Article

Dynamics of three species plankton model with Holling type-IV functional responses and control of toxic phytoplankton

Rashi Gupta
Department of Mathematics, National Institute of Technology, G E Road, Raipur – 492010, (C.G.), India
E-mail: rashigupta1234@gmail.com

Received 8 June 2019; Accepted 20 June 2019; Published 1 September 2019

Abstract
In this paper we proposed a minimal model of non-toxic phytoplankton - toxic phytoplankton - zooplankton dynamics with Holling type-II and Holling type-IV functional responses. We carried out the analytical study of spatial and non-spatial model for one dimensional system in detail and found out the condition for diffusive instability of a locally stable equilibrium. With the help of numerical simulations, we have observed that when the rate of inhibition of zooplankton growth by toxic material ingested in feeding on toxic phytoplankton is very large, then because of high toxic effect, the zooplankton goes to extinction.

keywords plankton; planktonic bloom; diffusion; spatio-temporal pattern.

1 Introduction
Phytoplankton are photo-synthesizing microscopic organisms of the plankton community and a key part of oceans, seas and freshwater basin ecosystems. The phytoplankton communities consist of accumulation of species with different morphological (size, shape) and physiological (nutrition mode, reproduction) characteristics and whose organization is a key to understand the dynamics of any ecosystem. A significant number of species of phytoplankton has been found that have the ability to produce toxic or inhibitory compounds (Chattopadhyay, 2002; Hallegrae, 1993; Sarkar, 2003; Steidinger, 1996). These are called toxin-producing phytoplankton (TPP), and distinguish them from non-toxic phytoplankton (NTP). These TPP are entirely different from other phytoplankton in biochemical nature.

Reduction in zooplankton density due to release of toxic substances by phytoplankton is one of the most indispensable parameters in this context (Fay, 1983; Keating, 1976; Kirk, 1992; Lefevre, 1952). Buskey and Stockwell (1993) have demonstrated in their field studies that micro- and meso-zooplankton populations are reduced during the blooms of a chrysophyte Aureococcus anophagefferens on the southern Texas coast. Toxicity may be a strong mediator of zooplankton feeding rate, as shown in both field studies (Estep, 1990; Hansen and Nielsen, 1990) and laboratory studies (Buskey, 1995; Huntley, 1986; Ives, 1987; Nejstgaard,
(1996). With the help of field observation and mathematical modelling, Roy et al. (2006, 2007) discovered the role of TPP in determining the dynamics and main-taining diversity of the overall phytoplankton and zooplankton species in the Bay of Bengal. Roy (2008) also studied the space-time framework for promo-tion of plankton diversity due to the presence of TPP. These observations indicate that the toxic substance as well as toxic phyto-plankton plays an important role in the growth of the zooplankton population and has a great impact on phytoplankton-zooplankton interactions. He investigated the effects of spatial interaction on plankton populations in the presence of toxic species.

In this paper, we propose and analyze a three-component mathematical model consisting of the NTP, TPP, and zooplankton populations for modelling the plankton dynamics in spatially distributed population with local diffusion. Here, we assume that the local growth of the prey is logistic and that the predator shows the Holling type II functional response for non-toxin-producing (NTP) and Holling type IV functional for TPP. We obtained the conditions for local stability of the model system in the absence and presence of diffusion. We also obtained the criteria for turing in-stability. We numerically simulated the model system using estimated parameter values. TPP provides a mechanism for switching of plankton dynamics from limit cycle to stability. Our observation indicates that TPP has a significant controlling command on zooplankton.

2 Model System
We formulate a mathematical model of those interacting groups: non-toxic phytoplankton, toxic phytoplankton and zooplankton under the following assumptions:

(i) Each of non-toxic phytoplankton and toxic phytoplankton population follow logistic growth.

(ii) The groups of phytoplankton exhibit Holling type-II and Holling type-IV functional response to the grazer zooplankton.

(iii) Toxic materials ingested on predation of Toxic Phytoplankton cause a significant inhibitory effect on zooplankton growth.

