW}{dt} = \frac{dx}{dt} + \frac{dy}{dt}$$

(3.1)

$\frac{dW}{dt} \le x(r+1) - mW \ \ where \ m = \min\{1, d\}$ and also $\frac{dx}{dt} \le rx\left(1 - \frac{x}{K}\right)$ then by a standard comparison theorem we have $\lim_{t \to \infty} Sup x(t) \le M \ where \ M = \{x(0), K\}$.

Therefore $\frac{dW}{dt} + mW \le M(r+1)$ (3.2)

Applying the theorem in differential inequalities (Birkhoff et al., 1962), we obtain

$0 \le W(x, y) \le \frac{M(r+1)}{m} + W(x(0), y(0)) / e^{mt}$ and for $t \to \infty, 0 \le W \le \frac{M(r+1)}{m}$.

Therefore all solutions of system (2.1) enter into the region

$$B = \left\{ (x, y) \in R_+^2 : W \le \frac{M(r+1)}{m} + \varepsilon, \ for \ any \ \varepsilon > 0 \right\}.$$

(3.3)

4 Permanency of the System

Theorem 3. If $\Delta(x, y) > 0$ then the solution of the system (2.1) is permanent.

Proof: Consider the Lyapunov function $h(x, y) = x^{p_1} y^{p_2}$, where p_1, p_2 are positive constants.

In positive octant $\Delta(x, y) = \dot{h}/h = p_1 F_1 / x + p_2 F_2 / y$. To show the system has permanence of the solution we required to show $\Delta(x, y) > 0$ at axial equilibrium point (E_2). Here it is clearly exist without any condition. Hence the system has a permanent solution with unconditionally.

5 Existences of the Equilibrium Points and Their Stability Analysis

In this section we will study the existence and stability behaviour of the system (2.1) at various equilibrium points. The equilibrium points of the system (2.1) are

(i) Trivial equilibrium : $E_1(0, 0)$

(ii) Persistent equilibrium : $E_2(k,0)$

(iii) Interior equilibrium : $E_3(x_2, y_2)$

where $x_2 = \dfrac{dL\theta + (d+\gamma)\theta y_2}{(L+y_2)e\beta - dL - (d+\gamma)y_2}$ and y_2 is the root of the following equation.

$$\rho_1 y^3 + \rho_2 y^2 + \rho_3 y + \rho_4 = 0 \tag{5.1}$$

where

$$\rho_1 = \beta(e\beta - d - \gamma)^2;$$
$$\rho_2 = \theta e\beta rk(d+\gamma) + \theta^2 e\beta r\gamma - \theta e^2\beta^2 rk + 2\beta L(e\beta - d)(e\beta - d - \gamma);$$
$$\rho_3 = L^2(e\beta - d)^2 + \theta^2 e\beta rd(2L+1) + \theta^2 e\beta Lr\gamma + \theta e\beta rdkL - \theta e^2\beta^2 rkL + \theta e\beta Lrkd - \theta e^2\beta^2 Lrk;$$
$$\rho_4 = \theta^2 L^2 e\beta rd + \theta e\beta Lrk\gamma + \theta e\beta L^2 rkd - \theta e^2\beta^2 L^2 rk;$$

The first two equilibrium points always exist and (5.1) has one and only positive root if $\eta^2 + 4\omega^3 > 0$, if $\eta^2 + 4\omega^3 = 0$ then (5.1) has two equal roots and if $\eta^2 + 4\omega^3 > 0$ then it has three distinct real roots, where $\eta = \rho_1^2\rho_4 - 3\rho_1\rho_2\rho_3 + 2\rho_2^3,\ \omega = \rho_1\rho_3 - \rho_2^2$. By Cardan's method, the roots of the equation (5.1) is $\dfrac{1}{\rho_1}(A - \dfrac{\omega}{A} - \rho_2)$, where A denotes the value of $\left[\dfrac{1}{2}\left(-\eta + \sqrt{\eta^2 + 4\omega^3}\right)\right]^{\frac{1}{3}}$. We obtain the remaining roots of the equation (5.1) by Cardan's method in a similar manner. Now, for positive root of y_2, one positive interior equilibrium point is attained provided that $(L+y_2)e\beta - dL - (d+\gamma)y_2 > 0$.

5.1 Stability and bifurcation analysis of each fixed point

The stability analysis of each equilibrium point finds discussion along with bifurcation analysis of the system (2.1). In this process, the following notations have been employed.

$\dot{X} = F(X,m) = \left(F_1(x,y), F_2(x,y)\right)^T$, where $X = (x,y)^T$, and the Jacobian matrix of the system $J \equiv AF(X,m)$.

5.1.1 Stability analysis at E_0

The Jacobian at this equilibrium point is denoted by J_0, and is defined as $J_0 = \begin{bmatrix} r & 0 \\ 0 & -d \end{bmatrix}$

The Latent values are r,-d. Hence the system (2.1) at equilibrium point E_0 is unstable.

