Article

Local dynamical properties and supercritical N-S bifurcation of a discrete-time host-parasitoid model with Allee effect

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Abstract

We explore thelocal dynamical properties and supercritical N-S bifurcation of the following Beddington model with Allee effect in $\mathbb{R}_+^2$:

$$x_{t+1} = x_t e^{r(1-x_t)-y_t},\ y_{t+1} = m x_t (1 - e^{-y_t}) \frac{y_t}{B + y_t},$$

where x_t (respectively y_t) denotes densities of host (respectively parasitoid) at time t, r and m respectively denotes number of eggs laid by host and parasitoid which survive through larvae, pupae, and adult stages, and B is constant. More specifically, we explored that model has three equilibria namely the trivial, boundary and positive equilibrium point. We studied the local dynamics along with topological classification about equilibria of the under consideration model. We also explored the existence of bifurcation about equilibria of the model. It is proved about boundary equilibrium point parasitoidgoes to extinction whilehost population undergoes a flip bifurcation to chaos by taking r as bifurcation parameter. It is explored that aboutpositive equilibrium point, model undergoes N-S bifurcation and in meantime invariant closed curve appears. In the perspective of the biology, these curves correspond to periodic or quasi-periodic oscillations between host and parasitoid populations. Finally theoretical results are verified numerically.

Keywords Beddington model; stability and bifurcation; Allee effect; numerical simulation.

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1 Introduction

Recently dynamics and bifurcation analysis of two-species models are widely explored (Beddington et al., 1975; Ufuktepe et al., 2015; Cao et al., 2013; Hu et al., 2011; Chen et al., 2013; Khan et al., 2017; Guckenheimer and Holmes, 1983; Kuznetsov, 2004) and reference cited therein. For instance, Beddington et al. (1975) have investigated the following discrete-time two species model:

$$\left.\begin{aligned} x_{t+1} &= x_t e^{r\left(1-\frac{x_t}{K}\right)-ay_t} \\ y_{t+1} &= cx_t(1-e^{-y_t}) \end{aligned}\right\}, \tag{1}$$

with carrying capacity is K. Ufuktepe et al. (2015) have explored the local dynamicsabout equilibria of the following Beddington model subject to Allee effect on the parasitoid population:

$$\left.\begin{aligned} x_{t+1} &= x_t e^{r(1-x_t)-y_t} \\ y_{t+1} &= mx_t(1-e^{-y_t})\frac{y_t}{B+y_t} \end{aligned}\right\}. \tag{2}$$

More precisely Ufuktepe et al. (2015) have proved that for all $r, m, B,$ model (2) has trivial and boundary equilibria, and also has the positive equilibria under restrictions to the parameters. Further Ufuktepe et al. (2015) have explored local dynamics, stable and unstable manifolds for boundary equilibria of the Beddington model, which is depicted in (2).

Here our purpose in this article is to explore the topological classification about trivial, boundary and positive equilibrium of model (2). We also explore the necessary and sufficient condition (s) under which positive equilibrium of (2) is a sink, repeller, saddle and non-hyperbolic. Further we explore Flip bifurcation about boundary equilibrium point, and N-S bifurcation if parameter m vary in the neighborhood of positive equilibrium point.

We arrange the rest of article as follows: Section 2 is about the local dynamical properties along with topological classification about equilibria of (2).In Section 3, we explore the existence of bifurcations about trivial, boundary and positive equilibrium of the model (2). Section 4 deals with the study of Flip bifurcation about boundary equilibrium point and N-S bifurcation about positive equilibrium point. Theoretical results are verified numerically in Section 5. Brief summary is given in Section 6.

2 Local Dynamical Properties About Equilibria: $O(0,0)$, $A(1,0)$ and $B(\theta, r(1-\theta))$ of Model (2)

In this Section, we explore local dynamical properties about $(0,0)$, $A(1,0)$ and $B(\theta, r(1-\theta))$ of model (2). From Theorem 2.1 of Ufuktepe et al. (2015), first we summarize existence result about equilibria in $\mathbb{R}_+^2$ as follows:

Lemma 2.1. In $\mathbb{R}_+^2$, model (2) has trivial, boundary and positive equilibria. Precisely

(i) $\forall\, r, m$ and B, model (2) has trial and boundary equilibria: $O(0,0)$, $A(1,0)$;

(ii) Suppose that

$$F(x) = -r + (r+m)x - mxe^{-r(1-x)}, \tag{3}$$

and

$$\theta = \frac{(B+r)\sqrt{r} + \sqrt{B+r}\sqrt{4m + r(4+B+r)}}{2\sqrt{r}(m+r)}, \tag{4}$$

then the following statements hold:

(ii.1) Model (2) has one positive equilibrium point $B(\theta, r(1-\theta))$ if and only if

$$m > 1, \tag{5}$$

and

$$\mathrm{B} = \mathrm{F}(\theta), \tag{6}$$

where

$$0 < \theta < 1; \tag{7}$$

(ii.2) Model (2) has two positive equilibria $C(l, r(1-l))$ if and only if (5) and following inequality hold:

$$\mathrm{B} < F(\theta), \tag{8}$$

where

$$0 < l < 1. \tag{9}$$

Hereafter local dynamical properties withtopological classification about $O(0,0)$, $A(1,0)$ and $B(\theta, r(1-\theta))$ is explored. Note that the Jacobin matrix $J_{(x,y)}$ about (x,y) of the model (2) is

$$J_{(x,y)} = \begin{pmatrix} (1-rx)e^{r(1-x)-y} & -xe^{r(1-x)-y} \\ \frac{my(1-e^{-y})}{B+y} & \frac{mxe^{-y}\left(y^2+B(-1+y+e^y)\right)}{(B+y)^2} \end{pmatrix}. \tag{10}$$

Moreover the characteristic equation of $J_{(x,y)}$ about (x,y) is

$$\lambda^2 - p\lambda + q = 0, \tag{11}$$

where

$$\left.\begin{aligned} p &= trace\, J_{(x,y)} \\ &= (1-rx)e^{r(1-x)-y} + \frac{mxe^{-y}\left(y^2+B(-1+y+e^y)\right)}{(B+y)^2} \\ q &= \det J_{(x,y)} \\ &= \frac{mx(1-rx)\left(y^2+B(-1+y+e^y)\right)e^{r(1-x)-2y}}{(B+y)^2} + \frac{mxy(1-e^{-y})e^{r(1-x)-y}}{B+y} \end{aligned}\right\}. \tag{12}$$

In the following two Lemmas, we will state the dynamics about $O(0,0)$ and $A(1,0)$ of model (2).

Lemma 2.2. For $O(0,0)$ one has

(i) $O(0,0)$ is never sink;

(ii) $O(0,0)$ is a saddle;

(iii) $O(0,0)$ is never source;

(iv) $O(0,0)$ is never non-hyperbolic.