We consider a reaction-diffusion model for non-toxic phytoplankton, toxic phytoplankton and zooplankton system.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_N(x,t)$</td>
<td>Concentrations of non-toxic phytoplankton at any location $x$ and time $t$.</td>
</tr>
<tr>
<td>$P_T(x,t)$</td>
<td>Concentrations of toxic phytoplankton at any location $x$ and time $t$.</td>
</tr>
<tr>
<td>$Z(x,t)$</td>
<td>Concentrations of zooplankton at any location $x$ and time $t$.</td>
</tr>
<tr>
<td>$k_1$</td>
<td>Carrying capacity of phytoplankton which is shared by non-toxic phytoplankton.</td>
</tr>
<tr>
<td>$k_2$</td>
<td>Carrying capacity of phytoplankton which is shared by toxic phytoplankton.</td>
</tr>
<tr>
<td>$r_1$</td>
<td>Constant intrinsic growth rate of non-toxic phytoplankton population.</td>
</tr>
<tr>
<td>$r_2$</td>
<td>Constant intrinsic growth rate of toxic phytoplankton population.</td>
</tr>
<tr>
<td>$w_1$</td>
<td>Rate at which non-toxic phytoplankton are consumed by zooplankton.</td>
</tr>
<tr>
<td>$w_2$</td>
<td>Rate at which toxic phytoplankton are consumed by zooplankton.</td>
</tr>
</tbody>
</table>
The maximum rate of gain in zooplankton growth due to predation of non-toxic phytoplankton at a rate $w_1$.

The rate of inhibition of (or reduction in) zooplankton growth by toxic material ingested in feeding on toxic phytoplankton.

Half saturation constant for non-toxic phytoplankton.

Half saturation constant for toxic phytoplankton.

Mortality rate of zooplankton due to natural death.

Diffusion coefficient of non-toxic phytoplankton.

Diffusion coefficient of toxic phytoplankton.

Diffusion coefficient of zooplankton.

Here, $\nabla^2 = \frac{\partial^2}{\partial x^2}$

Based on earlier assumptions, plankton dynamics may be written as follows:

$$\frac{\partial P_N}{\partial t} = r_N P_N \left(1 - \frac{P_N + \alpha P_T}{k_1}\right) - \frac{w_1 P_N Z}{m + P_N} + D_N \nabla^2 P_N,$$

$$\frac{\partial P_T}{\partial t} = r_T P_T \left(1 - \frac{P_T + \alpha P_N}{k_2}\right) - \frac{w_2 P_T Z}{P_T^2 + P_T + 1} + D_T \nabla^2 P_T,$$

$$\frac{\partial Z}{\partial t} = \frac{\xi_1 P_N Z}{m + P_N} - \frac{\xi_2 P_T Z}{P_T^2 + P_T + 1} - cZ + D_T \nabla^2 Z,$$

with non-zero initial conditions:

$P_N(x,0) > 0, P_T(x,0) > 0, Z(x,0) > 0, x \in [0, R]$.

and the zero-flux boundary conditions

$$\frac{\partial P_N}{\partial x} = \frac{\partial P_T}{\partial x} = \frac{\partial Z}{\partial x} = 0, x \in [0, R]$$

The zero flux boundary conditions are used for modeling the dynamics of spatially bounded aquatic ecosystem. Here, $r_1$ and $r_2$ are the intrinsic growth rate of non toxic phytoplankton and toxic phytoplankton in the absence of predation respectively; $\alpha_1$ and $\alpha_2$ are the interspecific competition coefficient for non toxic phytoplankton and toxic phytoplankton; $k_1$ and $k_2$ are the carrying capacity of phytoplankton populations; $w_1$ and $w_2$ are maximum rate of predation; $m$ and $\alpha$ are half-saturation constant for non toxic phytoplankton and toxic phytoplankton density respectively; $\xi_1$ is the rate at which non toxic phytoplankton is grazed and $\xi_2$ is the reduction rate in the growth of zooplankton due to toxic material ingested in feeding on toxic.
phytoplankton; \( c \) is the mortality rate of zooplankton, \( D_1, D_2 \) and \( D_3 \) are the diffusion coefficients of non-toxic phytoplankton, toxic phytoplankton and zooplankton density respectively.

### 3 Stability Analysis of Non-Spatial Model System

In this section, we restrict ourselves to the stability analysis of the model system in the absence of diffusion (\( D_1 = 0, D_2 = 0, D_3 = 0 \)) in which only the interaction part of the model system is taken into account. We find the non-negative equilibrium states of the model system and discuss their stability properties with respect to variation of several parameters. In this case, the model system reduces to the form

\[
\begin{align*}
\frac{dP_T}{dt} &= r_P P_T \left(1 - \frac{P_T + \alpha P_T}{k_T}\right) - \frac{w_P P_T Z}{m + P_T}, \\
\frac{dP_N}{dt} &= r_P P_N \left(1 - \frac{P_N + \alpha P_N}{k_N}\right) - \frac{w_P P_N Z}{m + P_N}, \\
\frac{dZ}{dt} &= \frac{\xi P_T Z}{m + P_N} - \frac{\xi Z}{\alpha} - cZ,
\end{align*}
\]

(4)

**Lemma 1**: \( \theta = \left((P_N(t), P_T(t), Z(t)) : 0 \leq P_N(t) \leq k_1, 0 \leq P_T(t) \leq k_2, 0 \leq P_N(t) + \frac{w_P}{\xi} z(t) \leq k_1 \left(1 + \frac{r}{\delta}\right)\right) \) is a region of attraction for all solutions initiating in the interior of the positive octant, where \( 0 < \delta \leq c \).