5.1.2 Stability analysis at E_1

The Jacobian matrix of the system (2.1) at equilibrium point E_1 is denoted by J_1 and is defined as

$$J_1 = \begin{bmatrix} -r & \dfrac{-\beta k}{\theta+k} \\ 0 & \dfrac{e\beta k}{\theta+k} - d \end{bmatrix}$$

The Latent values are $-r, \dfrac{e\beta k}{\theta+k} - d$. Therefore the system (2.1) is stable if $\beta < \beta'$ where $\beta' = \dfrac{d(\theta+k)}{ek}$.

If $\beta = \beta'$ then one Eigen value of J_1 is zero and the other eigen value is $-r$. Let the Eigen vectors of J_1 and J_1^T corresponding to the Latent value '0' are M and N respectively. M and N are defined as $M = \begin{bmatrix} \dfrac{-\beta k}{r(\theta+k)} M_2 \\ M_2 \end{bmatrix}$ and $N = \begin{bmatrix} 0_2 \\ N_2 \end{bmatrix}$ respectively. Here M_2, N_2 are two non zero real numbers. Now $N^T \left[F_\beta \left(X, \beta' \right) \right] = 0$, where $X_1 = (k, 0)$. Hence by Sotomayor theorem (Sotomayor, 1973) the system does not attain any saddle-node bifurcation around E_1. Again $N^T \left[AF_\beta \left(X_1, \beta' \right) M \right] = \dfrac{ekM_2 N_2}{\theta+k} \neq 0$ and $N^T \left[A^2 F \left(X_1, \beta' \right)(M, M) \right] \neq 0$, where $F_\beta = \dfrac{\partial F}{\partial \beta}, AF_\beta = \dfrac{\partial (AF)}{\partial \beta}$. Therefore, by the same theorem [25] the system experiences a transcritical bifurcation at $\beta = \beta'$ around the axial equilibrium E_1.

5.1.3 Stability analysis at E_2

The Jacobian matrix of the system around the equilibrium point E_2 (x_2, y_2) is defined as

$$J_2 = \begin{bmatrix} \dfrac{-rx_2}{k} + \dfrac{\beta x_2 y_2}{(\theta + x_2)^2} & \dfrac{-\beta x_2}{(\theta + x_2)} \\ \dfrac{\theta e \beta y_2}{(\theta + x_2)^2} & \dfrac{-\gamma L y_2}{(L + y_2)^2} \end{bmatrix}$$

In above matrix $a_{11} = \dfrac{-rx_2}{k} + \dfrac{\beta y_2 / \theta}{(\theta / x_2 + 1)(1 + x_2 / \theta)} \leq \dfrac{-rx_2}{k} + \dfrac{\beta y_2}{\theta} < 0, \text{if } \beta < \beta^s$

$$a_{12} = \frac{-\beta x_2}{(\theta + x_2)} < 0, a_{21} = \frac{\theta e \beta y_2}{(\theta + x_2)^2} > 0 \,\&\, a_{22} = \frac{-\gamma L y_2}{(L + y_2)^2} < 0$$

$tr(J_2) = a_{11} + a_{22} < 0 \,\&\, \det(J_2) = a_{11}a_{22} - a_{12}a_{21} > 0$ if $\beta < \beta^s$ Where $\beta^s = \dfrac{r\theta x_2}{ky_2}$

Therefore, the system (2.1) is locally asymptotically stable at positive equilibrium point $E_2(x_2, y_2)$ if $\beta<\beta^s$.

5.1.4 Bifurcation analysis

i) Saddle-node bifurcation: To account for the saddle node bifurcation we have to consider $\det(J_2)=0$ which gives $\beta=\beta^{SN}$ and one eigen value of J_2 will be zero. Here β^{SN} is the solution of the equation

$$\beta^2\left(k\theta exy(L+y)^2\right)+\beta\left(k\gamma xy^3(\theta+x)-k\gamma xy^2(\theta+x)(L+y)\right)+r\gamma xy(\theta+x)^3(L+y).$$

Let $\varepsilon_1\ and\ \varepsilon_2$ be the eigen vectors of J_2 and J_2^T corresponding to the eigen value zero. The first eigen vector ε_1 is defined as $\left[z_1\ and\ z_2\right]^T$ and $\varepsilon_2=\left[h_1, h_2\right]^T$, where $z_1=\frac{-a_{12}}{a_{11}}z_2=\frac{-a_{22}}{a_{21}}z_2, h_1=\frac{-a_{21}}{a_{11}}h_2=\frac{-a_{22}}{a_{12}}h_2$, and z_2, h_2 are any two non-zero real numbers.