Lemma 2.3. For $A(1,0)$ one has

(i) $A(1,0)$ is a sink if $0 < r < 2$;

(ii) $A(1,0)$ is never source;

(iii) $A(1,0)$ is a saddle if $r > 2$;

(iv) $A(1,0)$ is a non-hyperbolic if $r = 2$.

Now we will only study the local dynamics about $B(\theta, r(1-\theta))$of the model (2) in the case if (5), (6) and (7) hold. The characteristic equation of $J_{B(\theta,r(1-\theta)))}$ about $B(\theta, r(1-\theta))$ of the model (2) is given by

$$\lambda^2 - p(\theta, r(1-\theta))\lambda + q(\theta, r(1-\theta)) = 0, \tag{13}$$

where

$$\left.\begin{aligned} p &= 1 - r\theta + \frac{-r+r\theta+m\theta-m\theta(1-r+r\theta)e^{-r(1-\theta)}}{m\theta\left(1-e^{-r(1-\theta)}\right)} \\ q &= \frac{(1-r\theta)\left(-r+r\theta+m\theta-m\theta(1-r+r\theta)e^{-r(1-\theta)}\right)}{m\theta\left(1-e^{-r(1-\theta)}\right)} + r(1-\theta) \end{aligned}\right\}. \tag{14}$$

Hereafter by utilizing Theorem 1.1.1 of Kulenovic and Ladas (2003), we will only state condition(s) for equilibrium $B(\theta, r(1-\theta))$ of the model (2) to be locally asymptotically stable(sink), repeller, saddle and non-hyperbolic.

Lemma 2.4. For $B(\theta, r(1-\theta))$ of the model (2), one has

(i) $B(\theta, r(1-\theta))$ of the model (2) is sink if

$$0 < m < \min\left\{\frac{r^2(1-\theta)}{2-r+r\theta+(-2+r-r\theta+r^2\theta-r^2\theta^2)e^{-r(1-\theta)}}, \frac{r(1-\theta)(2-r\theta)}{\theta\left(2+r-3r\theta+(-2+r+r\theta-r^2\theta+r^2\theta^2)e^{-r(1-\theta)}\right)}\right\}, \tag{15}$$

and

$$m > \frac{r(1-\theta)(1-r\theta)}{\theta\left(2+r-2r\theta+(-2+r\theta-r^2\theta+r^2\theta^2)e^{-r(1-\theta)}\right)}; \tag{16}$$

(ii) $B(\theta, r(1-\theta))$ of the model (2) is a repeller if

$$\frac{r(1-\theta)(2-r\theta)}{\theta\left(2+r-3r\theta+(-2+r+r\theta-r^2\theta+r^2\theta^2)e^{-r(1-\theta)}\right)} < m < \frac{r(1-\theta)(1-r\theta)}{\theta\left(2+r-2r\theta+(-2+r\theta-r^2\theta+r^2\theta^2)e^{-r(1-\theta)}\right)}; \tag{17}$$

(iii) $B(\theta, r(1-\theta))$ of the model (2) is a saddle if

$$\left(1-r\theta+\frac{-r+r\theta+m\theta-m\theta(1-r+r\theta)e^{-r(1-\theta)}}{m\theta(1-e^{-r(1-\theta)})}\right)^2 +$$

$$4\left(\frac{(1-r\theta)\left(-r+r\theta+m\theta-m\theta(1-r+r\theta)e^{-r(1-\theta)}\right)}{m\theta\left(1-e^{-r(1-\theta)}\right)}+r(1-\theta)\right) > 0, \tag{18}$$

and

$$m < \frac{r(1-\theta)(2-r\theta)}{\theta\left(2+r-3r\theta+(-2+r+r\theta-r^2\theta+r^2\theta^2)e^{-r(1-\theta)}\right)}; \tag{19}$$

(iv) $B(\theta, r(1-\theta))$ of the model (2) is a non-hyperbolic if

$$m = \frac{r(1-\theta)(2-r\theta)}{\theta\left(2+r-3r\theta+(-2+r+r\theta-r^2\theta+r^2\theta^2)e^{-r(1-\theta)}\right)}, \tag{20}$$

or

$$m = \frac{r(1-\theta)(1-r\theta)}{\theta\left(2+r-2r\theta+(-2+r\theta-r^2\theta+r^2\theta^2)e^{-r(1-\theta)}\right)}, \tag{21}$$

and

$$m \geq \frac{r(1-\theta)}{\theta\left(2-r\theta+(-2-r+2r\theta)e^{-r(1-\theta)}\right)}. \tag{22}$$

In the rest of the section, we explore the topological classification about $B(\theta, r(1-\theta))$ of (2). Note that roots of the characteristic equations of $J_{B(\theta,r(1-\theta))}$ about $B(\theta, r(1-\theta))$ are

$$\lambda_{1,2} = \frac{-p\pm\sqrt{\Delta}}{2}, \tag{23}$$

where
$\Delta = p^2 - 4q,$

$$= \left(1-r\theta+\frac{-r+r\theta+m\theta-m\theta(1-r+r\theta)e^{-r(1-\theta)}}{m\theta\left(1-e^{-r(1-\theta)}\right)}\right)^2 - 4\left(\frac{(1-r\theta)\left(-r+r\theta+m\theta-m\theta(1-r+r\theta)e^{-r(1-\theta)}\right)}{m\theta\left(1-e^{-r(1-\theta)}\right)} + r(1-\theta)\right). \tag{24}$$

Now in the following two Lemmas, we will further classifying dynamics of (2) about $B(\theta, r(1-\theta))$ according to sign of discriminant Δ.

Lemma 2.5. If $\Delta \geq 0$ then for $B(\theta, r(1-\theta))$, one has

(i) $B(\theta, r(1-\theta))$ is locally asymptotically stable node if

$$0 < m < \frac{r(1-\theta)(2-r\theta)}{\theta\left(4+r-3r\theta+(-4+r+r\theta-r^2\theta+r^2\theta^2)e^{-r(1-\theta)}\right)}; \tag{25}$$

(ii) $B(\theta, r(1-\theta))$ is unstable node if

$$m > \frac{r(1-\theta)(2-r\theta)}{\theta\left(4+r-3r\theta+(-4+r+r\theta-r^2\theta+r^2\theta^2)e^{-r(1-\theta)}\right)}; \tag{26}$$

(iii) $B(\theta, r(1-\theta))$ is a non-hyperbolic if

$$m = \frac{r(1-\theta)(2-r\theta)}{\theta\left(4+r-3r\theta+(-4+r+r\theta-r^2\theta+r^2\theta^2)e^{-r(1-\theta)}\right)}. \tag{27}$$