The model system (4) possesses six non-negative real equilibrium points:

(i) Plankton-free equilibrium point \( E_0(0,0,0) \) always exists.

(ii) TPP and zooplankton-free equilibrium point \( E_1(k_1,0,0) \) exists on the boundary of the first octant.

(iii) NTP and zooplankton-free equilibrium point \( E_2(0,k_2,0) \) exists on the boundary of the first octant.

(iv) Zooplankton-free equilibrium point \( E_3(\hat{P}_N, \hat{P}_T, 0) \) is the planar equilibrium point on the \( P_N P_T \)-plane where \( \hat{P}_N = \frac{k_1 - \alpha k_2}{1 - \alpha \alpha_2} \) and \( \hat{P}_T = \frac{k_2 - \alpha k_1}{1 - \alpha \alpha_2} \) if \( \alpha < \frac{k_1}{k_2} < \frac{1}{\alpha_2} \).

(v) TPP-free equilibrium point \( E_4(\hat{P}_N, 0, \hat{Z}) \) is the planar equilibrium point on the \( P_N Z \)-plane where \( \hat{Z} = \frac{cm}{\xi_1 - c} \) and \( \hat{Z} = \frac{r_P}{w_P} \left(1 - \frac{P_N}{k_1}\right) \left(m + P_N\right) \) if \( \xi_1 > c, k_1(\xi_1 - c) > cm \).

(vi) The existence of interior equilibrium point \( E_5(\hat{P}_N^*, \hat{P}_T^*, Z^*) \).
In this case, $P_N^*$, $P_T^*$ and $Z^*$ are the positive solutions of the following three equations:

\[
\begin{align*}
\frac{r_1}{m + P_N} \left(1 - \frac{P_N + \alpha P_T}{k_1}\right) - \frac{w_1}{m + P_N} Z &= 0 \quad (5) \\
\frac{r_2}{m + P_N} \left(1 - \frac{P_T + \alpha P_N}{k_2}\right) - \frac{w_2}{m + P_N} Z &= 0 \quad (6) \\
\frac{\xi_1 P_N}{m + P_N} - \frac{\xi_2 P_T}{m + P_N} - c &= 0 \quad (7)
\end{align*}
\]

From Eq.(5), we get

\[
Z = \frac{r_1}{w_1} \left(1 - \frac{P_N + \alpha P_T}{k_1}\right) \left(\alpha + P_N\right) \quad (8)
\]

Clearly, $Z > 0$ if $k_1 > (P_N + \alpha P_T)$.

Putting the value of $Z$ from Eq.(8) in Eqs.(6) and (7), we obtain:

\[
\begin{align*}
F_1(P_N, P_T) &= \frac{r_2}{w_1} \left(1 - \frac{P_T + \alpha P_N}{k_2}\right) - \frac{r_1 w_2}{m + P_N} \left(1 - \frac{P_N + \alpha P_T}{k_1}\right) \left(\frac{P_T^2 + P_T + 1}{\alpha + P_T + 1}\right) = 0, \\
F_2(P_N, P_T) &= \frac{\xi_1 P_N}{m + P_N} - \frac{\xi_2 P_T}{m + P_N} \left(\frac{P_T^2 + P_T + 1}{\alpha + P_T + 1}\right) - c = 0
\end{align*}
\]

From Eq.(9), when $P_T = 0$ then $P_N = P_{N_e}$, where

\[
P_{N_e} > 0, \text{ if } r_2 w_1 < r_1 w_2 \quad (11)
\]

Putting $P_N = 0$ in Eq.(9), we note that $F_1(0, P_T)$ has a unique positive root $P_{T_e}$, which is the solution of the following equation:
\[ r_2 P_2^3 - (r_2 k_2 + r_2 d_1 k_2) P_2^2 + (k_2 d_2 r_2 + r_2 d_2 - \frac{r_1 w_2 k_2 d_1 d_2 k_2}{k_1 w_1}) P_2 - r_2 d_2 k_2 + \frac{r_1 w_2 d_1 d_2 k_2}{w_1} = 0 \]  