By simple calculation we can verify easily that $\varepsilon_2{}^T\left[F_\beta\left(X_2,\beta^{SN}\right)\right]\neq 0$ and $\varepsilon_2{}^T\left[A^2F\left(X_2,\beta^{SN}\right)(\varepsilon_1,\varepsilon_1)\right]\neq 0$. Therefore, by Sotomayor theorem [25] the system has saddle -node bifurcation at positive equilibrium $E_2(x_2, y_2)$ and also the system has neither transcritical nor any pitch-fork bifurcation at $E_2(x_2, y_2)$ since $\varepsilon_2{}^T\left[F_\beta\left(X_2,\beta^{SN}\right)\right]\neq 0$.

ii) Hopf-bifurcation: If $tr(J_2)=0$ which gives $\beta=\beta^{HB}$

where $\beta^{HB}=\frac{rx\left(L+y\right)^2+\gamma Ly}{k\left(L+y\right)^2}\times\frac{\left(\theta+x\right)^2}{xy}$ and

$$\det(J_2)=\frac{1}{k(\theta+x)^3(L+y)^2}[\beta^2\left(k\theta exy(L+y)^2\right)+\beta\left(k\gamma xy^3(\theta+x)-k\gamma xy^2(\theta+x)(L+y)\right)$$
$$+r\gamma xy(\theta+x)^3(L+y)]$$

If $tr(J_2)=0\ \&\ \det(J_2)>0$ then the Latent values of J_2 will be purely imaginary and by implicit function theorem, the system undergoes Hopf bifurcation at positive equilibrium E_2.

6 Formulation of the Discrete Mathematical Model

In this section we study the dynamics of discrete predator-prey model with intraspecific competition between predators, which has the following difference equations

$$x_{n+1} = x_n + r x_n \left(1 - \frac{x_n}{k}\right) - \frac{\beta x_n y_n}{\theta + x_n}$$
$$y_{n+1} = y_n + \frac{\beta x_n y_n}{\theta + x_n} - d y_n - \frac{\gamma y_n^2}{L + y_n} \tag{6.1}$$

where x and y are the population biomasses of the predator and prey generations n and n+1 respectively. In system (6.1), all the parameters are same as in system (2.1). The map given by equation (6.1) is a noninvertible map of the plane. The study of the dynamical properties of the above map allows us to have information about long-run behaviour of the predator–prey populations. Starting from the initial condition (x_0, y_0), the interaction of (6.1) uniquely determines a trajectory of the states of population output in the following form: $[x(n), y(n)] = T^n [x_0, y_0]$ where $n = 0,1,2,.........$

7 Fixed Points and Stability Analysis

The system (6.1) has two fixed points $E_1 and\ E_2$ and these are exactly same as the fixed points of continuous model (2.1). Next we have to study the stability of these fixed points.

To discuss the local stability of each fixed point, first we have to compute Jacobian matrix of the system (6.1) at each fixed point.

$$J = \begin{bmatrix} 1 - \frac{rx}{K} + \frac{\beta xy}{(\theta + x)^2} & -\frac{\beta x}{\theta + x} \\ \frac{e\beta\theta y}{(\theta + x)^2} & 1 - \frac{\gamma L y}{(L + y)^2} \end{bmatrix}$$

To study the stability of the fixed points of the system (6.1) we recall the following lemma.

Lemma 1. Let $\phi(\lambda) = \lambda^2 - L_1\lambda + L_2$ and $\phi(1) > 0$

Let λ_1, λ_2 are two roots of $\phi(\lambda) = 0$. Then:

i). $|\lambda_1| < 1$ and $|\lambda_2| < 1$ if and only if $\phi(-1) > 0$ and $\phi(0) < 1$

ii). $|\lambda_1| < 1$ and $|\lambda_2| > 1$ if and only if $\phi(-1) < 0$

iii). $|\lambda_1| > 1$ and $|\lambda_2| > 1$ if and only if $\phi(-1) > 0$ and $\phi(0) > 1$

iv). $\lambda_1 = -1$ and $\lambda_2 \neq 1$ if and only if $\phi(-1) = 0$ and $L_1 \neq 0, 2;$

v). λ_1, λ_2 are complex and $|\lambda_1| = 1, |\lambda_2| = 1$ if and only if $L_1^2 - 4L_2 < 0$ and $\phi(0) = 1$

A fixed point (x^*, y^*) is called a sink if $|\lambda_1| < 1$ and $|\lambda_2| < 1$, so it is locally asymptotically stable. (x^*, y^*)

is called a source if $|\lambda_1|>1$ and $|\lambda_2|>1$, so it is locally unstable. If (x^*, y^*) is called a saddle if $|\lambda_1|<1$ and $|\lambda_2|>1$ and (x^*, y^*) is called a non-hyperbolic if $|\lambda_1|=1, |\lambda_2|=1$.

Theorem 4. Suppose that the fixed point E_1

i).is a sink if $r>0$ and $\beta<\beta'$

ii).is a source if $r<0$ and $\beta>\beta'$

iii).is a saddle if $r>0$ and $\beta>\beta'$

iv).is non-hyperbolic if $r=1$ and $\beta=\beta'$ where $\beta' = \dfrac{d(\theta+K)}{eK}$.