Lemma 2.6. If $\Delta< 0$ then for $B\big(\theta, r(1-\theta)\big)$ one has

(i) $B\big(\theta, r(1-\theta)\big)$ is locally asymptotically stable focus if

$$0 < m < \frac{r(1-\theta)(1-r\theta)}{\theta\left(r(1-2\theta)+r\theta(1-r+r\theta)e^{-r(1-\theta)}\right)}; \tag{28}$$

(ii) $B\big(\theta, r(1-\theta)\big)$ is unstable focus if

$$m > \frac{r(1-\theta)(1-r\theta)}{\theta\left(r(1-2\theta)+r\theta(1-r+r\theta)e^{-r(1-\theta)}\right)}; \tag{29}$$

(iii) $B\big(\theta, r(1-\theta)\big)$ is non-hyperbolic if

$$m = \frac{r(1-\theta)(1-r\theta)}{\theta\left(r(1-2\theta)+r\theta(1-r+r\theta)e^{-r(1-\theta)}\right)}. \tag{30}$$

3 Existence of Bifurcations

Based on above theoretical discussion, the existence of bifurcations about $O(0,0)$, $A(1,0)$ and $B\big(\theta, r(1-\theta)\big)$ is explored in this Section. From Lemmas 2.2-2.3 and 2.5-2.6, one can conclude about the existence of bifurcations as follows:

(i) From Lemma 2.2, $O(0,0)$ is never non-hyperbolic and so no bifurcation exists about this equilibrium.

(ii) From Lemma 2.3, one can observe that $J_{A(1,0)}$ has one eigenvalue equal to -1 but other is not equal to 1 or -1 if $r=2$. Hence flip bifurcation exist about $A(1,0)$ and we can write the non-hyperbolic condition as follows:

$$F_{A(1,0)} = \{(r,m): r=2, r,m>0\}. \tag{31}$$

(iii) From Lemma 2.5, if $m = \frac{r(1-\theta)(2-r\theta)}{\theta\left(4+r-3r\theta+(-4+r+r\theta-r^2\theta+r^2\theta^2)e^{-r(1-\theta)}\right)}$ then none of the real eigenvalues of $J_{B(\theta,r(1-\theta))}$ about $B\big(\theta, r(1-\theta)\big)$ is -1. So no Flip bifurcation exists about $B\big(\theta, r(1-\theta)\big)$.

(iv) From Lemma 2.6, if $m = \frac{r(1-\theta)(1-r\theta)}{\theta\left(r(1-2\theta)+r\theta(1-r+r\theta)e^{-r(1-\theta)}\right)}$ then eigenvalues of $J_{B(\theta,r(1-\theta))}$ about $B\big(\theta, r(1-\theta)\big)$ are complex conjugates having modulus 1. So, there exist a N-S bifurcation when m varies in a neighborhood of $B\big(\theta, r(1-\theta)\big)$ and we can also rewrite the non-hyperbolic condition as follows:

$$N_{B(\theta,r(1-\theta))} = \left\{\begin{matrix}(r,m,\theta): \Delta< 0 \text{ and } m = \frac{r(1-\theta)(1-r\theta)}{\theta\left(r(1-2\theta)+r\theta(1-r+r\theta)e^{-r(1-\theta)}\right)}, \\ r,m>0, 0<\theta<1\end{matrix}\right\}. \tag{32}$$

4 Bifurcations Analysis

The detail analysis regarding flip bifurcation about $A(1,0)$ and N-S bifurcation about $B\big(\theta, r(1-\theta)\big)$ of system (2) are given in this Section. First we will study the flip bifurcation about $A(1,0)$ as follows: Recall Lemma 2.3, we can see that $J_{A(1,0)}$ has one eigenvalue equal to -1 but other is not equal to 1 or -1, when the parameters of the model (2) satisfying (31). So, about $A(1,0)$ model undergoes flip bifurcation if parameters of model (2) go through $F_{A(1,0)}$. If $(r,m) \in F_{A(1,0)}$ then its center manifold is $y=0$ and thus (2) becomes

$$x_{t+1} = x_t e^{r(1-x_t)}. \tag{33}$$

This indicates that parasitoid goes to extinction while host population undergoes a flip bifurcation to chaos by taking$r$ as bifurcation parameter.

Hereafter N-S bifurcation is explored about $B\big(\theta, r(1-\theta)\big)$ when $(r, m, \theta) \in N_{B(\theta, r(1-\theta))}$. By considering m in the neighborhood ofm^*, $i.e.$, $m = m^* + \varepsilon$ where $|\varepsilon| \ll 1$ then model (2) becomes:

$$\left.\begin{aligned} x_{t+1} &= x_t e^{r(1-x_t)-y_t} \\ y_{t+1} &= (m^* + \varepsilon)x_t(1 - e^{-y_t})\frac{y_t}{B+y_t} \end{aligned}\right\}. \tag{34}$$

The auxiliary equation of $J_{B(\theta, r(1-\theta))}$ about $B\big(\theta, r(1-\theta)\big)$ of (34) is given by

$$\kappa^2 - p(\varepsilon)\kappa + q(\varepsilon) = 0, \tag{35}$$

with

$$\left.\begin{aligned} p(\varepsilon) &= 1 - r\theta + \frac{-r+r\theta+(m^*+\varepsilon)\theta-(m^*+\varepsilon)\theta(1-r+r\theta)e^{-r(1-\theta)}}{(m^*+\varepsilon)\theta\left(1-e^{-r(1-\theta)}\right)} \\ q(\varepsilon) &= \frac{(1-r\theta)\left(-r+r\theta+(m^*+\varepsilon)\theta-(m^*+\varepsilon)\theta(1-r+r\theta)e^{-r(1-\theta)}\right)}{(m^*+\varepsilon)\theta\left(1-e^{-r(1-\theta)}\right)} + r(1-\theta) \end{aligned}\right\}. \tag{36}$$