(12)

It may be noted here that Eq.(12) has one or three positive roots. Eq. (12) can be re-written as

\[ P_2^3 + q_1 P_2^2 + q_2 P_2 + q_3 = 0 \]

(13)

where \( q_1 = (\alpha k_2), q_2 = (\alpha k_2 - \alpha + \frac{r_1 w_1 ma_2}{k_1 w_i r_2}), q_3 = \left( -\alpha + \frac{k_2 w_m a_i}{w_i r_2} \right). \)

Thus Eq.(12) has a unique real positive root \( P_{N_b} \) (other roots will be complex conjugate) if

\[ \frac{a_1^2}{4} + \frac{a_2^3}{27} > 0 \]

(14)

where \( a_1 = \frac{1}{3}(3q_2 - q_1^2), a_2 = \frac{1}{27}(2q_1^3 - 9q_1 q_2 + 27q_3). \)

Let \( F_{1P_N} \) and \( F_{1P_T} \) are the partial derivatives of \( F_1 \) with respect to \( P_N \) and \( P_T \) respectively. Now we have

\[ \frac{dP_N}{dP_T} = -\frac{F_{1P_T}}{F_{1P_N}}, \text{ where } F_{1P_N} \neq 0. \]

It may be noted that \( \frac{dP_N}{dP_T} < 0 \) if

either (i) \( F_{1P_N} > 0 \) and \( F_{1P_T} > 0, \)

or (ii) \( F_{1P_N} < 0 \) and \( F_{1P_T} < 0, \)

(15)

From Eq.(10), when \( P_T = 0 \) then \( P_N = P_{N_b} \) where:

\[ P_{N_b} = \frac{cm}{\xi_i - c} \]

(16)

\[ P_{N_b} > 0 \text{ if } \xi_i > c \]

(17)

Let \( F_{2P_N} \) and \( F_{2P_T} \) are the partial derivatives of \( F_2 \) with respect to \( P_N \) and \( P_T \) respectively. Now we have
\[
\frac{dP_N}{dP_T} = -\frac{F_{2_P}}{F_{2_P}}, \quad \text{where} \quad F_{2_P} \neq 0. \quad \text{It may be noted that} \quad \frac{dP_N}{dP_T} > 0 \quad \text{because}
\]

\[
F_{2_P} = \frac{m\xi}{(m + P_t)^2} > 0, \quad F_{2_P} = -\frac{P^2\xi\alpha(P_t + \alpha - 2)}{(P_t^2 + \alpha P_t + \alpha)^2} < 0
\]

(18)

From the above analysis, we note that the isoclines (9) and (10) intersect at a unique point \((P_N^*, P_T^*)\), if in addition to conditions (14), (15) and (17) the following condition holds:

\[P_{N_b} < P_{N_a}.\]

Knowing the values of \(P_N\) and \(P_T\), the value of \(Z^*\) can be calculated from Eq.(8). This completes the existence of equilibrium point \(E_5(P_N^*, P_T^*, Z^*)\).

Now, in order to investigate local behavior of the model system (1)-(3) at each equilibrium points, the variational matrix of the point is \((P_N, P_T, Z)\) computed as

\[
\begin{pmatrix}
P_N \frac{\partial f_1}{\partial P_N} + f_1 & P_N \frac{\partial f_1}{\partial P_T} & P_N \frac{\partial f_1}{\partial Z} \\
P_T \frac{\partial f_2}{\partial P_N} & P_T \frac{\partial f_2}{\partial P_T} + f_2 & P_T \frac{\partial f_2}{\partial Z} \\
Z \frac{\partial f_3}{\partial P_N} & Z \frac{\partial f_3}{\partial P_T} & Z \frac{\partial f_3}{\partial Z} + f_3
\end{pmatrix}
\]

(19)

Let \(V_j, j = 0,1,2,3,4,5\) denotes the variational matrix at \(E_j, j = 0,1,2,3,4,5\) respectively.

For \(E_0(0,0,0)\) we have

\[
V_0 = \begin{pmatrix}
r_1 & 0 & 0 \\
0 & r_2 & 0 \\
0 & 0 & -c
\end{pmatrix}
\]

The eigenvalues of matrix \(V_0\) are \(r_1, r_2, -c\). There is unstable manifold along \(P_N, P_T\) direction and stable manifold along \(Z\) direction. Therefore, the equilibrium point \(E_0\) is a saddle point.