Proof: To prove all these results first we have to compute the variation matrix of the system (6.1) at E_1 is

$$J_{E_1} = \begin{bmatrix} 1-r & \dfrac{-\beta K}{\theta+K} \\ 0 & 1+\dfrac{e\beta K}{\theta+K}-d \end{bmatrix}$$

The eigen values of above Jacobian matrix are $\lambda_1 = 1-r, \lambda_2 = 1+\dfrac{e\beta K}{\theta+K}-d$.

By using Lemma1 we can easily verify that E_1 is a sink if $1-r < i.e., r>0$ and $1+\dfrac{e\beta K}{\theta+K}-d<1 i.e., \beta<\beta'=\dfrac{d(\theta+K)}{eK}$; E_1 is a source if $r<0$ and $\beta>\beta'$; E_1 is a saddle if $r>0$ and $\beta>\beta'$ and is non-hyperbolic $r=1$ and $\beta=\beta'$.

8 Local Stability and Hopf-bifurcation around Interior Point

We now investigate the local stability of positive equilibrium (x^*, y^*). The variation matrix at the positive equilibrium (x^*, y^*) is

$$J_{E_2} = \begin{bmatrix} 1-\dfrac{rx^*}{K}+Mx^* & \dfrac{-\beta x^*}{\theta+x^*} \\ e\theta M & 1-LN \end{bmatrix} \quad \text{where } M = \frac{\beta y^*}{(\theta+x^*)^2}, N = \frac{\gamma y^*}{(L+y^*)^2}$$

The characterstic equation of this Jacobian matrix is $\phi(\lambda) = \lambda^2 - B\lambda + C = 0$

Where $B = 2 - \frac{rx^*}{K} + Mx^* - LN$, $C = 1 - \frac{rx^*}{K} + Mx^* - LN + \frac{rx^*}{K}LN - LMNx^* + \frac{e\theta M\beta x^*}{\theta + x^*}$

Theorem 5: If $\frac{rx^*}{K} + LMNx^* + LN > Mx^* + \frac{rx^*}{K}LN + \frac{e\theta M\beta x^*}{\theta + x^*} > \frac{2rx^*}{K} + 2LMNx^* + 2LN - 3$ then the fixed point E_2 is locally asymptotically stable.

Proof: In order to prove E_2 is locally stable by using Lemma1 we have to verify $\phi(1) > 0, \phi(-1) > 0 \, and \, C < 1.$

$$\phi(1) = \frac{rx^*}{K}LN - LMNx^* + \frac{e\theta M\beta x^*}{\theta + x^*}$$

If $\frac{rx^*}{K}LN + \frac{e\theta M\beta x^*}{\theta + x^*} > LMNx^*$ then $\phi(1) > 0$

$$\phi(-1) = 3 - \frac{2rx^*}{K} + 2Mx^* - 2LN + \frac{rx^*}{K}LN - LMNx^* + \frac{e\theta M\beta x^*}{\theta + x^*}$$

If $Mx^* + \frac{rx^*}{K}LN + \frac{e\theta M\beta x^*}{\theta + x^*} > \frac{2rx^*}{K} + 2LN + LMNx^* - 3$ then $\phi(-1) > 0$

$$C < 1 => \frac{rx^*}{K} - Mx^* + LN - \frac{rx^*}{K}LN + LMNx^* - \frac{e\theta M\beta x^*}{\theta + x^*} > 0$$

Which is true if $\frac{rx^*}{K} + LN + LMNx^* > Mx^* + \frac{rx^*}{K}LN + \frac{e\theta M\beta x^*}{\theta + x^*}$

From above results E_2 is locally asymptotically stable if () is exist.

Corollary 1: The fixed point E_2 is unstable if and only if the following conditions holds

$\frac{rx^*}{K} + LMNx^* + LN < Mx^* + \frac{rx^*}{K}LN + \frac{e\theta M\beta x^*}{\theta + x^*}$, or

$Mx^* + \frac{rx^*}{K}LN + \frac{e\theta M\beta x^*}{\theta + x^*} < \frac{2rx^*}{K} + 2LMNx^* + 2LN - 3$.

Corollary 2: Suppose that $Mx^* + \frac{rx^*}{K}LN + \frac{e\theta M\beta x^*}{\theta + x^*} > \frac{2rx^*}{K} + 2LMNx^* + 2LN - 3$ then the system (6.1) undergoes a Hopf-bifurcation when β passes through a critical value β_c where $\det j = 1 \, at \, \beta = \beta_c$.

9 Stochastic Stability of The Deterministic System at Positive Equilibrium Point

To investigate the environmental fluctuations on model (2.1), it is understood that the stochastic perturbations are of white noise type and that they are proportional to the distances of $x(t)$ and $y(t)$ respectively. So system (2.1) results

$$\begin{aligned}\frac{dx}{dt}&=rx\left(1-\frac{x}{k}\right)-\frac{\beta xy}{\theta+x}+\sigma_1\left(x-x_2\right)d\xi_t^1\\ \frac{dy}{dt}&=\frac{e\beta xy}{\theta+x}-dy-\frac{\gamma y^2}{L+y}+\sigma_2\left(y-y_2\right)d\xi_t^2\end{aligned} \tag{9.1}$$

where σ_1,σ_2 are real constants, $\xi_t^i=\xi_i\left(t\right),i=1,2$ are independent of each other standard Wiener processes. The system (9.1) has the same equilibrium as the system (2.1).

