Article

Chaotic behavior of harvesting Leslie-Gower predator-prey model

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Abstract
This article deals with the study of some qualitative properties of a harvesting Leslie-Gower predator-prey model. Particularly, we explore the existence, uniqueness, boundedness of positive equilibrium point and local stability analysis of positive equilibrium point. Moreover, it is shown that there exists period-doubling bifurcation and Nimark-sacker bifurcation for the unique positive steady-state of given system. In order to control the bifurcation we introduce a feedback strategy. For further confirmation of complexity and chaotic behavior largest Lyapunov exponents are plotted.

Keywords predator-prey model; period-doubling bifurcation; Nimark-Sacker bifurcation; chaos control.

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1 Introduction

The dynamics of predator and its prey is considerable importance in both mathematical ecology and ecology (Rana, 2015). Volterra (1931) proposed the following predator-prey model:

$$\left.\begin{aligned}\frac{dH}{dt} &= (a - bP)H,\\ \frac{dP}{dt} &= (cH - d)P,\end{aligned}\right\} \qquad (1)$$

where $(a, b, c, d) \in R^{+}$. Moreover, Lotka (1920) obtained the same system during the chemical reaction observation, thus system (1) is known as Lotka-Volterra population model. The dynamics of model (1) is not individually reasonable as a narrative of the interface. An other serious deficiency is that the relative rate of increase of the predator is unbounded above in second part of system (1). In order to overcome these deficiencies, Leslie (1948) formulate the following set of equations:

$$\left.\begin{aligned}\frac{dH}{dt} &= (r - aP - bH)H,\\ \frac{dP}{dt} &= \left(s - d\frac{P}{H}\right)P,\end{aligned}\right\} \qquad (2)$$

where H is the density of prey population and P is the density of predator population at time t. The Leslie-Gower system suggested for predator-prey system is consider as a classic predator-prey model (Elhassanein, 2014). According to the Leslie-Gower equations, there is a reciprocal relationship between decline of a predator population and its preferred food availability per capita. This supposition can be seen from the proportional relation between number of prey and the carrying capacity of predator environment. This result increases in the number of both prey and predator indefinitely, which are not predictable in the Lotka-Volterra system (1). Recently, Korobeinikov (2001) investigate the global dynamic of system (2) and shows that system could not admit oscillatory behavior (Kumar, 2005; Huang at el., 2006) shows that there exists limit cycle in the predator-prey model incorporating Holling type-II or III functional responses. Many scholars investigate the dynamics of model (2). Aguirre et al. (2009) discussed the existence of two limit cycles of system (2) with additive Allee effect. Aziz-Alaoui and Daher Okiye (2003) proposed Holling-type II schemes and modified Leslie-Gower model, and explored the globally stability and boundedness of model. Nindjin et al. (2006) further introduced a time delay scheme for the model defined by Yafia et al. (2008). The bifurcation analysis and limiting behavior of delay model is discussed in Nindjin et al. (2008). Moreover, Leslie-Gower-type food chain system is discussed in Aziz-Alaoui (2002) and Chen et al. (2009). In Huo and Li (2004) prey refuge is introduced and showed that the persistent property of model is independent of refuge. In addition, the Periodic behavior of non-autonomous case of system (2) is studied by Huo and Li (2004). Gakkhar and Singh (2006) explored the dynamical behavior of a Leslie-Gower predator-prey system with seasonally varying parameters. A further impulsive effect is considered by Song and Li (2008). The investigation of harvest of population and biological results are much more important for human needs, such as in wildlife, forestry and fishery management (Makinde, 2007). For these purposes, Zhang at el. (2011) proposed the following considerable predator and prey system which imperilled to constant effort harvesting according to both Leslie-Gower model and this hypothesis and commercial significance;

$$\left.\begin{aligned} \frac{dH}{dt} &= (r - aP - bH)H - cH, \\ \frac{dP}{dt} &= \left(s - d\frac{P}{H}\right)P - kP, \end{aligned}\right\} \tag{3}$$

where P and H are, respectively, the densities of predator species and prey species at time t. Moreover, s and r are the intrinsic growth rate of predator and prey, a is the predation rate, b measures the strength of competition among individual prey, d is a measure of the food quantity that the prey provides converted to predator birth, $\frac{P}{H}$ is the LeslieGower term which measures the loss in the predator population due to rarity of its favorite food, c and k are the constant effort harvesting of prey and predator, respectively. To ensure the sustainable development of both species, it is assumed that $0 < c < r$ and $0 < k < s$. Darti at el. (2015) discussed the local stability analysis of system (3) by introducing a nonstandard finite difference scheme. In order to implement the method of piecewise constant arguments for continuous systems, we assume the average growth at the regular time interval for both papulation. Then system (3) can be written as:

$$\left.\begin{aligned} \frac{1}{H(t)}\frac{dH(t)}{dt} &= (r - aP([t]) - bH([t])) - c, \\ \frac{1}{P(t)}\frac{dP(t)}{dt} &= \left(s - d\frac{P([t])}{H([t])}\right) - k, \end{aligned}\right\} \tag{4}$$

for $0 < t < 1$ the integer part of t is $[t]$. Furthermore, one can integrate system (4) for $t \in [n{:}\, n+1]$ for $n = 0,1,2,..$, and whence obtain the following system;

$$\left.\begin{aligned} H(t) &= H_n e^{[(r - aP_n - bH_n - c)(t-n)]}, \\ P(t) &= P_n e^{\left[\left(s - d\frac{P_n}{H_n} - k\right)(t-n)\right]}. \end{aligned}\right\} \tag{5}$$

Applying $t \to n+1$, we have the following discrete-time Leslie-Gower model:

$$\left.\begin{aligned} H_{n+1} &= H_n e^{(r-aP_n-bH_n-c)}, \\ P_{n+1} &= P_n e^{\left(s-d\frac{P_n}{H_n}-k\right)}. \end{aligned}\right\} \quad (6)$$

Next, we investigate the boundedness of system (6).

2 Boundedness

In order to explore the boundedness of system (6), we need the following remark .

Remark 2.1 [Yang, (2006)] Let X_t exists for $x_0 > 0$, and $x_{t+1} \le x_t e^{A[1-Bx_t]}$ for every $t \in [t_1, \infty]$, where B is positive constant. Then

$$\lim_{n\to\infty} x_{t+1} \le \frac{1}{AB} e^{(A-1)}.$$

Now, one can state the following theorem for the uniform boundedness of system (6), which is direct consequence of Remark 2.1.

Theorem 2.1 Any positive solution$(H_n, P_n)$of system (6) is uniformly bounded.

Proof Let (H_n, P_n) be any positive solution, then from system (6) one has;

$$H_{n+1} \le H_n e^{r[1-\frac{b}{r}H_n]},$$

for all $n = 0,1,2,\ldots$. Assuming that $H_0 > 0$ and by applying Remark 2.1, one can get the following result.

$$\lim_{n\to\infty} Sup H_n \le \frac{1}{b} e^{(r-1)} = U_1. \quad (7)$$

Moreover, from second part of system (6), we have:

$$P_{n+1} \le P_n e^{s[1-\frac{d}{U_1 s}P_n]}.$$

Let $P_0 > 0$ and again by applying Remark (2.1), one can get the following result.

$$\lim_{n\to\infty} Sup P_n \le \frac{U_1}{d} e^{(s-1)} = U_2. \quad (8)$$

Thus it follows that

$$\lim_{n\to\infty} Sup(H_n, P_n) \le U,$$

where $U = max\{U_1, U_2\}$. Thus proof is completed.

3 Existence of Positive Fixed Point and Local Stability

It is easy to see that system (6) has two fixed points, the boundary fixed point $(H_1, 0)$, where $H_1 = \frac{r-c}{b}$ and the unique positive fixed point $(H_*, P_*) = \left(\frac{d(r-c)}{bd+a(s-k)}, \frac{(r-c)(s-k)}{bd+a(s-k)}\right)$. Let

$$F_J(H_*,P_*) = \begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix}$$

be the Jacobian matrix evaluated at (H_*,P_*), then characteristic polynomial of Jacobian matrix is:

$\mathbb{P}(\lambda) = \lambda^2 - B_1\lambda + B_2,$ (9)

where

$$B_1 = w_{11} + w_{22},$$

and

$$B_2 = w_{11}w_{22} - w_{12}w_{21}.$$

In order to discuss the stability of fixed points, we have the following Lemma.