The variational matrix for \(E_1(k_1,0,0)\) is given by
The eigenvalues of the matrix $V_1$ are $-r_1, r_2 \left(1 - \frac{\alpha_2 k_1}{k_2}\right)$, $\left(\frac{\xi_1 k_1}{k_1 + m} - c\right)$. There is a stable manifold along $P_N$ direction and unstable manifold along $P_T, Z -$ direction provided $k_2 < k_1 < \frac{cm}{(\xi_1 - c)}$, $\xi_1 > c$. It is a saddle point if the inequality opposes.

The variational matrix for $E_2(0, k_2, 0)$ is given by

$$V_2 = \begin{pmatrix} r_1 \left(1 - \frac{\alpha_2 k_2}{k_1}\right) & 0 & 0 \\ -r_2 \alpha_2 & -r_2 & -\frac{w_2 \alpha k_2}{k_2^2 + k_2 \alpha + \alpha} \\ 0 & 0 & -c - \frac{\xi_2 k_2 \alpha}{k_2^2 + k_2 \alpha + \alpha} \end{pmatrix}$$

The eigenvalues of the matrix $V_2$ are $r_1 \left(1 - \frac{\alpha_2 k_2}{k_1}\right), -r_2$, and $-c - \frac{\xi_2 k_2 \alpha}{k_2^2 + k_2 \alpha + \alpha}$. There is unstable manifold along $P_N$ direction if $k_1 > \alpha_2 k_2$ and stable manifold along $P_T$ and $Z$ direction. $E_2$ is a saddle point.

The variational matrix for $E_3(P_N, P_T, 0)$ is given by

$$V_3 = \begin{pmatrix} -\frac{r_1 P_N}{k_1} & -\alpha_1 R_P N & -\frac{w_1 P_N}{m + P_N} \\ -\frac{r_2 \alpha_2 P_T}{k_2} & -\frac{r_2 P_T}{k_2} & -\frac{w_2 \alpha P_T}{P_T^2 + P_T \alpha + \alpha} \\ 0 & 0 & -c - \frac{\xi_3 P_N}{m + P_N} - \frac{\xi_3 P_T}{P_T^2 + P_T \alpha + \alpha} \end{pmatrix}$$
The equilibrium point \( E_3(P_N, P_T, 0) \) is stable or unstable in the positive direction orthogonal to the \( P_N \) plane, i.e. \( Z \)-direction depending on whether \( \lambda_3 = c + \frac{\xi_1 P_N}{m + P_N} - \frac{\xi_2 P_T \alpha}{P_T^2 + P_T \alpha + \alpha} \) is negative or positive respectively.

The variational matrix about the equilibrium point \( E_4(P_N, 0, Z) \) is given by

\[
V_4 = \begin{pmatrix}
P_N \left( -\frac{r_i}{k_i} + \frac{w_i Z}{(P_N + m)^2} \right) & -\frac{\alpha_i r_i P_N}{k_i} & -\frac{w_i P_N}{m + P_N} \\
0 & r_2 \left( 1 - \frac{\alpha_2 P_N}{k_2} \right) - w_2 Z & 0 \\
\frac{m \xi_1 Z}{(P_N + m)^2} & -\xi_2 Z & 0
\end{pmatrix}
\]

The equilibrium point \( E_4(P_N, 0, Z) \) is stable or unstable in the positive direction orthogonal to the \( P_N \) plane, i.e. \( P_T \)-direction depending on whether \( r_2 \left( 1 - \frac{\alpha_2 P_N}{k_2} \right) - w_2 Z \) is negative or positive respectively, if \( w_i k_i Z < r_i (P_N + m)^2 \).

The variational matrix about the equilibrium point \( E_5(P_N^*, P_T^*, Z^*) \) is given by

\[
V_5 = \begin{pmatrix}
P_N^* \left( -\frac{r_i}{k_i} + \frac{w_i Z^*}{(P_N^* + m)^2} \right) & -\frac{\alpha_i r_i P_N^*}{k_i} & -\frac{w_i P_N^*}{m + P_N^*} \\
-\frac{r_2 \alpha_2 P_T^*}{k_2} & -\frac{r_2 P_T^*}{k_2} \frac{(w_2 Z^* \alpha^2 - w_2 Z^* \alpha P_T^*)}{(P_T^* + P_T^* \alpha + \alpha)^2} & -\frac{w_2 \alpha P_T^*}{P_T^* + P_T^* \alpha + \alpha} \\
\frac{m \xi_1 Z^*}{(P_N^* + m)^2} & \frac{\xi_2 Z^* \alpha P_T^* - \xi_2 Z^* \alpha^2}{(P_T^* + P_T^* \alpha + \alpha)^2} & \frac{\xi_2 P_N^*}{m + P_N^*} - \frac{\xi_2 P_T^* \alpha}{P_T^* + P_T^* \alpha + \alpha} - c
\end{pmatrix}
\]

\[
= \begin{pmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{pmatrix}
\]