The roots of auxiliary equations of $J_{B(l, r(1-l))}$ about $B\big(\theta, r(1-\theta)\big)$ are

$$\begin{aligned} \kappa_{1,2} &= \frac{p(\varepsilon) \pm \iota\sqrt{4q(\varepsilon) - p^2(\varepsilon)}}{2}, \\ &= \frac{1-r\theta}{2} + \frac{-r+r\theta+(m^*+\varepsilon)\theta-(m^*+\varepsilon)\theta(1-r+r\theta)e^{-r(1-\theta)}}{2(m^*+\varepsilon)\theta\left(1-e^{-r(1-\theta)}\right)} \pm \iota\frac{\sqrt{\Omega}}{2}, \end{aligned} \tag{37}$$

where

$$\begin{aligned} \Omega = {} & 4\left(\frac{(1-r\theta)\left(-r+r\theta+(m^*+\varepsilon)\theta-(m^*+\varepsilon)\theta(1-r+r\theta)e^{-r(1-\theta)}\right)}{(m^*+\varepsilon)\theta\left(1-e^{-r(1-\theta)}\right)} + r(1-\theta)\right) - \\ & \left(1 - r\theta + \frac{-r+r\theta+(m^*+\varepsilon)\theta-(m^*+\varepsilon)\theta(1-r+r\theta)e^{-r(1-\theta)}}{(m^*+\varepsilon)\theta\left(1-e^{-r(1-\theta)}\right)}\right)^2. \end{aligned} \tag{38}$$

and

$$\left|\kappa_{1,2}\right| = \sqrt{q(\varepsilon)}\ , \quad \frac{d|\kappa_{1,2}|}{d\varepsilon}\Big|_{\varepsilon=0} = \frac{r(1-\theta)(1-r\theta)}{2m^{*2}\theta\left(1-e^{-r(1-\theta)}\right)} \neq 0. \tag{39}$$

Additionally it is required that when $\varepsilon = 0, \kappa_{1,2}^n \neq 1, n = 1,2,3,4,$ which is equivalent to $p(0) \neq -2,0,1,2.$ Because $p(0)^2 - 4q(0) < 0$ and $q(0) = 1.$ Thus $p(0)^2 < 4$ and hence $p(0) \neq \pm 2.$ So we only need to require that $p(0) \neq 0,1.$ By computation one gets

$$r \neq 2 - r\theta + r^2\theta^2,\ \ 1 + r^2\theta^2. \tag{40}$$

Let $u_t = x_t - x^*,\ v_t = y_t - y^*$ then transforming equilibrium $B\big(\theta, r(1-\theta)\big)$to $O(0,0)$ where $x^* = \theta, y^* = r(1-\theta)$. By calculating one gets

$$\left.\begin{aligned} u_{t+1} &= (u_t + x^*)e^{r_1(1-u_t-x^*)-v_t-y^*} - x^* \\ v_{t+1} &= (m^* + \varepsilon)(u_t + x^*)\left(1 - e^{-v_t-y^*}\right)\frac{v_t+y^*}{B+v_t+y^*} - y^* \end{aligned}\right\}. \tag{41}$$

Hereafter normal form of (41) is studied when $\varepsilon = 0$. Expanding (41) as a Taylor series up to third-order about $(u_t, v_t) = (0,0)$ one gets:

$$\left.\begin{aligned} u_{t+1} = {} & m_{11}u_t + m_{12}v_t + m_{13}u_t^2 + m_{14}u_tv_t + m_{15}v_t^2 + m_{16}u_t^3 + m_{17}u_t^2v_t + \\ & m_{18}u_tv_t^2 + m_{19}v_t^3 + o((|u_t| + |v_t|)^3) \\ v_{t+1} = {} & m_{21}u_t + m_{22}v_t + m_{23}u_tv_t + m_{24}v_t^2 + m_{25}u_tv_t^2 + m_{26}v_t^3 + \\ & o((|u_t| + |v_t|)^3) \end{aligned}\right\}, \tag{42}$$

where

$$\left.\begin{array}{l} m_{11}=1-rx^*,\quad m_{12}=-x^*,\quad m_{13}=-\frac{-r(2-rx^*)}{2},m_{14}=rx^*-1,m_{15}=x^*,\\ m_{16}=\frac{r^2(3-rx^*)}{6},\ m_{17}=\frac{r(2-rx^*)}{2},\ m_{18}=\frac{1-rx^*}{2},\quad m_{19}=\frac{-x^*}{6},\\ m_{21}=m^*\left(1-e^{-y^*}\right)\frac{y^*}{B+y^*},\quad m_{22}=m^*x^*\left(\frac{B\left(1-e^{-y^*}\right)}{(B+y^*)^2}+\frac{y^*e^{-y^*}}{B+y^*}\right),\\ m_{23}=m^*\left(\frac{B\left(1-e^{-y^*}\right)}{(B+y^*)^2}+\frac{y^*e^{-y^*}}{B+y^*}\right),\\ m_{24}=\frac{1}{2}m^*x^*\left(\frac{2Be^{-y^*}}{(B+y^*)^2}-\frac{2B\left(1-e^{-y^*}\right)}{(B+y^*)^3}-\frac{y^*e^{-y^*}}{B+y^*}\right),\\ m_{25}=\frac{1}{2}m^*\left(\frac{2Be^{-y^*}}{(B+y^*)^2}-\frac{2B\left(1-e^{-y^*}\right)}{(B+y^*)^3}-\frac{y^*e^{-y^*}}{B+y^*}\right),\\ m_{26}=\frac{1}{6}m^*x^*\left(\frac{-3Be^{-y^*}}{(B+y^*)^2}-\frac{6Be^{-y^*}}{(B+y^*)^3}+\frac{6B\left(1-e^{-y^*}\right)}{(B+y^*)^4}+\frac{y^*e^{-y^*}}{B+y^*}\right). \end{array}\right\}. \quad (43)$$

Now let

$$\eta=\frac{1-r\theta}{2}+\frac{-r+r\theta+m^*\theta-m^*\theta(1-r+r\theta)e^{-r(1-\theta)}}{2m^*\theta\left(1-e^{-r(1-\theta)}\right)}, \quad (44)$$

and

$$\xi=\frac{\sqrt{\Omega}}{2}. \quad (45)$$

And an invertible matrix T defined by

$$T=\begin{pmatrix} m_{12} & 0\\ \eta-m_{11} & -\xi\end{pmatrix}. \quad (46)$$

By computing T^{-1} one gets:

$$T^{-1}=\begin{pmatrix} \frac{1}{m_{12}} & 0\\ \frac{\eta-m_{11}}{\xi m_{12}} & \frac{-1}{\xi}\end{pmatrix}. \quad (47)$$