Lemma 3.1 Let $\mathbb{F}(\lambda) = \lambda^2 - B_1\lambda + B_2$, and $\mathbb{F}(1) > 0$ moreover, λ_1, $\lambda_2$are root of $\mathbb{F}(\lambda) = 0$, then:

(i) $|\lambda_1| < 1$and $|\lambda_2| < 1 \Leftrightarrow \mathbb{F}(-1) > 0$ and $B_2 < 1$;

(ii) $|\lambda_1| < 1$ and $|\lambda_2| > 1$ or ($|\lambda_1| > 1$ and $|\lambda_2| < 1$) $\Leftrightarrow \mathbb{F}(-1) < 0$;

(iii) $|\lambda_1| > 1$ and $|\lambda_2| > 1 \Leftrightarrow \mathbb{F}(-1) > 0$ and $B_2 > 1$;

(iv) $\lambda_1 = -1$ and $|\lambda_2| \neq 1 \Leftrightarrow \mathbb{F}(-1) = 0$ and $B_1 \neq 0, 2$;

(v) λ_1 and λ_2 are complex and $|\lambda_1| = 1$ and $|\lambda_2| = 1 \Leftrightarrow {B_1}^2 - 4B_2 < 0$ and $B_2 = 1$.

As λ_1 and λ_2 are eigenvalue of (6) we have the following Topological type results.The point $(H_*,P_*)$is known as sink if $|\lambda_1| < 1$ and $|\lambda_2| < 1$, thus the sink is locally asymptotic stable. (H_*,P_*) is known as source if $|\lambda_1| > 1$ and $|\lambda_2| > 1$. Source is always unstable. (H_*,P_*) is known as saddle if $|\lambda_1| < 1$ and $|\lambda_2| > 1$or ($|\lambda_1| > 1$ and $|\lambda_2| < 1$) is known as non-hyperbolic either $|\lambda_1| = 1$ and $|\lambda_2| = 1$.

At first, we explore the stability analysis of boundary fixed point $(H_1, 0) = (\frac{r-c}{b}, 0)$. The Jacobian matrix at fixed point $(H_1, 0) = (\frac{r-c}{b}, 0)$ is given by;

$$F_J\left(\frac{r-c}{b}, 0\right) = \begin{bmatrix} 1-r+c & \frac{a(r-c)}{b} \\ 0 & e^{(s-k)} \end{bmatrix}$$

The characteristic polynomial of Jacobian matrix is given by;

$$\mathbb{F}(\lambda) = \lambda^2 - (\mathrm{e}^{s-k} + c - r + 1)\lambda + (1 - r + c)\mathrm{e}^{s-k}.$$

Hence, $\mathbb{F}(\lambda) = 0$ has two roots namely, $\lambda_1 = e^{(s-k)}$ and $\lambda_2 = 1 - (r - c)$. In addition, $r > c$ and $s > k$, implies that $|\lambda_1| > 1$ and $|\lambda_2| < 1$ if and only $0 < r - c < 2$ and $|\lambda_2| > 1$ if and only if $r - c > 2$. Hence the boundary fixed point $(H_1, 0) = (\frac{r-c}{b}, 0)$ is source if and only if $r - c > 2$ and saddle point if and only if $0 < r - c < 2$. Next, our objective is to explore the local stability of the unique positive fixed point

$(H_*, P_*) = \left(\frac{d(r-c)}{bd+a(s-k)}, \frac{(r-c)(s-k)}{bd+a(s-k)}\right)$.

Let equation(9) be the characteristic polynomial of Jacobian matrix evaluated at $(H_*, P_*) = \left(\frac{d(r-c)}{bd+a(s-k)}, \frac{(r-c)(s-k)}{bd+a(s-k)}\right)$, where

$$B_1 = 2 + k - s + \frac{bd(c-r)}{bd + a(s-k)}$$

and

$$B_2 = 1 + (1 + c - r)(k - s) + \frac{bd(c-r)}{a(s-k)+bd}.$$

Moreover, $\mathbb{F}(1) = (c-r)(k-s) > 0$ and $\mathbb{F}(-1) = 4 + (2 + c - r)(k - s) + \frac{2bd(c-r)}{a(s-k)+bd}$. Hence, one has the following Lemma for the local stability of unique positive fixed point $(H_*, P_*) = \left(\frac{d(r-c)}{bd+a(s-k)}, \frac{(r-c)(s-k)}{bd+a(s-k)}\right)$.

Proposition 3.1 Let $(H_*, P_*) = \left(\frac{d(r-c)}{bd+a(s-k)}, \frac{(r-c)(s-k)}{bd+a(s-k)}\right)$ be the unique positive fixed point of (6), then the following results hold:

(i) (H_*, P_*) is interior of unit circle if the given below condition satisfied:

$$(k-s)(1+c-r) < \frac{bd(r-c)}{bd+a(s-k)} < \frac{4+(k-s)(2+c-r)}{2}.$$

(ii) (H_*, P_*) is the exterior of the unit disk if the given below conditions hold:

$$\frac{bd(r-c)}{bd+a(s-k)} < \frac{4+(k-s)(2+c-r)}{2} \;\; and \;\; (k-s)(1+c-r) > \frac{bd(r-c)}{bd+a(s-k)}.$$

(iii) (H_*, P_*) is saddle point if the following condition holds:

$$\frac{bd(r-c)}{bd+a(s-k)} > \frac{4+(k-s)(2+c-r)}{2}.$$

(iv) (H_*, P_*) is non-hyperbolic if one of the given below condition holds:

$$(iv.1) \quad r = 2 + c + \frac{4a(k-s)}{a(k-s)^2 + bd(s-k-2)} \;\; and \;\; s - k - 2 \neq \frac{bd(c-r)}{bd - ak + as},$$

$$s - k \neq \frac{bd(c-r)}{bd-ak+as}.$$

$$(iv.2)(k-s) < \frac{bd(r-c)}{bd + a(s-k)} < (k-s) + 4 \;\; and \;\; \mathrm{a} = \frac{bd\left(1 + \Delta + \frac{1}{r-1-c}\right)}{\Delta^2}.$$

4 Bifurcation Analysis of Positive Equilibrium

In this section, we explore period-doubling and Neimark-Sacker bifurcations of unique positive equilibrium point of system (6). For similar theory of bifurcation analysis of discrete-time systems can be found in past references (Din, 2014a, b; Din, 2017; Khan, 2014; Elsadany at el., 2012; He and Lai, 2011; Liu and Xiao, 2007; Jing and Yang, 2006; Sun and Cao, 2007; Zhang at el., 2007; Nedorezov, 2015). According to Lemma 3.1, one of the characteristic root of (9) evaluated at positive fixed point is -1 and other is neither -1 nor 1, if part (iv.1) of proposition 3.1 holds. Thus (6) undergoes period-doubling bifurcation when the parameters vary in the least neighborhood of the following set:

$$B_{S1}=\left\{(a,r,s,d,k)\in\Re^{+}:r=2+c+\frac{4a\Delta}{a\Delta^2-bd(2+\Delta)},A_1\neq 0,2\right\},$$

where $\Delta=k-s$. Moreover, the characteristic root of (9) evaluated at positive fixed point are complex conjugate with absolute 1, if the condition (iv.2) of proposition 3.1 holds. Thus (6) undergoes Neimark-sacker bifurcation when the parameters vary in the least neighborhood of the following set:

$$B_{S2}=\left\{(a,r,s,d,k)\in\Re^{+}:\Delta<\frac{bd(r-c)}{bd-a\Delta}<4+\Delta,a=\frac{bd\left(1+\Delta+\frac{1}{r-1-c}\right)}{\Delta^2}\right\}.$$

First, we study the flip bifurcation of positive equilibrium of system (6). The unique positive equilibrium point $(H_*,P_*)=\left(\frac{d(r-c)}{bd+a(s-k)},\frac{(r-c)(s-k)}{bd+a(s-k)}\right)$ of system (6) undergoes flip bifurcation when parameters vary in a small neighborhood of B_{S1}. Let $r_1=2+c+\frac{4a(k-s)}{a(k-s)^2+bd(s-k-2)}$ and taking parameters $(a,r_1,s,d.k)\in B_{S1}$ arbitrarily, then in terms of parameters $(a,r_1,s,d.k)$, system (6) can be described by the following two dimensional map: $\begin{pmatrix}H\\P\end{pmatrix}\to\begin{pmatrix}He^{(r_1-aP-bH-c)}\\Pe^{\left(s-d\frac{P}{H}-k\right)}\end{pmatrix}$. (10)