In the following theorem, we are able to find conditions for the positive equilibrium point \( E_5(P_N^*, P_T^*, Z^*) \) to be locally asymptotically stable.
Theorem 2.1. Suppose that the positive equilibrium point $E_5(P_N^*, P_T^*, Z^*)$ of the model system (4) exists. The equilibrium point $E_5(P_N^*, P_T^*, Z^*)$ is locally asymptotically stable if the following conditions hold:

(i) $r_1(P_N^* + m)^2 > w_1 Z^* r_1 P_T^* (P_T^* + P_T^* \alpha + \alpha)^2 > k_2 (w_2 Z^* \alpha P_T^* - w_2 Z^* \alpha^2)$

(ii) $\frac{r_2 P_T^*}{k_2} \left( \frac{w_2 Z^* \alpha^2 - w_2 Z^* \alpha P_T^*}{(P_T^* + P_T^* \alpha + \alpha)^2} \right) > \frac{\alpha_1 r_1 w_1 P_T^* (m + P_N^*)}{k_1 (P_T^* + P_T^* \alpha + \alpha)} \cdot P_N^* \left( \frac{r_1}{k_1} + \frac{w_1 Z^*}{(P_N^* + m)^2} \right) > \frac{w_1 r_5 P_T^* P_T^*}{k_2 (P_T^* + P_T^* \alpha + \alpha)}$,

(iii) $\alpha_1 r_5 r_1 r_5 P_T^* (P_N^* + m)^3 > m w_1 \xi_z k_1 k_2, m w_1 \xi_z w_1 \alpha r_1 > \frac{w_1 r_5 P_T^*}{k_2} \left( \frac{\xi_z Z^* \alpha P_N^* - \xi_z Z^* \alpha^2}{(P_T^* + P_T^* \alpha + \alpha)^2} \right) > \frac{w_1 r_5 P_T^* P_T^*}{k_1 k_2}$

3 Stability Analysis of Spatial Model System

In this section, we study the effect of diffusion on the model system about the interior equilibrium point. In order to derive the condition of stability for the equilibrium point with diffusion, we have considered the linearized form of the model system (1) about $E_5(P_N^*, P_T^*, Z^*)$ with small perturbations $U(x; t), V(x; t)$ and $W(x; t)$ as $P_N = P_N^* + U, P_T = P_T^* + V, Z = Z^* + W$.

The linearized form of the equations is obtained as

$$
\begin{align*}
\frac{\partial U}{\partial t} &= a_{11} U + a_{12} V + a_{13} W + D_1 \frac{\partial^2 U}{\partial x^2} \\
\frac{\partial V}{\partial t} &= a_{21} U + a_{22} V + a_{23} W + D_2 \frac{\partial^2 V}{\partial x^2} \\
\frac{\partial W}{\partial t} &= a_{31} U + a_{32} V + a_{33} W + D_3 \frac{\partial^2 W}{\partial x^2}
\end{align*}
$$

where

$$
\begin{align*}
a_{11} &= -\frac{w_1 Z^* P_N^*}{(\alpha + P_N^*)^2}, a_{12} = -a_{13} \frac{r_1 P_N^*}{k_1}, a_{13} = -w_1 P_N^*, \\
a_{21} &= -r_1 P_N^* \frac{r_2 P_T^*}{k_2}, a_{22} = -\frac{(w_2 Z^* \alpha^2 - w_2 Z^* \alpha P_T^*)}{(P_T^* + P_T^* \alpha + \alpha)^2}, a_{23} = -r_1 P_T^* \frac{r_2 P_T^*}{k_2} \frac{\alpha}{(P_T^* + P_T^* \alpha + \alpha)}, \\
a_{31} &= -m w_1 \xi_z Z^* \frac{w_1 P_N^*}{(m + P_N^*)^2}, a_{32} = -\frac{\xi_z Z^* \alpha P_T^*}{(P_T^* + P_T^* \alpha + \alpha)^2}, a_{33} = -\frac{\xi_z P_T^* \xi_z P_T^* \alpha}{(P_T^* + P_T^* \alpha + \alpha) - C}
\end{align*}
$$
Let us suppose that the model system (20) has the solution which is of the form
\[
\begin{pmatrix}
U \\
V \\
W
\end{pmatrix}
= 
\begin{pmatrix}
A \\
B \\
C
\end{pmatrix}
\exp(\lambda t) \cos(n\pi x / R),
\]