The stochastic differential system (9.1) may then centred at its positive equilibrium E_2 by the change of variables

$$u_1=x-x_2,\; u_2=y-y_2 \tag{9.2}$$

The linearized Stochastic Differential Equations around E_2 take the form

$$du\left(t\right)=f\left(u\left(t\right)\right)dt+g\left(u\left(t\right)\right)d\xi\left(t\right) \tag{9.3}$$

where $u\left(t\right)=\left(u_1\left(t\right),u_2\left(t\right)\right)^T$, $f\left(u\left(t\right)\right)=J_2$, which defined in section (5.1.3) and $g\left(u(t)\right)=diag(\sigma_1u_1,\sigma_2u_2)$.

Let $C^{1,2}\left(\left[0,+\infty\right)\times\Re^2,\Re^+\right)$ be the family of nonnegative functions. $W\left(t,u\right)$ defined on $\left[0,+\infty\right)\times\Re^2$ is a continuously differentiable function with respect to t and twice with respect to u.

We define the differential operator L for a function $W\left(t,u\right)$ by

$$LW(t,u)=\frac{\partial W\left(t,u\right)}{\partial t}+f^T\left(u\right)\frac{\partial W\left(t,u\right)}{\partial u}+\frac{1}{2}Tr\left[g^T\left(u\right)\frac{\partial^2W\left(t,u\right)}{\partial u^2}g\left(u\right)\right] \tag{9.4}$$

$\frac{\partial W}{\partial u}=col\left(\frac{\partial W}{\partial u_1},\frac{\partial W}{\partial u_2},\frac{\partial W}{\partial u_3}\right),\frac{\partial^2W\left(t,u\right)}{\partial u^2}=\left(\frac{\partial^2W}{\partial u_j\partial u_i}\right)\quad i,j=1,2$ and 'T' means transposition.

With reference to the book by Afanas'ev et al. 1996; the following theorem holds.

Theorem 6. Suppose there exist a function $W(t,u)\in C^{1,2}\left(\left[0,+\infty\right)\times\Re^2,\Re^+\right)$ satisfying the following inequalities

$$K_1 |u|^p \le W(t,u) \le K_2 |u|^p ; \; LW(t,u) \le -K_3 |u|^p , \tag{9.5}$$

where K_1, K_2, K_3 *and* p are positive constants.

Then the trivial solution of (9.3) is exponentially p - stable for $t \ge 0$ Moreover, if in (9.5), $p = 2$ the trivial solution of (9.3) is globally asymptotically stable in probability.

Theorem 7. Suppose that $\sigma_1^2 \le 2\left(\frac{rx_2}{k} - \frac{\beta x_2 y_2}{(\theta + x_2)^2}\right)$, $\sigma_2^2 \le 2\frac{\gamma L y_2}{(L + y_2)^2}$ hold. Then, the trivial solution of (9.3) is asymptotically mean square stable.

Proof: Let us consider the Lyapunov function

$$W(u) = \frac{1}{2}\left[w_1 u_1^2 + w_2 u_2^2\right] \tag{9.6}$$

where w_1, w_2 are nonnegative constants to be chosen in the following . It is easy to check that inequalities (9.5) hold true with $p = 2$.

$$\begin{aligned} LW(u) = w_1 &\left[\left(\frac{-rx_2}{k} + \frac{\beta x_2 y_2}{(\theta + x_2)^2}\right)u_1 - \frac{\beta x_2}{(\theta + x_2)}u_2\right]u_1 + w_2\left[\frac{\theta e \beta y_2}{(\theta + x_2)^2}u_1 - \frac{\gamma L y_2}{(L + y_2)^2}u_2\right]u_2 \\ &+ \frac{1}{2}Tr\left[g^T(u)\frac{\partial^2 W(t,u)}{\partial u^2}g(u)\right] \end{aligned} \tag{9.7}$$

with $\frac{1}{2}Tr\left[g^T(u)\frac{\partial^2 W(t,u)}{\partial u^2}g(u)\right] = \frac{1}{2}\left[w_1\sigma_1^2 u_1^2 + w_2\sigma_2^2 u_2^2\right]$

If in (9.7) we choose $\frac{\beta x_2}{(\theta + x_2)}w_1 = \frac{\theta e \beta y_2}{(\theta + x_2)^2}w_2$, then

$$LW(u) = -\left(\frac{rx_2}{k} - \frac{\beta x_2 y_2}{(\theta + x_2)^2} - \frac{1}{2}\sigma_1^2\right)w_1 u_1^2 - \left(\frac{\gamma L y_2}{(L + y_2)^2} - \frac{1}{2}\sigma_2^2\right)w_2 u_2^2$$

According to Theorem (1), we conclude that the trivial solution of system (9.3) is globally asymptotically stable.