Taking $\bar{r}$ as small bifurcation parameter and a perturbation of map (11) can be described by the following map:

$$\begin{pmatrix}H\\P\end{pmatrix}\to\begin{pmatrix}He^{((r_1+\bar{r})-aP-bH-c)}\\Pe^{\left(s-d\frac{P}{H}-k\right)}\end{pmatrix}, \quad (12)$$

where $|\bar{r}|\ll 1$, which is a small perturbation parameter. Taking $x=H-H_*$ and $y=P-P_*$, then map (13) is converted into the following form;

$$\begin{pmatrix}x\\y\end{pmatrix}\to\begin{pmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}+\begin{pmatrix}f_1(x,y,\bar{r})\\f_2(x,y,\bar{r})\end{pmatrix}, \quad (14)$$

where

$$f_1(x,y,\bar{r})=a_{13}x^2+a_{14}xy+a_{15}y^2+b_1x^3+b_2x^2y+b_3xy^2+b_4y^3+e_1\bar{r}x+e_2\bar{r}y$$
$$+e_3\bar{r}xy+e_4\bar{r}y^2+e_5\bar{r}x^2+O((|x|+|y|+|\bar{r}|)^4),$$
$$f_2(x,y,\bar{r})=a_{23}x^2+a_{24}x^3+a_{25}xy+d_1x^2y+d_2y^2+d_3xy^2+d_4y^3+e_6\bar{r}xy+e_7\bar{r}y^2$$
$$+e_8\bar{r}x^2+O((|x|+|y|+|\bar{r}|)^4),$$

$$a_{11}=1+\frac{bd(c-r_1)}{a(s-k)+bd},a_{12}=\frac{ad(c-r_1)}{a(s-k)+bd},a_{21}=\frac{(s-k)^2}{d},a_{22}=1+k-s,$$

$$a_{13}=\frac{1}{2}b\left(\frac{bd(r_1-c)}{a(s-k)+bd}-2\right),a_{14}=a\left(\frac{bd(r_1-c)}{a(s-k)+bd}-1\right),a_{15}=\frac{da^2(r_1-c)}{2a(s-k)+2bd},$$

$$a_{15}=\frac{da^2(r_1-c)}{2a(s-k)+2bd},b_1=\frac{b^2(bd(3+c-r_1)+3a(s-k))}{6a(s-k)+6bd},$$

$$b_2=\frac{1}{2}ab\left(2+\frac{bd(c-r_1)}{a(s-k)+bd}\right),b_3=\frac{a^2(bd(1+c-r_1)+a(s-k))}{2a(s-k)+2bd},$$

$$b_4=\frac{a^3d(c-r_1)}{6a(s-k)+6bd},a_{23}=\frac{(k-s)^2(2+k-s)(a(s-k)+bd)}{2d^2(c-r_1)},$$

$$a_{24}=\frac{(k-s)^2(6+6k+k^2-2(3+k)s+s^2)(a(s-k)+bd)^2}{6d^3(c-r)^2},$$

$$a_{25}=\frac{(k-s)(2+k-s)(a(s-k)+bd)}{d(c-r_1)},d_2=\frac{(2+k-s)(a(s-k)+bd)}{2c-2r_1},$$

$$d_1=\frac{(k-s)(1-s+k)(4+k-s)(a(s-k)+bd)^2}{2d^2(c-r_1)^2},$$

$$d_3=\frac{(2+4k+k^2-2(2+k)s+s^2)(a(s-k)+bd)^2}{2d(c-r_1)^2},$$

$$d_4=\frac{(3+k-s)(a(s-k)+bd)^2}{6(c-r_1)^2},e_1=\frac{bd}{a(k-s)-bd},$$

$$e_2=\frac{da}{a(k-s)-bd},e_3=\frac{bda}{a(s-k)+bd},e_4\frac{a^2d}{2a(s-k)+2bd},$$

$$e_5=\frac{b^2d}{2a(s-k)+2bd},e_6=\frac{a_{25}}{c-r_1},e_7=\frac{d_2}{c-r_1},e_8=\frac{a_{23}}{c-r_1}.$$

Next, we consider the following translation:

$$\binom{x}{y}=T\binom{u}{v}, \qquad (15)$$

where $T=\begin{pmatrix}a_{12} & a_{12}\\ -1-a_{11} & \lambda_2-a_{11}\end{pmatrix}$ be a nonsingular matrix along with transformation (15), the map (16) can be written as:

$$\binom{u}{v}\to\begin{pmatrix}-1 & 0\\ 0 & \lambda_2\end{pmatrix}\binom{u}{v}+\binom{f(u,v,\bar{r})}{g(u,v,\bar{r})}, \qquad (17)$$

where

$$f(u,v,\bar{r})=\left(\frac{\phi\, b_1}{a_{12}\varphi}-\frac{a_{24}}{\varphi}\right)x^3+\left(\frac{\phi\, b_2}{a_{12}\varphi}-\frac{d_1}{\varphi}\right)yx^2+\left(\frac{\phi\, b_3}{a_{12}\varphi}-\frac{d_3}{\varphi}\right)xy^2$$

$$+\left(\frac{\phi\, b_4}{a_{12}\varphi}-\frac{d_4}{\varphi}\right)y^3+\left(\frac{\phi\, a_{15}}{a_{12}\varphi}-\frac{d_2}{\varphi}\right)y^2+\left(\frac{\phi\, a_{14}}{a_{12}\varphi}-\frac{a_{25}}{\varphi}\right)xy$$

$$+\left(\frac{\phi\, a_{13}}{a_{12}\varphi}-\frac{a_{23}}{\varphi}\right)x^2+\frac{\phi\, e_1\bar{r}x}{a_{12}\varphi}+\frac{\phi\, e_2\bar{r}y}{a_{12}\varphi}+\left(\frac{\phi\, e_3}{a_{12}\varphi}-\frac{e_6}{\varphi}\right)\bar{r}yx$$

$$+\left(\frac{\phi\, e_4}{a_{12}\varphi}-\frac{e_7}{\varphi}\right)\bar{r}y^2+\left(\frac{\phi\, e_5}{a_{12}\varphi}-\frac{e_8}{\varphi}\right)\bar{r}x^2+O((|u|+|v|+|\bar{r}|)^4),$$

$$g(u,v,\bar{r})=-\left(\frac{\psi\, b_1}{a_{12}\varphi}-\frac{a_{24}}{\varphi}\right)x^3-\left(\frac{\psi\, b_2}{a_{12}\varphi}-\frac{d_1}{\varphi}\right)yx^2-\left(\frac{\psi\, b_3}{a_{12}\varphi}-\frac{d_3}{\varphi}\right)xy^2$$

$$-\left(\frac{\psi\, b_4}{a_{12}\varphi}-\frac{d_4}{\varphi}\right)y^3-\left(\frac{\psi\, a_{14}}{a_{12}\varphi}-\frac{a_{25}}{\varphi}\right)xy-\left(\frac{\psi\, a_{15}}{a_{12}\varphi}-\frac{d_2}{\varphi}\right)y^2$$

$$-\left(\frac{\psi\, a_{13}}{a_{12}\varphi}-\frac{a_{23}}{\varphi}\right)x^2-\frac{\psi\, e_1\bar{r}x}{a_{12}\varphi}-\frac{\psi\, e_2\bar{r}y}{a_{12}\varphi}-\left(\frac{\psi\, e_3}{a_{12}\varphi}-\frac{e_6}{\varphi}\right)\bar{r}yx$$

$$-\left(\frac{\psi\, e_4}{a_{12}\varphi}-\frac{e_7}{\varphi}\right)\bar{r}y^2-\left(\frac{\psi\, e_5}{a_{12}\varphi}-\frac{e_8}{\varphi}\right)\bar{r}x^2+O((|u|+|v|+|\bar{r}|)^4),$$

where,

$$x=a_{12}(u+v),\ y=\psi u+\phi v,\lambda_2+1=\varphi.$$

Let $W^c(0,0,0)$ be the center manifold of (17) evaluated at (0,0) in a small neighborhood of $\bar{r}=0$, then $W^c(0,0,0)$ can be approximated as follows:

$$W^c(0,0,0)=\{(u,v,\bar{r})\in\mathbb{R}^3: v=m_1u^2+m_2u\bar{r}+m_3\bar{r}^2+O((|u|+|\bar{r}|)^3)\},$$

where

$$m_1=\frac{\psi^3 a_{15}-a_{12}{}^3a_{23}+\psi\, a_{12}{}^2(a_{13}-a_{25})+\psi^2 a_{12}(a_{14}-d_2)}{\varphi\, a_{12}(-1+\lambda_2)},$$

$$m_2=\frac{\psi\,(\psi\, e_2+a_{12}e_1)}{\varphi\, a_{12}(-1+\lambda_2)},m_3=0.$$