Using following transformation

$$\begin{pmatrix} u_t\\ v_t\end{pmatrix}=\begin{pmatrix} m_{12} & 0\\ \eta-m_{11} & -\xi\end{pmatrix}\begin{pmatrix} X_t\\ Y_t\end{pmatrix}, \quad (48)$$

equation (42) gives

$$\begin{pmatrix} X_{t+1}\\ Y_{t+1}\end{pmatrix}=\begin{pmatrix} \eta & -\xi\\ \xi & \eta\end{pmatrix}\begin{pmatrix} X_t\\ Y_t\end{pmatrix}+\begin{pmatrix} G(X_t,Y_t)\\ H(X_t,Y_t)\end{pmatrix}, \quad (49)$$

where

$$\left.\begin{array}{c} G(X_t,Y_t)=l_{11}X_t^2+l_{12}X_tY_t+l_{13}Y_t^2+l_{14}X_t^3+l_{15}X_t^2Y_t+l_{16}X_tY_t^2+l_{17}Y_t^3+\\ o((|u_t|+|v_t|)^3)\\ H(X_t,Y_t)=l_{21}X_t^2+l_{22}X_tY_t+l_{23}Y_t^2+l_{24}X_t^3+l_{25}X_t^2Y_t+l_{26}X_tY_t^2+l_{27}Y_t^3+\\ o((|u_t|+|v_t|)^3) \end{array}\right\}, \quad (50)$$

and

$$\left.\begin{aligned}
&l_{11}=m_{12}m_{13}+(\eta-m_{11})m_{14}+\frac{m_{15}}{m_{12}}(\eta-m_{11})^2, l_{12}=-\xi\left(m_{14}+\frac{2(\eta-m_{11})m_{15}}{m_{12}}\right),\\
&l_{13}=\frac{\xi^2m_{15}}{m_{12}}, l_{14}=m_{16}{m_{12}}^2+m_{12}m_{17}(\eta-m_{11})+m_{18}(\eta-m_{11})^2+\frac{m_{19}}{m_{12}}(\eta-m_{11})^3\\
&l_{15}=-\xi\left(m_{12}m_{17}+2m_{18}(\eta-m_{11})+3\frac{(\eta-m_{11})^2m_{19}}{m_{12}}\right),\\
&l_{16}=\xi^2\left(m_{18}+\frac{3m_{19}(\eta-m_{11})}{m_{12}}\right), l_{17}=-\frac{\xi^3m_{19}}{m_{12}},\\
&l_{21}=\frac{\eta-m_{11}}{\xi}\left(m_{12}m_{13}+(\eta-m_{11})m_{14}+\frac{(\eta-m_{11})^2}{m_{12}}m_{15}+m_{12}m_{23}-(\eta-m_{11})m_{24}\right),\\
&l_{22}=-(\eta-m_{11})m_{14}-\frac{2(\eta-m_{11})^2}{m_{12}}m_{15}+m_{12}m_{23}+2m_{24}(\eta-m_{11}),\\
&l_{23}=\frac{\xi(\eta-m_{11})}{m_{12}}m_{15}-\xi m_{24},\\
&l_{24}=\frac{\eta-m_{11}}{\xi}\begin{pmatrix}m_{16}{m_{12}}^2+(\eta-m_{11})m_{12}(m_{17}-m_{25})+(\eta-m_{11})^2(m_{18}-m_{26})+\\ \frac{(\eta-m_{11})^3}{m_{12}}m_{19}\end{pmatrix},\\
&l_{25}=(\eta-m_{11})\begin{pmatrix}-m_{12}m_{17}-2(\eta-m_{11})m_{18}-\frac{3(\eta-m_{11})^2}{m_{12}}m_{19}+2m_{12}m_{25}+\\ 3(\eta-m_{11})m_{26}\end{pmatrix},\\
&l_{26}=\xi\left((\eta-m_{11})m_{18}+\frac{3(\eta-m_{11})^2}{m_{12}}m_{19}-m_{12}m_{25}-3(\eta-m_{11})m_{25}\right),\\
&l_{27}=\xi^2\left(m_{26}-\frac{\eta-m_{11}}{m_{12}}m_{19}\right),
\end{aligned}\right\}.(51)$$

In addition,

$$\left.\begin{aligned}
G_{X_tX_t}|_{(0,0)}=2l_{11}, G_{X_tY_t}|_{(0,0)}=l_{12}, G_{Y_tY_t}|_{(0,0)}=2l_{13}, G_{X_tX_tX_t}|_{(0,0)}=6l_{14}, G_{X_tX_tY_t}|_{(0,0)}=2l_{15}.\\
G_{X_tY_tY_t}|_{(0,0)}=2l_{16}, G_{Y_tY_tY_t}|_{(0,0)}=6l_{17}, H_{X_tX_t}|_{(0,0)}=2l_{21}, H_{X_tY_t}|_{(0,0)}=l_{22}, H_{Y_tY_t}|_{(0,0)}=2l_{23}\\
H_{X_tX_tX_t}|_{(0,0)}=6l_{24}, H_{X_tX_tY_t}|_{(0,0)}=2l_{25}, H_{X_tY_tY_t}|_{(0,0)}=2l_{26}, H_{Y_tY_tY_t}|_{(0,0)}=6l_{27}
\end{aligned}\right\}.(52)$$

Now it is required that $\Psi\neq 0$, if (49) undergo N-S bifurcation (see Cao et al., 2013; Huet al., 2011; Chen et al., 2013; Khan et al., 2017; Guckenheimer and Holmes, 1983; Kuznetsov, 2004):

$$\Psi=-Re\left(\frac{(1-2\bar{\kappa})\bar{\kappa}^2}{1-\kappa}\tau_{11}\tau_{20}\right)-\frac{1}{2}\|\tau_{11}\|^2-\|\tau_{02}\|^2+Re(\bar{\kappa}\tau_{21}), \quad (53)$$

where

$$\left.\begin{aligned}
\tau_{02}&=\frac{1}{8}\left[G_{X_tX_t}-G_{Y_tY_t}+2H_{X_tY_t}+\iota\left(H_{X_tX_t}-H_{Y_tY_t}+2G_{X_tY_t}\right)\right]\Big|_{(0,0)}\\
\tau_{11}&=\frac{1}{4}\left[G_{X_tX_t}+G_{Y_tY_t}+\iota\left(H_{X_tX_t}+H_{Y_tY_t}\right)\right]\Big|_{(0,0)}\\
\tau_{20}&=\frac{1}{8}\left[G_{X_tX_t}-G_{Y_tY_t}+2H_{X_tY_t}+\iota\left(H_{X_tX_t}-H_{Y_tY_t}-2G_{X_tY_t}\right)\right]\Big|_{(0,0)}\\
\tau_{21}&=\frac{1}{16}\begin{bmatrix}G_{X_tX_tX_t}+G_{Y_tY_tY_t}+H_{X_tX_tY_t}+H_{Y_tY_tY_t}+\\ \iota\left(H_{X_tX_tX_t}+H_{X_tY_tY_t}-G_{X_tX_tY_t}-G_{Y_tY_tY_t}\right)\end{bmatrix}\Bigg|_{(0,0)}
\end{aligned}\right\}.(54)$$

By calculating one gets

$$\left.\begin{aligned}
\tau_{02}&=\frac{1}{4}[l_{11}-l_{13}+l_{22}+\iota(l_{21}-l_{23}+l_{12})]\\
\tau_{11}&=\frac{1}{2}[l_{11}+l_{13}+\iota(l_{21}+l_{23})]\\
\tau_{20}&=\frac{1}{4}[l_{11}-l_{13}+l_{22}+\iota(l_{21}-l_{23}-l_{12})]\\
\tau_{21}&=\frac{1}{8}\begin{bmatrix}3l_{14}+3l_{17}+l_{25}+3l_{27}+\\ \iota(3l_{24}+l_{26}-l_{15}-3l_{17})\end{bmatrix}
\end{aligned}\right\}. \quad (55)$$