10 Numerical and Computer Simulation

Example1. For the fixed parameter values $r = 0.8, k = 50, e = 0.628, \theta = 10, L = 5, d = 0.06, \gamma = 0.225$ and varying β values, the system moves from stable to unstable or unstable to stable. If β varies from 0.4 to 0.66 the system is stable, from 0.67 to 1.8 the system is unstable and the system is stable when greater than 1.8 (Illustrated in Fig. 1, Fig. 2 & Fig. 3). Also, we observed when $\beta = \beta^{HB} = 1.8$ the system will go under Hopf bifurcation since for $\beta = 1.8$, the model (2.1) have the equilibrium point (1.8451, 5.3368), $tr(J_2) \approx 0$ and $\det(J_2) = 0.1174 > 0$.

Example 2: Taking the parameter values as $r = 0.2; k = 50; \beta = 0.214; e = 0.628; \gamma = 0.05; \theta = 10; L = 0.5; d = 0.06;$ with noise $\sigma_1 = 0.01; \sigma_1 = 0.2;$ Fig4 shows, the system (9.1) is asymptotically stable for intensities of white noise sufficiently large.

Example 3: Fig5 to Fig7 illustrate that for different values of γ and keeping other parameter values are made unchanging in system (2.1), the system goes unstable to stable when γ increasing. This indicates the intra-specific rivalry between predators for prey diminishes the development of the predator population and allows for biologically tenable oscillations and the presence of stable coexistence equilibrium.

Example 4: If $r = 0.5; k = 500; \beta = 0.314; e = 0.628; \gamma = 0.2; \theta = 0.02; L = 50; d = 0.06;$

For above parameter values the system (2.1) has three equilibrium points (as shown in Fig8), of these one equilibrium point is spiral source at (0.00879, 0.04584), at this the Latent values of the Jacobian matrix are complex with positive real part, second equilibrium point is saddle at (81.9137, 109.0935), at this point the Latent values are opposite signs and third equilibrium point is nodal sink at (417.9748, 109.1915), at this point the Latent values of the Jacobian matrix are negative real part.

If $\beta = 0.514$ then only one equilibrium point is exist (as shown in Fig9) and which is spiral source at (0.00458, 0.0239), at this point the Latent values of the Jacobian matrix are complex with positive real part and it is unstable. That means when β is increased from 0.314 to 0.514, two equilibrium points saddle node and nodal sink are disappear and the other equilibrium point remains same. This indicates at $\beta = 0.514$ the system (2.1) experiences saddle node bifurcation at positive equilibrium.

11 Concluding Remarks

The majority of the Prey-Predator models imagine intra-specific competition between predators without saturation constant. This system has been shown to possess transcritical bifurcation at $\beta = \beta'$ around the axial equilibrium. Also, it has saddle-node bifurcation at positive equilibrium point $\beta = \beta^{SN}$ and the system experiences Hopf bifurcation. Discuss the stochastic stability of the system by constructing a suitable function. Also, these analytical results were supported by suitable numerical examples.

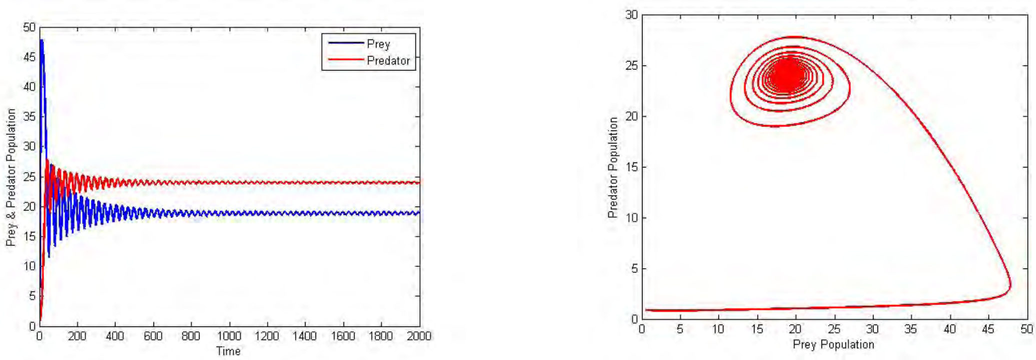

Fig. 1 Shows the trajectories and phase graphs of the model (2.1) with $\beta = 0.4$