Hence, the map restricted to the center manifold $W^c(0,0,0)$ is given by

$$F: u\to -u+k_1u^2+k_2u\bar{r}+k_3u^2\bar{r}+k_4u\bar{r}^2+k_5u^3+O((|u|+|\bar{r}|)^4),$$

where

$$k_1=\left(\frac{\phi\, a_{13}}{a_{12}\varphi}-\frac{a_{23}}{\varphi}\right)a_{12}{}^2+\left(\frac{\phi\, a_{14}}{a_{12}\varphi}-\frac{a_{25}}{\varphi}\right)\psi\, a_{12}+\left(\frac{\phi\, a_{15}}{a_{12}\varphi}-\frac{d_2}{\varphi}\right)\psi^2,$$

$$k_2=\frac{\phi\, e_1}{\varphi}+\frac{\phi\, e_2\psi}{a_{12}\varphi},$$

$$k_3=2\left(\frac{\phi\, a_{13}}{a_{12}\varphi}-\frac{a_{23}}{\varphi}\right)a_{12}{}^2m_2+\left(\frac{\phi\, e_5}{a_{12}\varphi}-\frac{e_8}{\varphi}\right)a_{12}{}^2+\left(\frac{\phi\, a_{14}}{a_{12}\varphi}-\frac{a_{25}}{\varphi}\right)\psi\, a_{12}m_2$$

$$+2\left(\frac{\phi\, a_{15}}{a_{12}\varphi}-\frac{d_2}{\varphi}\right)\psi\,\phi\, m_2+\left(\frac{\phi\, e_4}{a_{12}\varphi}-\frac{e_7}{\varphi}\right)\psi^2+\frac{\phi^2e_2m_1}{a_{12}\varphi}$$

$$+\left(\left(\frac{\phi\, a_{14}}{a_{12}\varphi}-\frac{a_{25}}{\varphi}\right)\phi\, m_2+\left(\frac{\phi\, e_3}{a_{12}\varphi}-\frac{e_6}{\varphi}\right)\psi\right)a_{12},$$

$$k_4=2\left(\frac{\phi\, a_{13}}{a_{12}\varphi}-\frac{a_{23}}{\varphi}\right)a_{12}{}^2m_3+\left(\frac{\phi\, a_{14}}{a_{12}\varphi}-\frac{a_{25}}{\varphi}\right)\psi\, a_{12}m_3+\frac{\phi^2e_2m_2}{a_{12}\varphi}+\frac{\phi\, e_1m_2}{\varphi}$$

$$+\left(\frac{\phi\, a_{14}}{a_{12}\varphi}-\frac{a_{25}}{\varphi}\right)\phi\, m_3a_{12}+2\left(\frac{\phi\, a_{15}}{a_{12}\varphi}-\frac{d_2}{\varphi}\right)\psi\,\phi\, m_3,$$

$$k_5=\left(\frac{\phi\, b_3}{a_{12}\varphi}-\frac{d_3}{\varphi}\right)\psi^2a_{12}+\left(\frac{\phi\, a_{14}}{a_{12}\varphi}-\frac{a_{25}}{\varphi}\right)\psi\, a_{12}m_1+\left(\frac{\phi\, a_{14}}{a_{12}\varphi}-\frac{a_{25}}{\varphi}\right)\phi\, m_1a_{12}$$

$$+\left(\frac{\phi\, b_1}{a_{12}\varphi}-\frac{a_{24}}{\varphi}\right)a_{12}{}^3+\left(\frac{\phi\, b_2}{a_{12}\varphi}-\frac{d_1}{\varphi}\right)\psi\, a_{12}{}^2+2\left(\frac{\phi\, a_{13}}{a_{12}\varphi}-\frac{a_{23}}{\varphi}\right)a_{12}{}^2m_1$$

$$+\left(\frac{\phi\, b_4}{a_{12}\varphi}-\frac{d_4}{\varphi}\right)\psi^3+2\left(\frac{\phi\, a_{15}}{a_{12}\varphi}-\frac{d_2}{\varphi}\right)\psi\,\phi\, m_1.$$

Next, we define the following two nonzero real numbers:

$$l_1 = \left(\frac{\partial^2 f}{\partial u \partial \tilde{r}} + \frac{1}{2}\frac{\partial F}{\partial \tilde{r}}\frac{\partial^2 F}{\partial u^2}\right)_{(0,0)} = \frac{\phi\, e_1}{\varphi} + \frac{\phi\, e_2 \psi}{a_{12}\varphi} \neq 0,$$

$$l_2 = \left(\frac{1}{6}\frac{\partial^3 F}{\partial u^3} + \left(\frac{1}{2}\frac{\partial^2 F}{\partial u^2}\right)^2\right)_{(0,0)} = k_5 + k_1^2 \neq 0.$$

Due to aforemention analysis, we have the following result about period-doubling bifurcation of system (6).

Theorem 4.1 If $l_2 \neq 0$, then system (6) undergoes period-doubling bifurcation at the unique positive equilibrium (H_*, P_*), when parameter r varies in small neighborhood of r_1. Furthermore, if $l_2 > 0$, then the period-two orbits that bifurcate from (H_*, P_*)) are stable and if $l_2 < 0$, then these orbits are unstable.

Next, we discuss the Neimark-Saker bifurcation for system (6) at unique positive fixed point (H_*, P_*). We explore the conditions for which system (6) have a non-hyperbolic positive fixed point along with a pair of complex conjugate root of (9) with $|\lambda_{1,2}| = 1$. $\mathbb{P}(\lambda) = 0$ has two complex conjugate roots with $|\lambda_{1,2}| = 1$ if the condition(iv.2) of proposition 3.1 is satisfied: In order to observe the Neimark-Sacker bifurcation, one can choose the parameters (a, r, s, d, k) from the set B_{S2}, whence the variation of parameters in the neighborhood of B_{S2}, results in the Neimark-Sacker bifurcation for unique positive equilibrium point (H_*, P_*). Assuming system (6) with parameters (a_1, r, s, d, k), which refers to the following map:

$$\begin{pmatrix} H \\ P \end{pmatrix} \to \begin{pmatrix} He^{(r - a_1 P - bH - c)} \\ Pe^{\left(s - d\frac{P}{H} - k\right)} \end{pmatrix}. \qquad (18)$$

It is trivial to see that mapping (18) has a unique positive equilibrium point (H_*, P_*) such that (H_*, P_*) is unique positive equilibrium fixed of system (3). Since $(a_1, r, s, d, k) \in B_{S2}$ and $a_1 = \frac{bd\left(1 + \Delta + \frac{1}{r-1-c}\right)}{\Delta^2}$. Taking $\tilde{a}$ as bifurcation parameter and considering the perturbation of (18) as follows:

$$\begin{pmatrix} H \\ P \end{pmatrix} \to \begin{pmatrix} He^{(r - (a_1 + \tilde{a})P - bH - c)} \\ Pe^{\left(s - d\frac{P}{H} - k\right)} \end{pmatrix}, \qquad (19)$$

where $|\tilde{a}| \ll 1$ is taken as small perturbation parameter. Next we consider the transformation $x = H - \frac{d(c-r)}{bd + a(s-k)}$, $y = P - \frac{(c-r)(k-s)}{bd + a(s-k)}$ so that map (18) is transferred into the following form:

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} g_1(x,y) \\ g_2(x,y) \end{pmatrix}, \qquad (20)$$

where

$$g_1(x,y) = a_{13}x^2 + a_{14}xy + a_{15}y^2 + b_1x^3 + b_2x^2y + b_3xy^2 + b_4y^3 + O((|x| + |y|),$$

$$g_2(x,y) = a_{23}x^2 + a_{24}x^3 + a_{25}xy + d_1x^2y + d_2y^2 + d_3xy^2 + d_4y^3 + O((|x| + |y|),$$

and $a_{11}, a_{12}, a_{21}, a_{22}, a_{13}, a_{14}, a_{15}, b_1, b_2, b_3, b_4, a_{23}, a_{24}, a_{25}, d_1, d_2, d_3, d_4$ are given in (21) by replacing r into $r_1 + \tilde{r}$. The characteristic equation of variational matrix of system (20) valued at the equilibrium (0,0) can be described as follows:

$$\lambda^2 - p(\tilde{a})\lambda + q(\tilde{a}) = 0, \qquad (22)$$

where

$$p(\tilde{a}) = 2 + k - s + \frac{bd(c-r)}{bd - (a_1 + \tilde{a})(k-s)},$$
$$q(\tilde{a}) = 1 + k(1 + c - r) + (r - c - 1)s + \frac{bd(c-r)}{bd - (a_1 + \tilde{a})(k-s)}.$$

Since $(a_1, r, s, d, k) \in B_{S2}$, the zeros of (22) are complex numbers λ_1, λ_2 such that $\lambda_1 = \overline{\lambda}_2$ and $|\lambda_{1,2}| = 1$. Then it follows that:

$$\lambda_1, \lambda_2 = \frac{p(\tilde{a})}{2} \pm \frac{i}{2}\sqrt{4q(\tilde{a}) - p^2(\tilde{a})}.$$

Then we obtain

$$|\lambda_1| = |\lambda_2| = \sqrt{q(\tilde{a})}, \left(\frac{d\sqrt{q(\tilde{a})}}{d\tilde{a}}\right)_{\tilde{a}=0} = \frac{bd(c-r)(k-s)}{2\left(bd - a_1(k-s)\right)^2} \frac{1}{\sqrt{1 + (k-s)(1+c-r) + \frac{bd(c-r)}{bd-(k-s)a_1}}} > 0.$$

Since $(a_1, r, s, d, k) \in B_{S2}$, it follows that $-2 < p(0) = 2 + k - s + \frac{bd(c-r)}{bd-(k-s)a_1} < 2$. Next, assuming that $p(0) \neq 0,1$, that is, $a_1 \neq \frac{bd(2+c+k-r-s)}{(k-s)(2+k-s)}$ and $a_1 \neq \frac{bd(1+c+k-r-s)}{(k-s)(1+k-s)}$. Thus $p(0) \neq \pm 2, 0, -1$ gives $\lambda_1^m, \lambda_2^m \neq 1$ for all $m = 1,2,3,4$ at $\tilde{a} = 0$. Hence, zeros of (22) do not lie in the intersection of the unit circle with the coordinate axes when $\tilde{a} = 0$ and if the following conditions are satisfied:

$$a_1 \neq \frac{bd(2+c+k-r-s)}{(k-s)(2+k-s)}, a_1 \neq \frac{bd(1+c+k-r-s)}{(k-s)(1+k-s)}. \qquad (23)$$

The canonical form of (20) at $\tilde{a} = 0$ can be obtain by taking $\gamma = \frac{p(0)}{2}$, $\delta = \frac{1}{2}\sqrt{4q(0) - p^2(0)}$ and assuming the following transformation:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_{12} & 0 \\ \gamma - a_{11} & -\delta \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}. \qquad (24)$$

By using transformation (24), one has the following canonical form of system (20):

$$\begin{pmatrix} u \\ v \end{pmatrix} \to \begin{pmatrix} \gamma & -\delta \\ \delta & \alpha \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} \tilde{f}(u,v) \\ \tilde{g}(u,v) \end{pmatrix}, \qquad (25)$$

where

$$\tilde{f}(u,v) = \frac{a_{13}}{a_{12}}x^2 + \frac{a_{14}}{a_{12}}xy + \frac{a_{15}}{a_{12}}y^2 + \frac{b_1}{a_{12}}x^3 + \frac{b_2}{a_{12}}x^2y + \frac{b_3}{a_{12}}xy^2 + \frac{b_4}{a_{12}}y^3 + O((|u| + |v|)^4),$$

$$\tilde{g}(u,v) = \left(\frac{(\gamma - a_{11})a_{13}}{a_{12}\delta} - \frac{a_{23}}{\delta}\right)x^2 + \left(\frac{(\gamma - a_{11})a_{14}}{a_{12}\delta} - \frac{a_{25}}{\delta}\right)xy + \left(\frac{(\gamma - a_{11})a_{15}}{a_{12}\delta} - \frac{d_2}{\delta}\right)y^2$$
$$+ \left(\frac{(\gamma - a_{11})b_1}{a_{12}\delta} - \frac{a_{24}}{\delta}\right)x^3 + \left(\frac{(\gamma - a_{11})b_2}{a_{12}\delta} - \frac{d_1}{\delta}\right)x^2y + \left(\frac{(\gamma - a_{11})b_3}{a_{12}\delta} - \frac{d_3}{\delta}\right)xy^2$$
$$+ \left(\frac{(\gamma - a_{11})b_4}{a_{12}\delta} - \frac{d_4}{\delta}\right)y^3 + O((|u| + |v|)^4),$$

$x = a_{12}u$ and $y = (\gamma - a_{11})u - \delta v$. whence, one has the following nonzero real number:

$$L = \left(\left[-Re\left(\frac{(1-2\lambda_1)\lambda_2^2}{1-\lambda_1}\xi_{20}\xi_{11}\right) - \frac{1}{2}|\xi_{11}|^2 - |\xi_{02}|^2 + Re(\lambda_2\xi_{21})\right]\right)_{\tilde{a}=0},$$

where

$$\xi_{20} = \frac{1}{8}\left[\tilde{f}_{uu} - \tilde{f}_{vv} + 2\tilde{g}_{uv} + i\left(\tilde{g}_{uu} - \tilde{g}_{vv} - 2\tilde{f}_{uv}\right)\right],$$

$$\xi_{11} = \frac{1}{4}\left[\tilde{f}_{uu} + \tilde{f}_{vv} + i(\tilde{g}_{uu} + \tilde{g}_{vv})\right],$$

$$\xi_{02} = \frac{1}{8}\left[\tilde{f}_{uu} - \tilde{f}_{vv} - 2\tilde{g}_{uv} + i\left(\tilde{g}_{uu} - \tilde{g}_{vv} + 2\tilde{f}_{uv}\right)\right],$$

$$\xi_{21} = \frac{1}{16}\left[\tilde{f}_{uuu} + \tilde{f}_{uvv} + \tilde{g}_{uuv} + \tilde{g}_{vvv} + i\left(\tilde{g}_{uuu} + \tilde{g}_{uvv} - \tilde{f}_{uuv} - \tilde{f}_{vvv}\right)\right].$$

One can notice that, the sufficient condition for existence of Neimark-Sacker bifurcation is that L must be nonzero Kuznetsov (1997). Due to aforemention analysis, we have the following significance for direction and existence of Neimark-Sacker bifurcation, see Guckenheimer and Holmes (1983), Robinson (1999), and Wiggins (2003).

Theorem 4.2 There exists Neimark-Sacker bifurcation at (H_*, P_*) whenever $a$varies in a least neighborhood of $a_1 = \frac{bd\left(1+k+\frac{1}{-1-c+r}-s\right)}{(k-s)^2}$. In addition, if $L < 0, (L > 0)$, respectively, then an attracting or repelling invariant closed curve bifurcates from the equilibrium point for $a > a_1$,

$(a < a_1)$, respectively.

5 Hybrid Control of Period-Doubling Bifurcation

In order to control the period-doubling bifurcation in the system (6), we apply the method of hybrid control. This method is considered as control strategy see also Luo at el. (2003, 2004), Chen and Yu (2005), and Elabbasy at el. (2007).

Consider the following controlled system corresponding to system (6);

$$\begin{pmatrix} H_{n+1} \\ P_{n+1} \end{pmatrix} \rightarrow \begin{pmatrix} \theta\left(H_n e^{(r-aP_n-bH_n-c)}\right) + (1-\theta)H_n \\ \theta(P_n e^{\left(s-d\frac{P_n}{H_n}-k\right)}) + (1-\theta)P_n \end{pmatrix}, (26)$$

where $0 < \theta < 1$. The original system (6) and the corresponding controlled system(26) has the same fixed point, the variational matrix at positive fixed point (H_*, P_*) of controlled system can be written as:

$$\begin{pmatrix} 1 + \frac{bd(c-r)\theta}{bd+a(-k+s)} & \frac{ad(c-r)\theta}{bd+a(-k+s)} \\ \frac{(k-s)^2\theta}{d} & 1-(k-s)\theta \end{pmatrix}.$$

The following result gives condition for local asymptotic stability of positive equilibrium (H_*, P_*) of the controlled system (26).