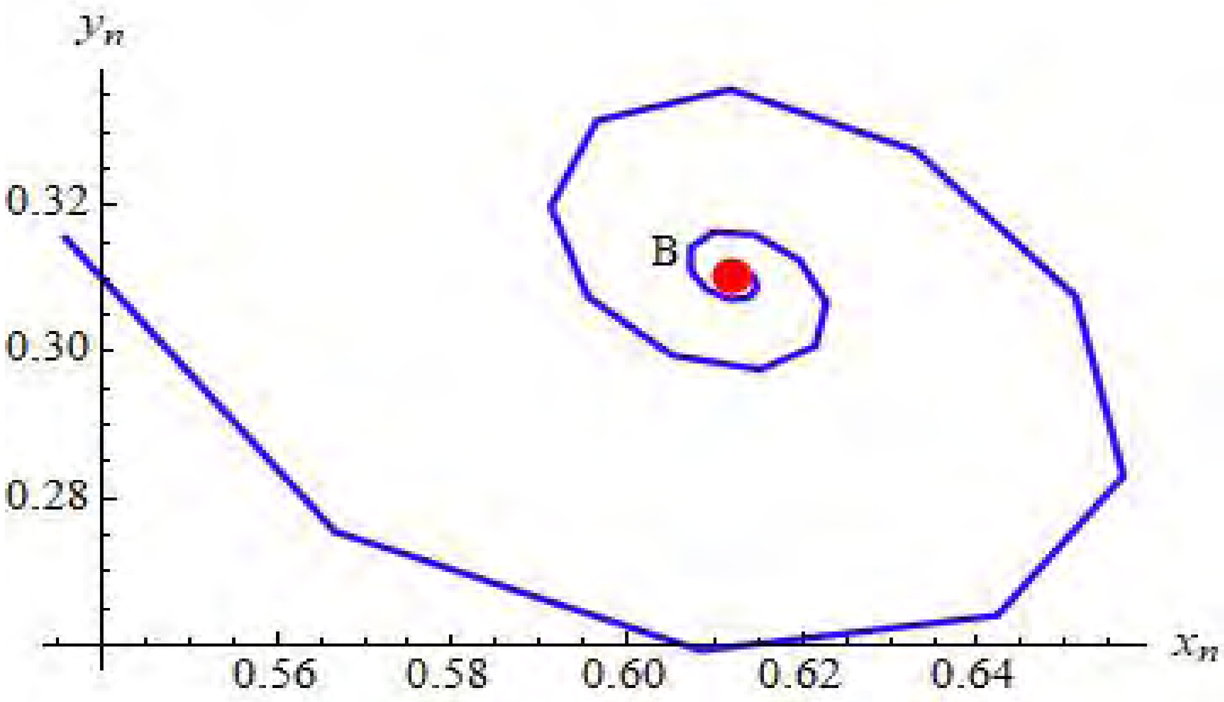
By bifurcation theory given Guckenheimer and Holmes (1983); Kuznetsov (2004), one has the following result:

Theorem 4.1. If $\Psi\neq 0$ then model (2) undergoes N-S bifurcation about $B(\theta, r(1-\theta))$ as parameters m goes through $N_{B(\theta,r(1-\theta))}$. Additionally attracting (respectively repelling) close invariant curve bifurcates from

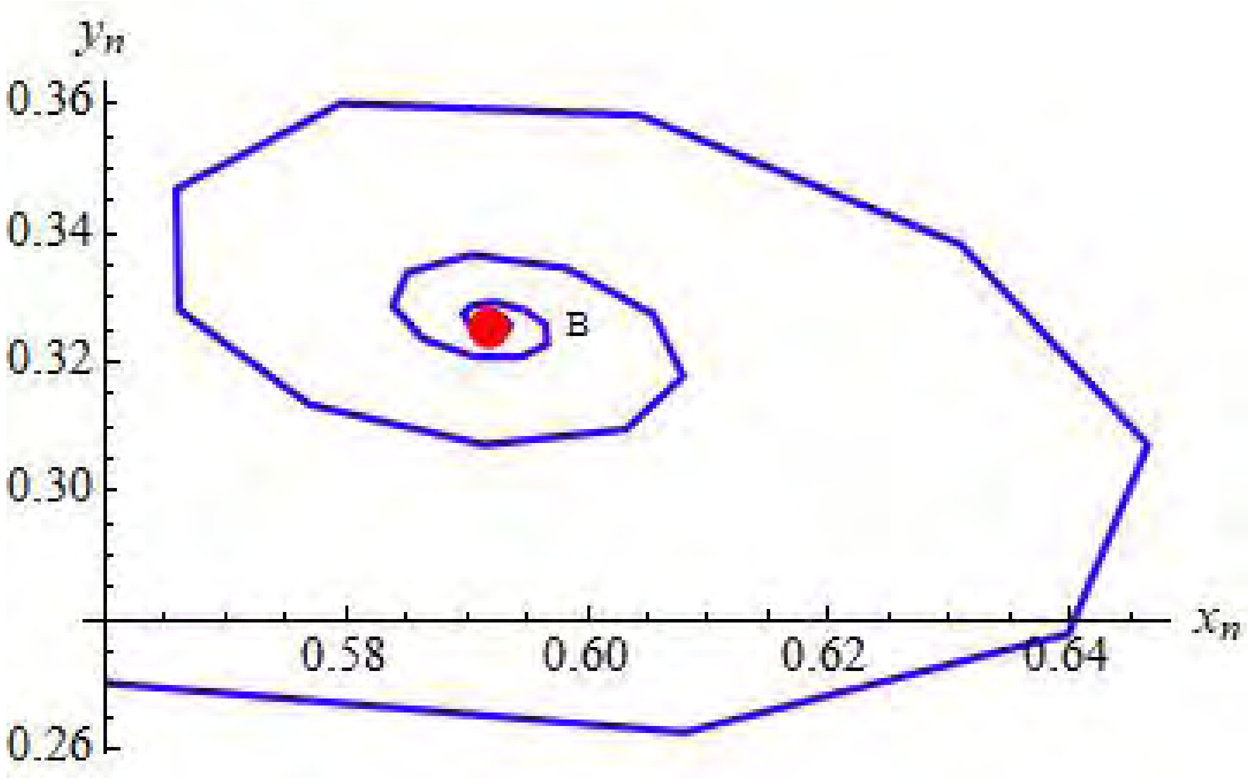
$B(\theta, r(1-\theta))$ if $\Psi < 0$ (respectively $\Psi > 0$).