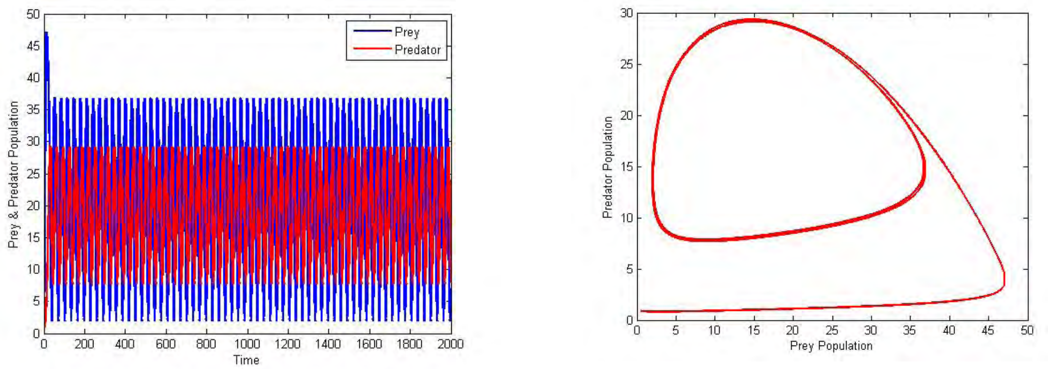

Fig. 2 Shows the trajectories and phase graphs of the model (2.1) with $\beta = 0.67$

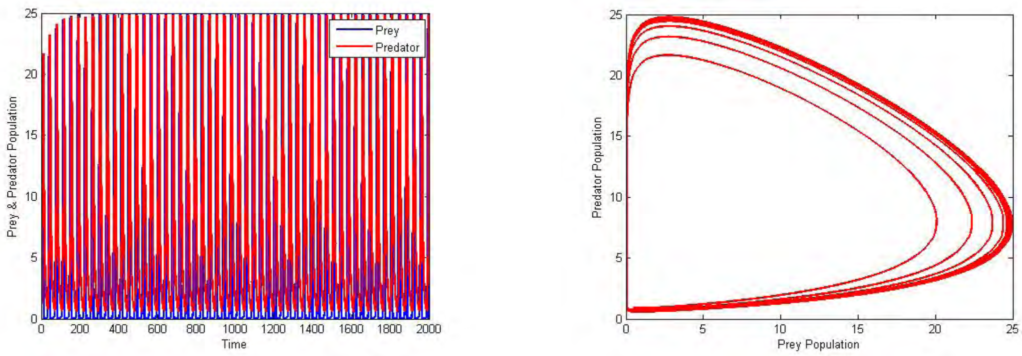

Fig. 3 Shows the trajectories and phase graphs of the model (2.1) with $\beta = 1.8$

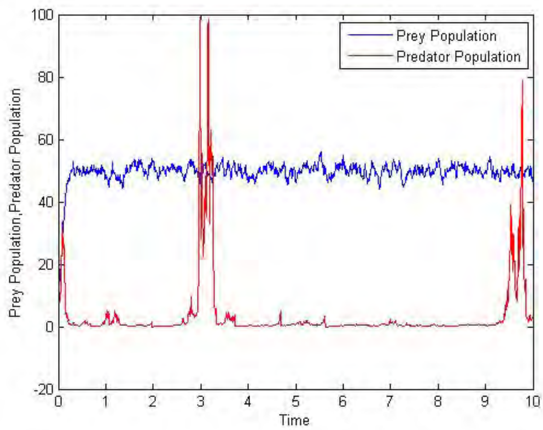

Fig. 4 Shows the trajectories of the model (9.1) with noise.

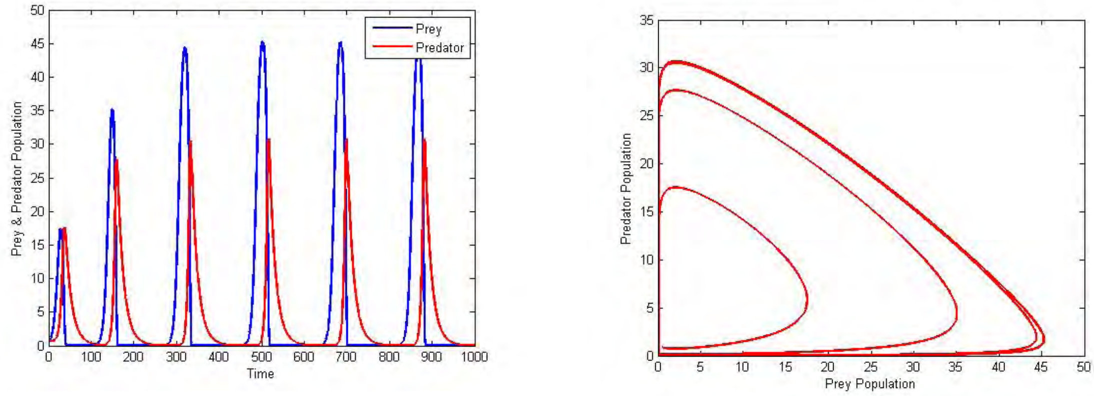

Fig. 5 Shows the trajectories and phase graphs of the model (2.1) with $\gamma = 0.01$