Theorem 5.1 The equilibrium population (H_*, P_*) of control system (26) is locally asymptotically stable if and only if the following condition hold:

$$\left|2 + (k - s)\theta + \frac{bd(c - r)\theta}{bd + a(s - k)}\right| < 1 + \theta^2(r - c)(s - k) + \frac{bd(c + k - r - s)\theta - a(s - k)^2}{bd + a(s - k)} + 1 < 2.$$

6 Numerical Simulation

Example 6.1 Consider the particular values of the parameters, $c = 2,\ s = 6,\ k = 5,\ d = 29.4,\ b = 60,\ a = 4$ and $r \in [3.5, 6]$, in this case we have the following form of system (6);

$$\left.\begin{aligned} H_{n+1} &= H_n e^{(r - 4P_n - 40H_n - 2)}, \\ P_{n+1} &= P_n e^{\left(6 - 29.4\frac{P_n}{H_n} - 6\right)}. \end{aligned}\right\} \quad (27)$$

Moreover, according to aforemention particular values of biological constants, system (27) has unique positive equilibrium point $(H_*,\ P_*) = (0.02494343891, 0.0008484162896)$. Thus both the papulation undergoes period-doubling bifurcation and the bifurcation diagrams are given in (Fig. 1 and Fig. 2). In addition, the maximum lypunov exponent is shown in (Fig. 3).

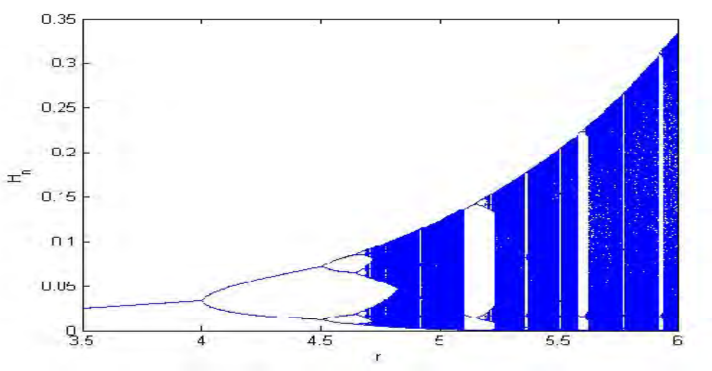

Fig. 1 Bifurcation diagram for H_n.

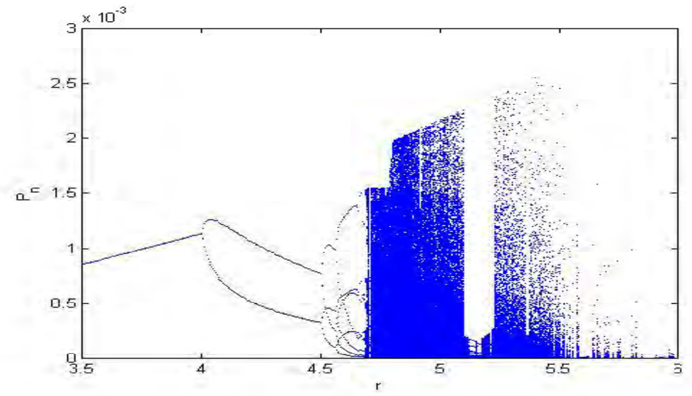

Fig. 2 Bifurcation diagram for P_n.

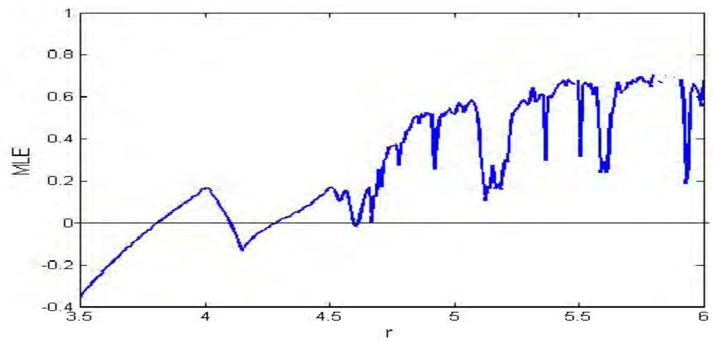

Fig. 3 MLE.

Example 6.2 Let $b = 1.8,\ c = 0.99,\ d = 0.4,\ s = 6,\ k = 5,\ r = 3$ and $a \in (0.1,2.9)$. Then, the system (6) has the following mathematical form:

$$\left.\begin{aligned} H_{n+1} = &\quad H_n e^{(3-aP_n-1{,}8H_n-0.99)}, \\ P_{n+1} = &\quad P_n e^{\left(6-0.4\frac{P_n}{H_n}-5\right)}. \end{aligned}\right\} \qquad (28)$$

In this case $(H_*, P_*) = (0.5289473684, 1.322368421)$. According to these parametric values, the plots of H_n and P_n are given in (Fig. 4 and Fig. 5), which shows that both papulation undergoes Neimark-sacker bifurcation. Moreover, the maximum lypunov exponents are shown in (Fig. 6). Moreover, phase portrait for different values of the bifurcation parameter a are shown in (Fig. 7, Fig. 8, Fig. 9, Fig. 10, Fig. 11, Fig. 12, Fig. 13 and Fig. 14), which confirm the existence of the Neimark-sacker bifurcation when a passes through $a = 0.51063$ (see Fig. 8) and for $a = 2.87$ a chaotic attractor shows that there exist chaos in system (6), (see Fig. 13 and Fig. 14).

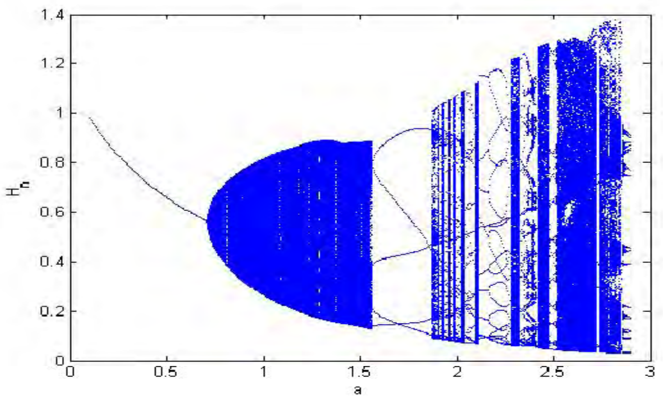

Fig. 4 Bifurcation diagram for H_n.

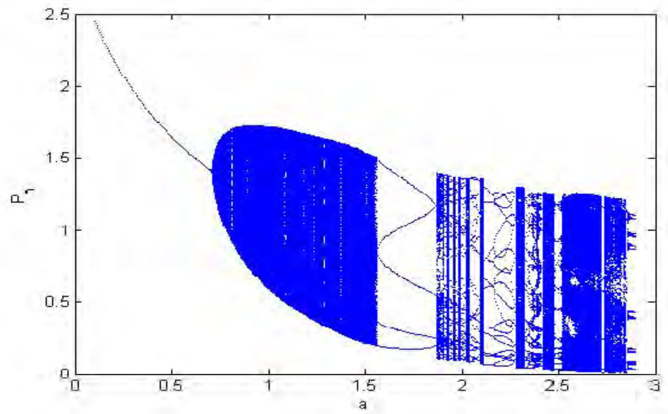

Fig. 5 Bifurcation diagram for P_n.