5 Numerical Simulations

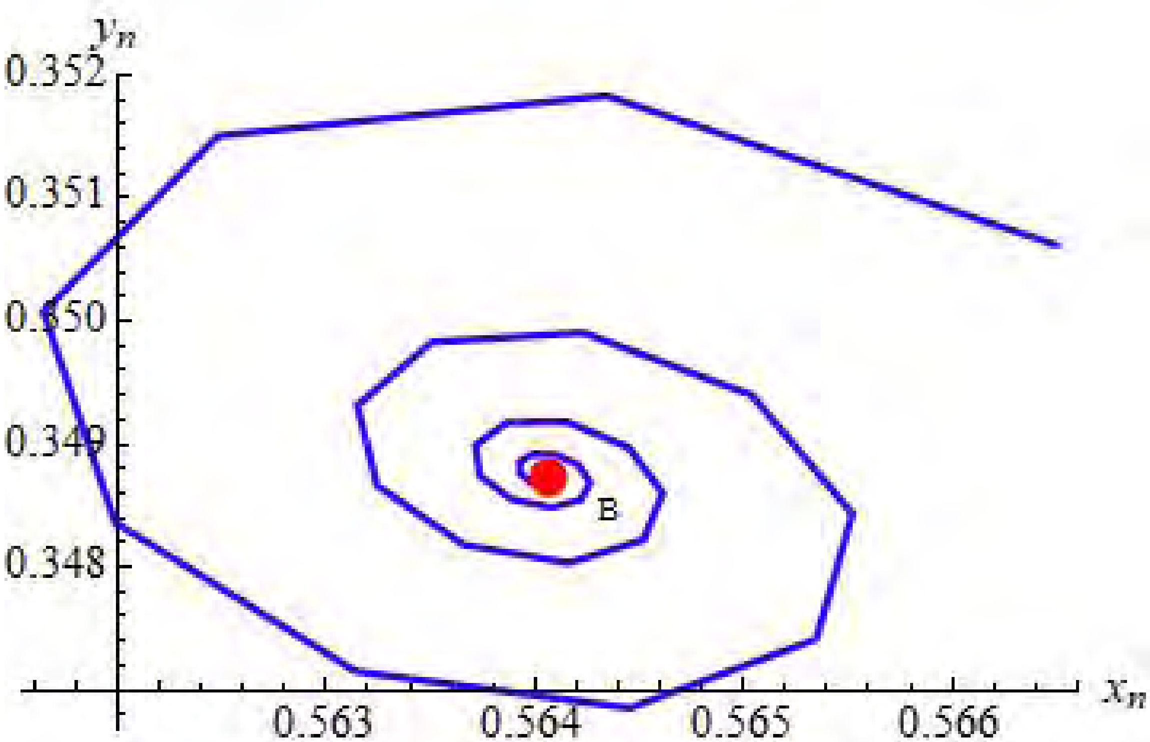
Here we will provide simulations in order to verify obtained results. Fixing parameters $r = 0.8, l = 0.41$ then from (iii) of Lemma 2.6 one gets $m = 3.0696247901516025$. From theoretical point of view equilibrium $B(\theta, r(1-\theta))$ of (2) is locally asymptotically sable focus if $m < 3.0696247901516025$. For example if $m = 1.9 < 3.0696247901516025$ then it is clear from Fig. 1A that $B(\theta, r(1-\theta))$ is locally asymptotically focus. Similarly for chosen bifurcation values, if $m = 1.9 < 3.0696247901516025$ then one can easily see that $B(l, r(1-l))$ is locally asymptotically sable focus (see Fig. 1B-N). But if $m > 3.0696247901516025$ then $B(\theta, r(1-\theta))$ is unstable focus and meanwhile stable invariant close curves appear. The appearance of these curve implies that (2) undergoes a supercritical N-S bifurcation if m varies in neighborhood of $B(\theta, r(1-\theta))$ (see Fig. 2A-N). Moreover bifurcation diagrams along with Maximum Lyapunov Exponent are presented in Fig. 3. Finally, bifurcation diagrams in $3D$ are presented in Fig. 4.



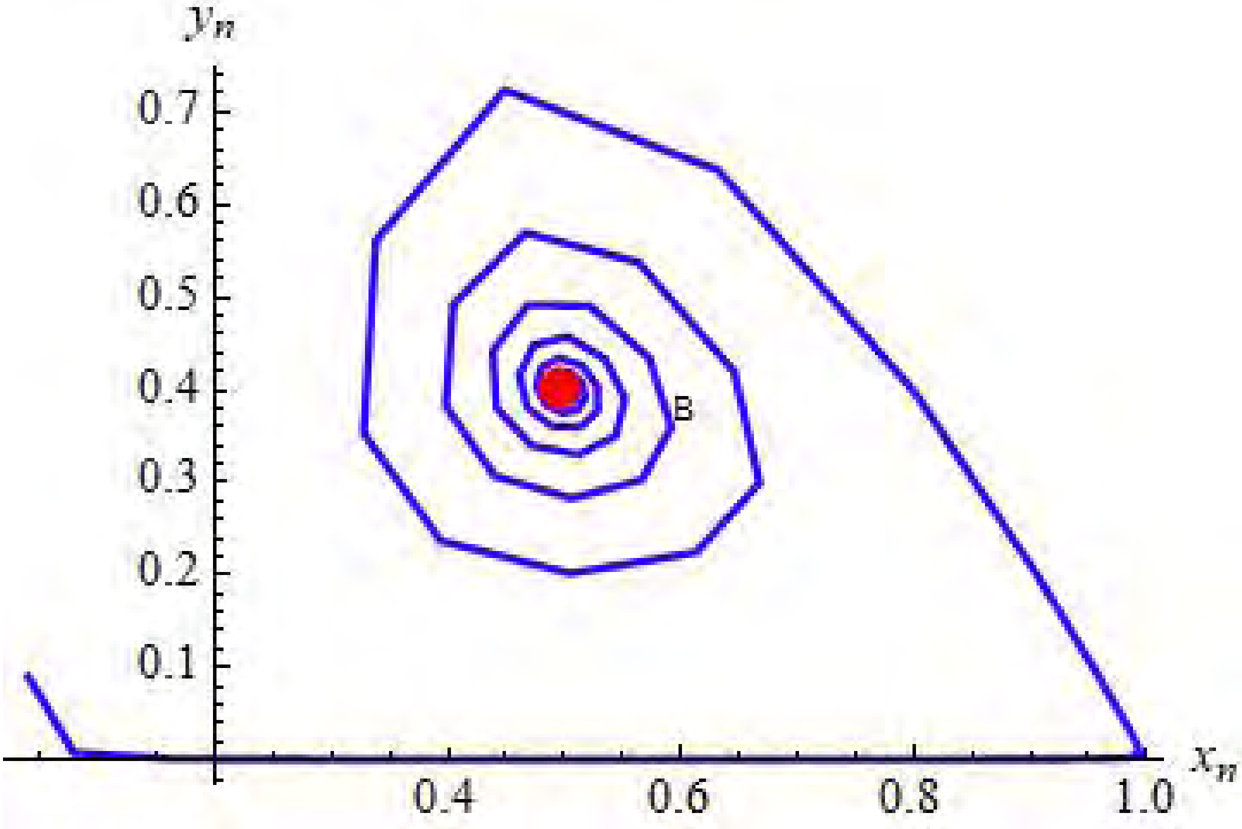
(A) $m = 1.9$ with (0.4,0.09)