where \(A, B, C\) are sufficiently small constants. \(R / n\pi\) is the critical wavelength and \(k = n\pi / R\) is wave number, \(R\) is the length of the system, \(2\pi / n\) is the period of cosine and \(\lambda\) is the frequency respectively. The characteristic equation of the linearized system is given by
\[
\lambda^3 + \rho_1 \lambda^2 + \rho_2 \lambda + \rho_3 = 0
\]

where
\[
\rho_1 = -\text{Tr}(M) + (D_1 + D_2 + D_3)k^2
\]
\[
\rho_2 = R(M) + (D_1D_2 + D_1D_3 + D_2D_3)k^4 - [a_{11}(D_2 + D_3) + a_{22}(D_1 + D_3) + a_{33}(D_1 + D_3)]k^2,
\]
\[
\rho_3 = P(k^2)
\]

With
\[
P(k^2) = b_0k^6 - b_4k^4 + b_2k^2 - \text{Det}(M),
\]

where
\[
b_0 = D_1D_2D_3,
\]
\[
b_1 = a_{11}D_2D_3 + a_{22}D_1D_3 + a_{33}D_1D_2
\]
\[
b_2 = D_1(a_{22}a_{33} - a_{23}a_{32}) + D_2(a_{11}a_{33} - a_{13}a_{31}) + D_3a_{12}a_{23},
\]

and
\[
M = \begin{pmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{pmatrix}, \text{Tr}(M) = a_{11} + a_{22} + a_{33},
\]
\[
R(M) = a_{11}(a_{22} + a_{33}) - a_{12}a_{33} + a_{22}a_{33} - a_{23}a_{32},
\]
\[
\text{Det}(M) = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{22}a_{33}a_{13}
\]

**Theorem 3.1**: The positive equilibrium point \(E_\delta(P_N^*, P_T^*, Z^*)\) is locally asymptotically stable in the presence of diffusion if and only if:
\( \rho_1 > 0 \)  
(26)

\( \rho_2 > 0 \)  
(27)

\( \rho_1 \rho_2 - \rho_1 > 0 \)  
(28)

From Eq. (22) and using the Routh-Hurwitz criterion, the above theorem follows immediately.

4 Turing Instability

The Turing instability occurs if at least one of the roots of the above Eq.(22) has a positive root or positive real part or in other words, \( \text{Re}(\lambda) > 0 \) for some \( k > 0 \). Irrespective of the sign of \( \rho_1 \) and \( \rho_2 \), the diffusion-driven instability occurs when \( \rho_3 = P(k^2) < 0 \). Hence the condition for diffusive instability is given by

\[
P(k^2) = b_7k^6 - b_7k^4 + b_2k^2 - \text{Det}(M) < 0
\]

(29)

\( P \) is a cubic polynomial in \( k^2 \). The critical values of \( P(k^2) \) occurs at \( k^2 = k_{cr}^2 \), where

\[
k_{cr}^2 = \frac{b_1 \pm \sqrt{h_1^2 - 3b_0b_2}}{3b_0}
\]

(30)

For positive value of critical points \( k^2 = k_{cr}^2 \) we require:

\[
b_1^2 - 3b_0b_2 > 0 \text{ and } b_1 > 0 \text{ or } b_2 < 0
\]

(31)

\[
r_1 = 0.4632; \quad r_2 = 0.4425; \quad \alpha_1 = 0.002; \quad \alpha_2 = 0.001; \quad k_1 = 505; \quad k_2 = 505; \\
w_1 = 0.6625; \quad w_2 = 0.435; \quad \xi_1 = 0.516; \quad \xi_2 = 0.198; \quad m = 45; \quad \alpha = 30; \quad c = 0.109
\]

(32)

For this set of parameter values given in Eq.(32), we have obtained the equilibrium point \( (N^*_T, P^*_T, Z^*_T) \)

\[
=(37.2907; 15.6704; 53.2830). \text{ For } D_1 = D_2 = 0.0001, D_3 = 10, \text{ and the above set of parameter values given in Eq.(32), we have obtained the critical values } k_{cr}^2 = (-17.01; 8.798) \text{ and corresponding }
\]

\[
P(k_{cr}^2) = (6.497; -2.441) \text{ (cf. Fig. 1). The graph of } P(k^2) \text{ versus } k^2 \text{ has been plotted for different values of } D_3. \text{ The positive values of } k^2 \text{ for which } \rho_3 = P(k^2) < 0, \text{ the plankton system (1) is unstable.}
Fig. 1 The graph of the function $P(k^2)$ versus $k^2$ for the set of parametric values given in Eq.(32) With $D_1 = D_2 = 0.0001$ for (a) $D_3 = 10$ and (b) $D_3 = 10, 20, 30$