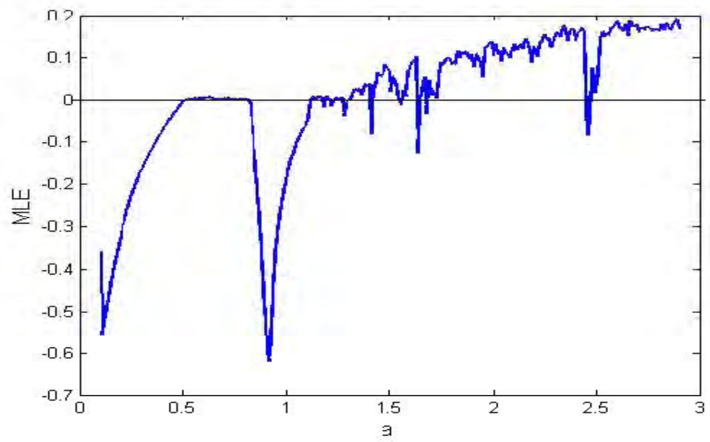

Fig. 6 MLE

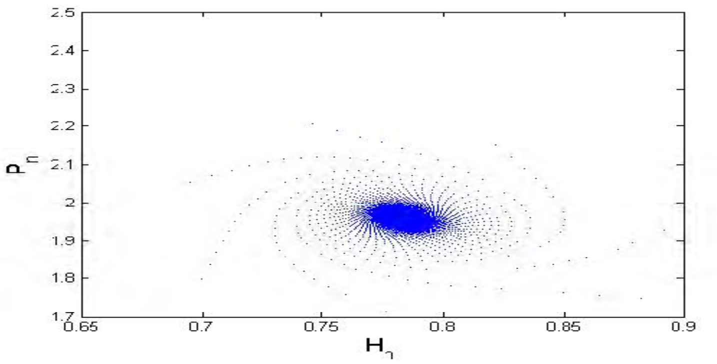

Fig. 7 Phase portrait for $a = 0.5103$

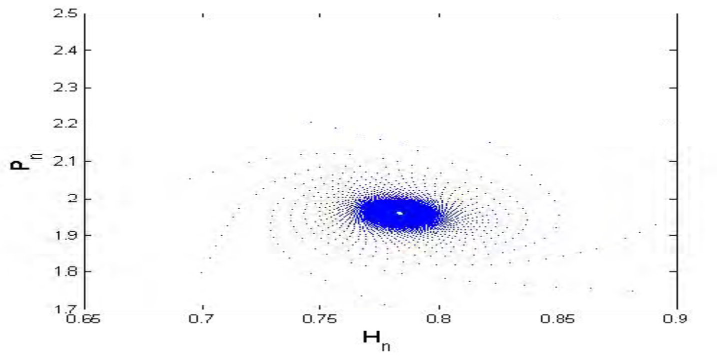

Fig. 8 Phase portrait for $a = 0.51063$

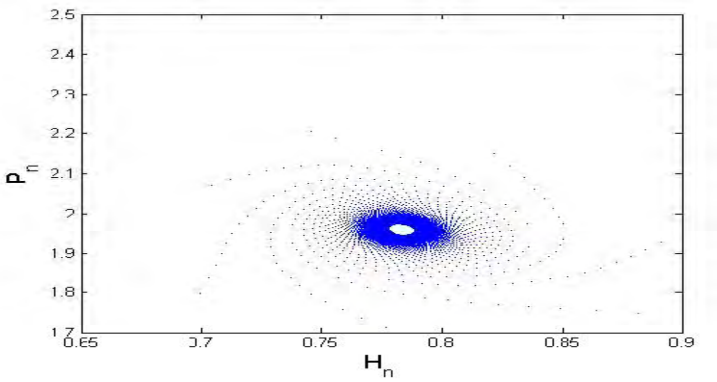

Fig. 9 Phase portrait for $a = 0.5107$

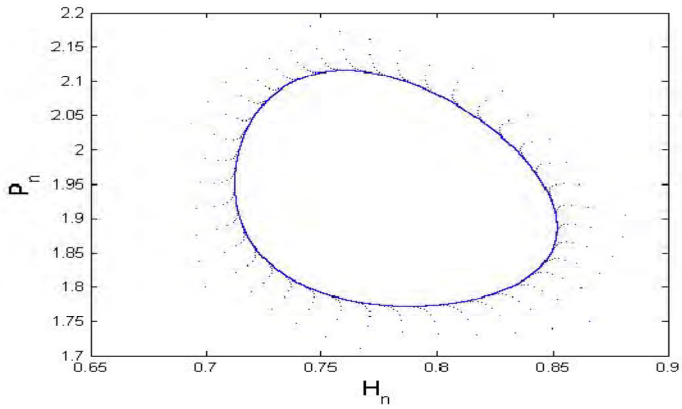

Fig. 10 Phase portrait for $a = 0.5207$

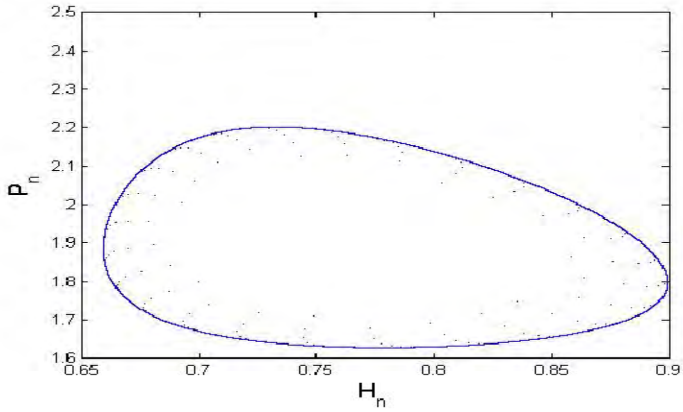

Fig. 11 Phase portrait for $a = 0.5407$

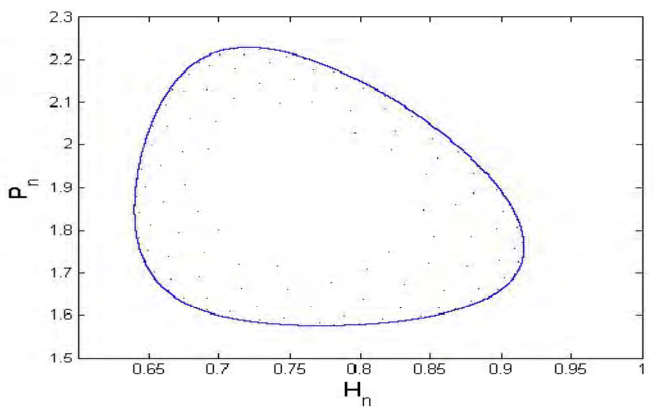

Fig. 12 Phase portrait for $a = 0.5508$

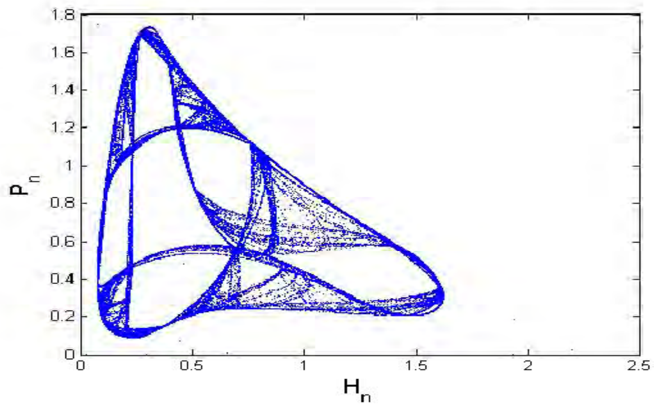

Fig. 13 Phase portrait for $a = 1.870$

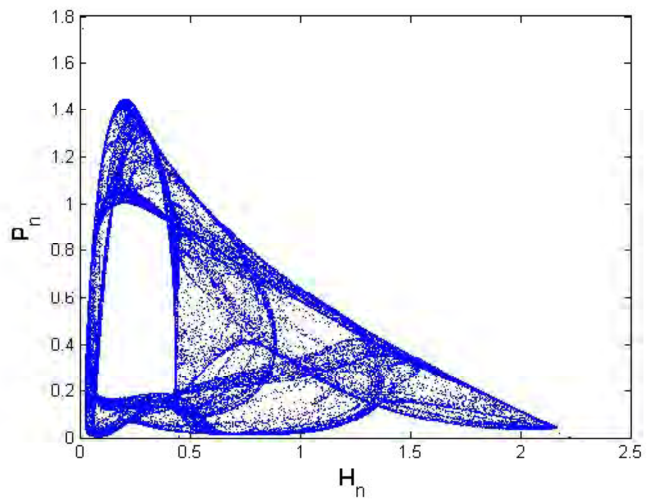

Fig. 14 Phase portrait for $a = 2.87$

Example 6.3 Let $c = 2,\ s = 6,\ k = 5,\ d = 29.4,\ b = 60,\ a = 4,\ r = 5$ and $\theta \in [0,1]$, the controlled system (26) is locally asymptotically stable if and only if $0 < \theta < 0.668941$. Moreover, one can see that the uncontrolled system (6) undergoes period-doubling bifurcation (see Fig. 1 and Fig.2) for aforemention parametric values, but on other hand in the controlled system (26) the period-doubling bifurcation is controlled (see Fig. 15).