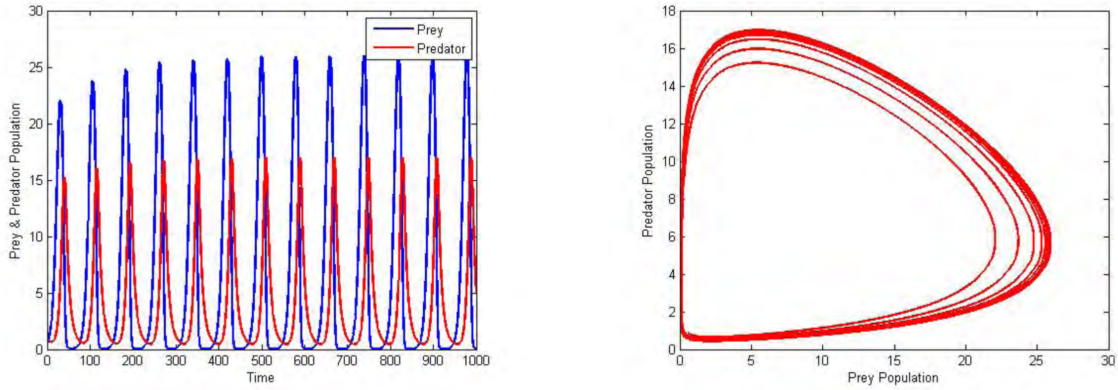

Fig. 6 Shows the trajectories and phase graphs of the model (2.1) with $\gamma = 0.1$

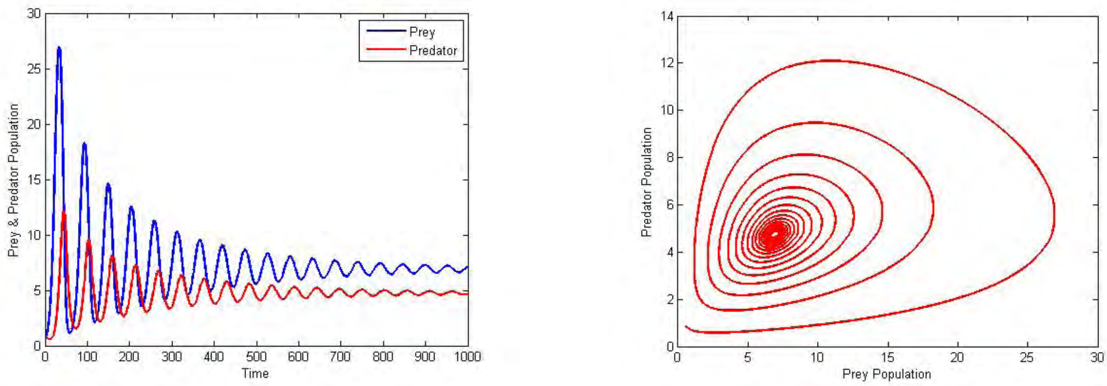

Fig. 7 Shows the trajectories and phase graphs of the model (2.1) with $\gamma = 0.2$

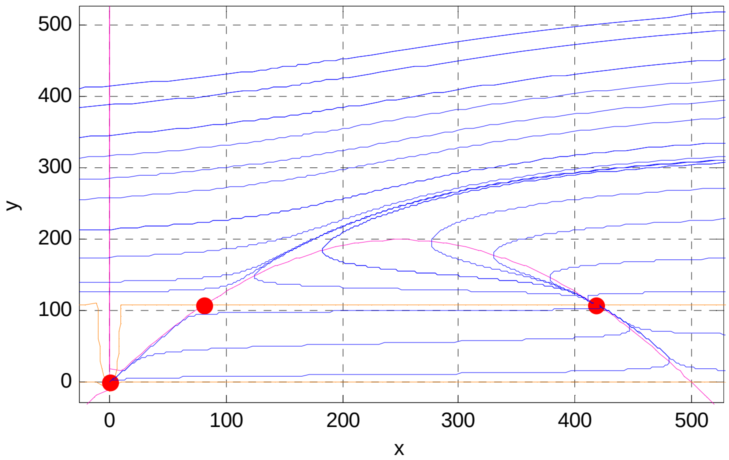

Fig. 8 Illustrates the three equilibrium points of the system (2.1)

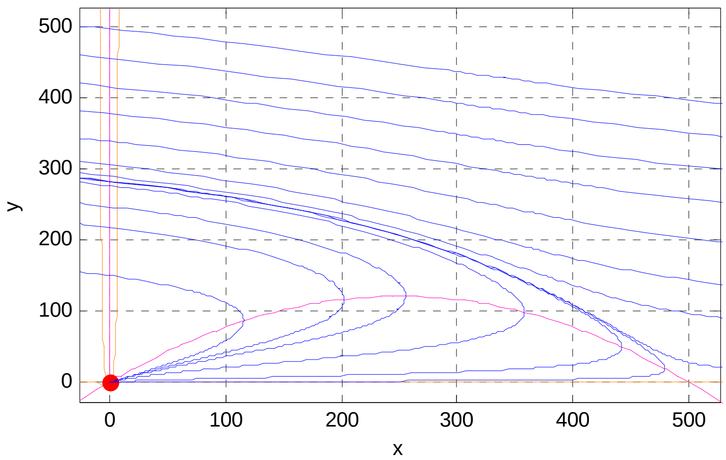

Fig. 9 The equilibrium point E_3

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