5 Numerical Simulation

In this section, we perform numerical simulations to understand the mechanism that will control the growth of TPP. For that purpose, we have plotted the spatio-temporal patterns and time series to observe the controlling parameters of TPP growth. The dynamics of the model system (1) is studied with the help of numerical simulation for one-dimensional case. To investigate the spatio-temporal dynamics of the model system (1), we have solved it numerically using semi-implicit (in time) finite difference method. The step lengths $\Delta x$ and $\Delta t$ of the numerical grid are chosen sufficiently small so that the results are numerically stable. We choose the following set of parameters (other set of parameters may also exists) for the model system (1):

$r_1 = 0.3; r_2 = 0.2; \alpha_1 = 0; \alpha_2 = 0; k_1 = 300; k_2 = 250; w_1 = 0.6625; w_2 = 0.435;

\xi_1 = 0.516; \xi_2 = 0.198; m = 45; \alpha = 30; c = 0.109, D_1 = D_2 = 0.001, D_3 = 0.01$

with initial condition

$P_1(x,0) = P_1^* + \epsilon_1 \sin \left( \frac{2\pi(x-x_0)}{S} \right),$

$P_2(x,0) = P_2^* + \epsilon_1 \sin \left( \frac{2\pi(x-x_0)}{S} \right),$

$Z(x,0) = Z^* + \epsilon_1 \sin \left( \frac{2\pi(x-x_0)}{S} \right),$

(34)

where

$\epsilon_1 = 5 \times 10^{-4}, x_0 = 0.1, S = 0.2$ and

$(P_N^*, P_T^*, Z^*) = (28.3728, 34.7472, 30.0831).$

The model system (1) with a fixed set of parameter values given in Eq.(33) and initial condition (34) asserting that the density of TPP remain high in whole domain and system shows the limit cycle behavior (cf. Fig. 2 & 3).
Fig. 2 Time Series of the model system (1) for the fixed set of parameter values given in Eq. (33).
When the rate of inhibition of zooplankton growth by toxic material ingested in feeding on TPP is very large ($\xi_2 = 5$), then because of high toxic effect, the zooplankton goes to extinction.

Fig. 3 Spatio-temporal pattern of NTP, TPP and zooplankton of the model system (1) for the fixed set of parameter values given in Eq. (33).

Fig. 4 Time series of the model system (1) when $\xi_2 = 5$ and rest of the parametric values are same as Eq.(33).
Fig. 5 Spatio-temporal pattern of NTP, TPP and zooplankton of the model system (1) for the fixed set of parameter values given in Eq. (33).
6 Discussions and Conclusions
In this paper, a simple mathematical model of NTP-TPP-zooplankton system in which zooplankton population reduces due to release of toxic chemical by phytoplankton or due to toxic phytoplankton being eaten by zooplankton has been proposed and analysed. We have investigated the model both analytically and numerically. We have investigated the effects of spatial interaction and spatio-temporal pattern formation. Numerical analysis demonstrates the following conclusions:
(i) The model system (1) with a fixed set of parameter values given in Eq.(33) and initial condition (34) asserting that the density of TPP remain high in whole domain and system shows the limit cycle behaviour.
(ii) When the rate of inhibition of zooplankton growth by toxic material ingested in feeding on TPP is very large $\xi_2 = 5$, then because of high toxic effect the zooplankton goes to extinction.

The results obtained suggest that toxic substances or toxic phytoplankton may serve as a key factor in the termination of planktonic blooms. Here we conclude that TPP has an inhibitory effect on zooplankton and high abundance of TPP is not favorable for the persistence of zooplankton.

7 References
Buskey EJ, Hyatt CJ. 1995. Effects of the Texas (USA) brown tide alga on planktonic grazers. Marine Ecology Progress Series, 126: 285-292
Hansen FC., Trophic interaction between zooplankton and Phaeocystis cf. Globosa. Helgol Meeresunters, 49: 283-293
Nejstgaard JC, Solberg PT. 1996, Repression of copepod feeding and fecundity by the toxic haptophyte Prymnesium patelliferum. Sarsia, 81: 339-344