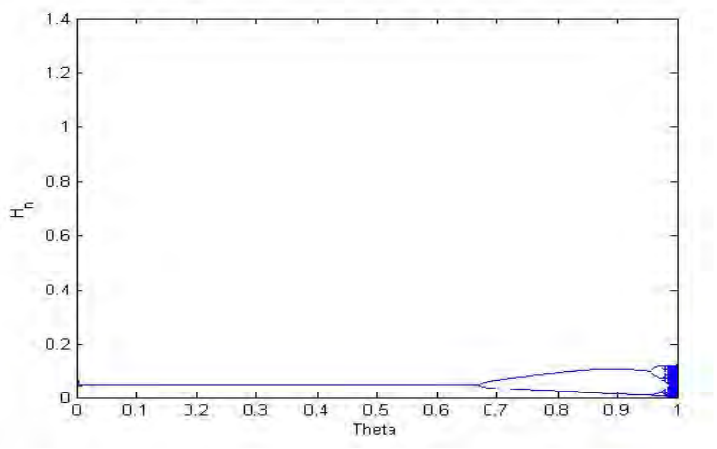

Fig. 15 Bifurcation diagram for H_n.

Example 6.4 In order to explore the controllability of Neimark-sacker bifurcation, we fixed the parameters $b = 1.8,\ c = 0.99,\ d = 0.4,\ s = 6,\ k = 5,\ r = 3,\ a = 1.8$ and $\theta \in (0,1)$, then controlled system (26) is locally asymptotically stable if and only if $0 < \theta < 0.8$. Moreover, Neimark-saker bifurcation is controlled for the maximum range of controlled parameter θ (see Fig. 16 and Fig. 17). Moreover, In order to confirm the stability of system (26), the plot of H_n and P_n along with a phase portrait are shown in (Fig. 18, Fig. 19 and Fig. 20).

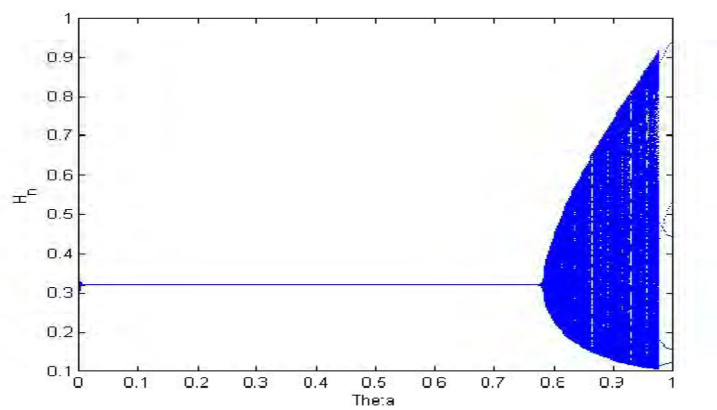

Fig. 16 Bifurcation diagram for H_n

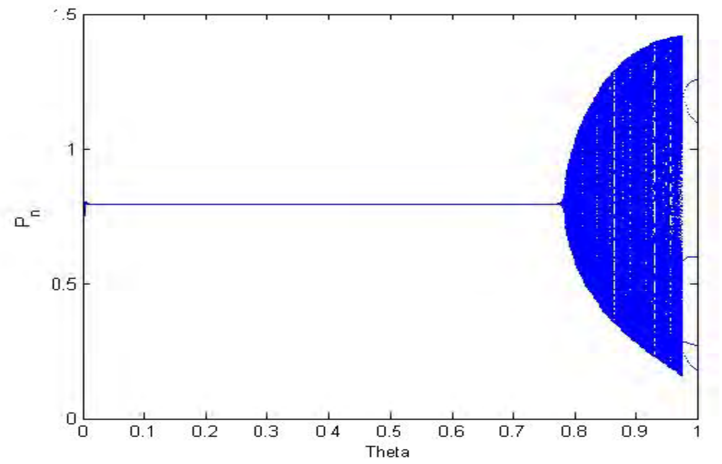

Fig. 17 Bifurcation diagram for P_n

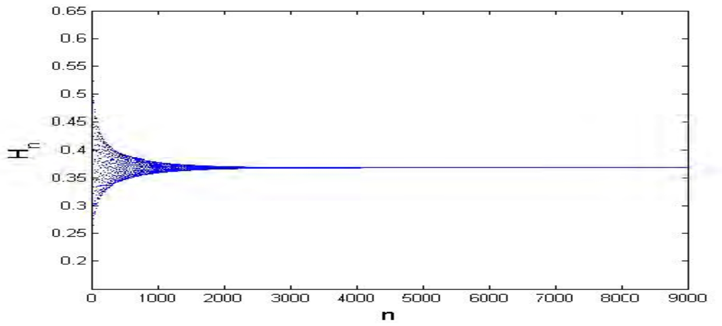

Fig. 18 Plot of H_n for controlled system (26)

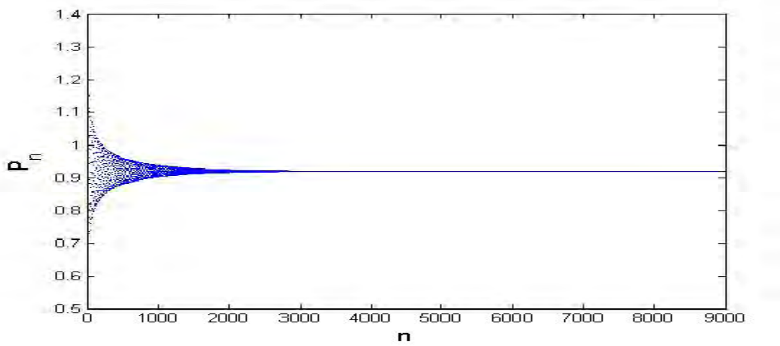

Fig. 19 Plot of P_n for controlled system (26)

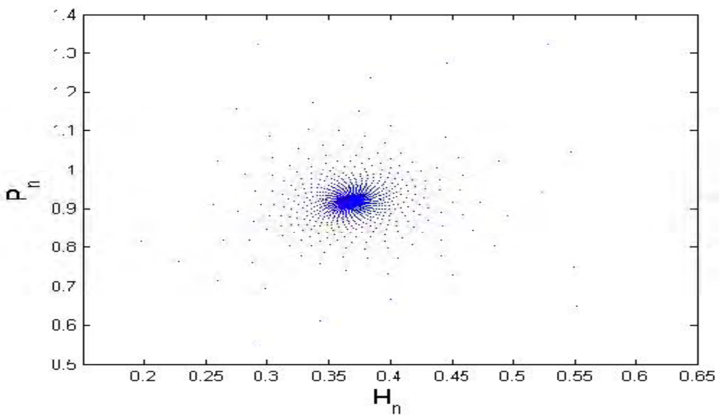

Fig. 20 Phase portrait for controlled system (26)

7 Concluding Remarks

In this paper, we study aharvesting Leslie-Gower predator-prey model. We discretize the system of differential equations by implemented the method of piecewise constant arguments for continuous systems. In particular, we discuss the boundedness and existence of unique positive equilibrium point of discrete time version of a harvesting Leslie-Gower predator-prey model (3). Parametric conditions are successfully calculated for local asymptotic stability of model (6), existence and direction of Neimark-sacker and period-doubling bifurcation are computed explicitly and it is shown that system (6) undergoes period-doubling and Neimark-sacker bifurcations when parameter r varies in the neighborhood of $r_1 = 2 + c + \frac{4a(k-s)}{a(k-s)^2+bd(s-k-2)}$ and parameter a varies in the neighborhood of $a_1 = \frac{bd\left(1+\Delta+\frac{1}{r-1-c}\right)}{\Delta^2}$. Moreover, in order to control the bifurcations we have implemented the hybrid control technique, hence stability of fixed point is restored for wide range of both parameters a and r. Finally, some numerical computations with graphical analysis are provided to shows the correctness of theoretical investigation.

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