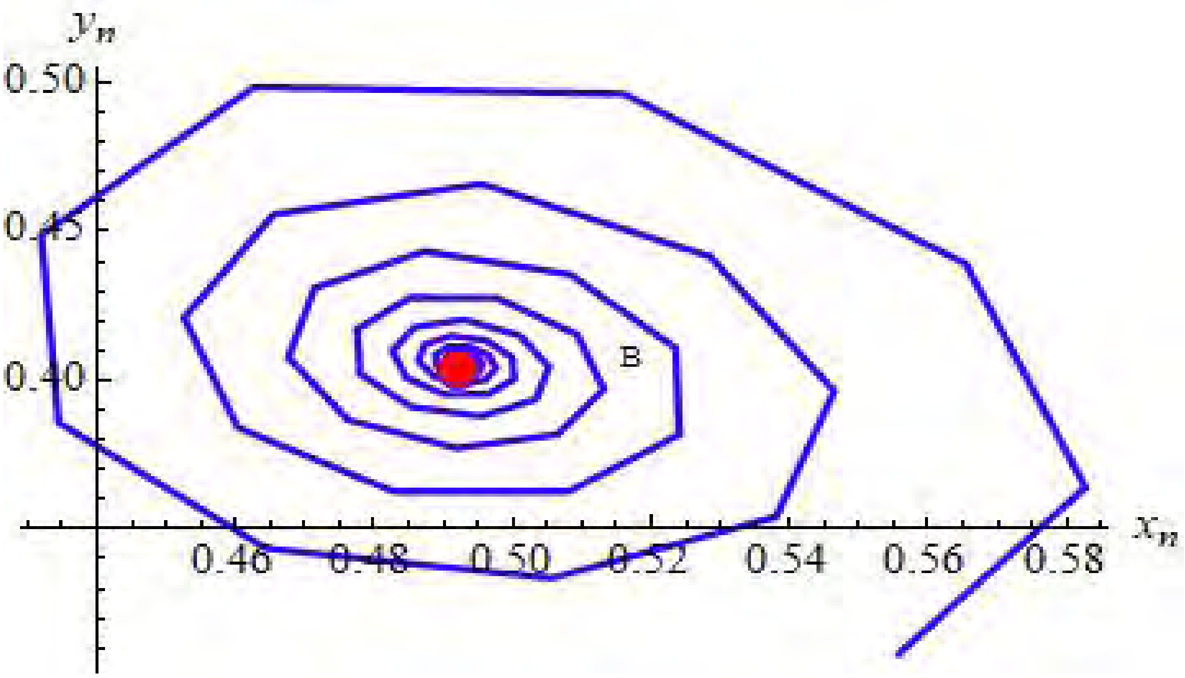
(B) $m = 1.978987$ with (0.54,0.09)



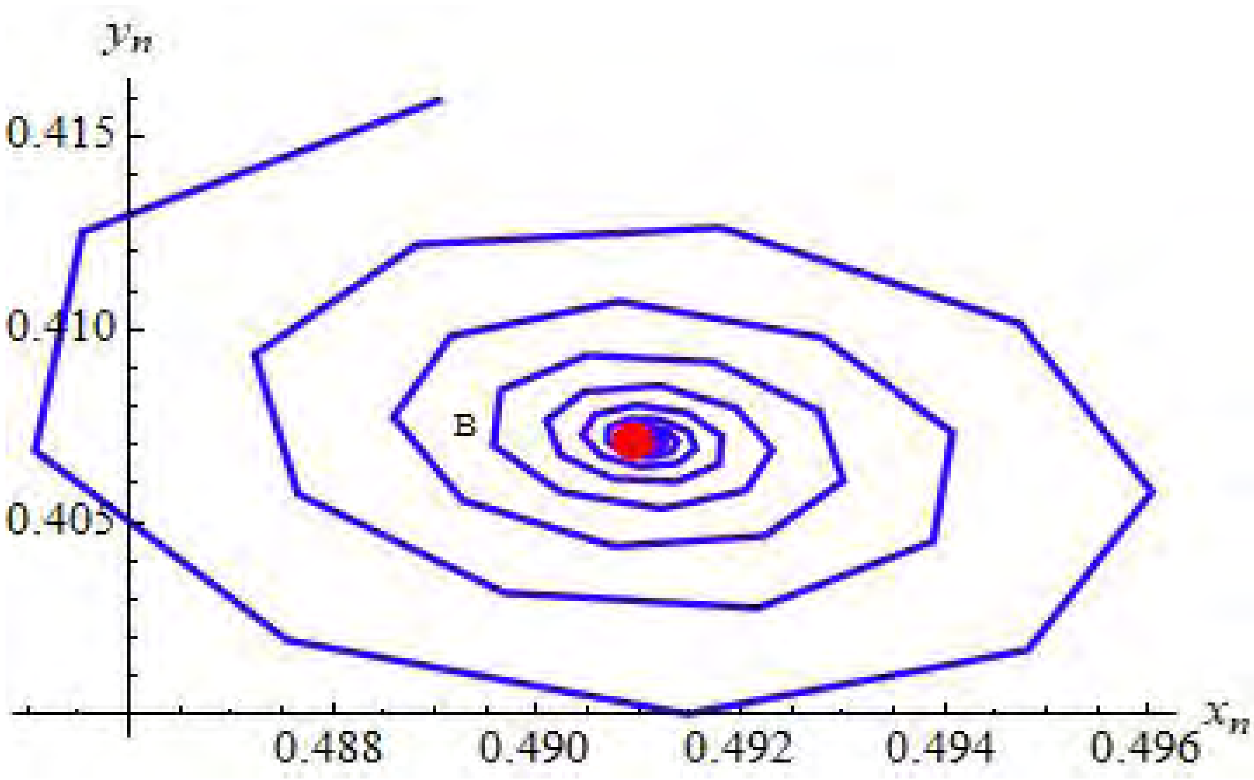
(C) $m = 2.1$ with (0.74,0.09)



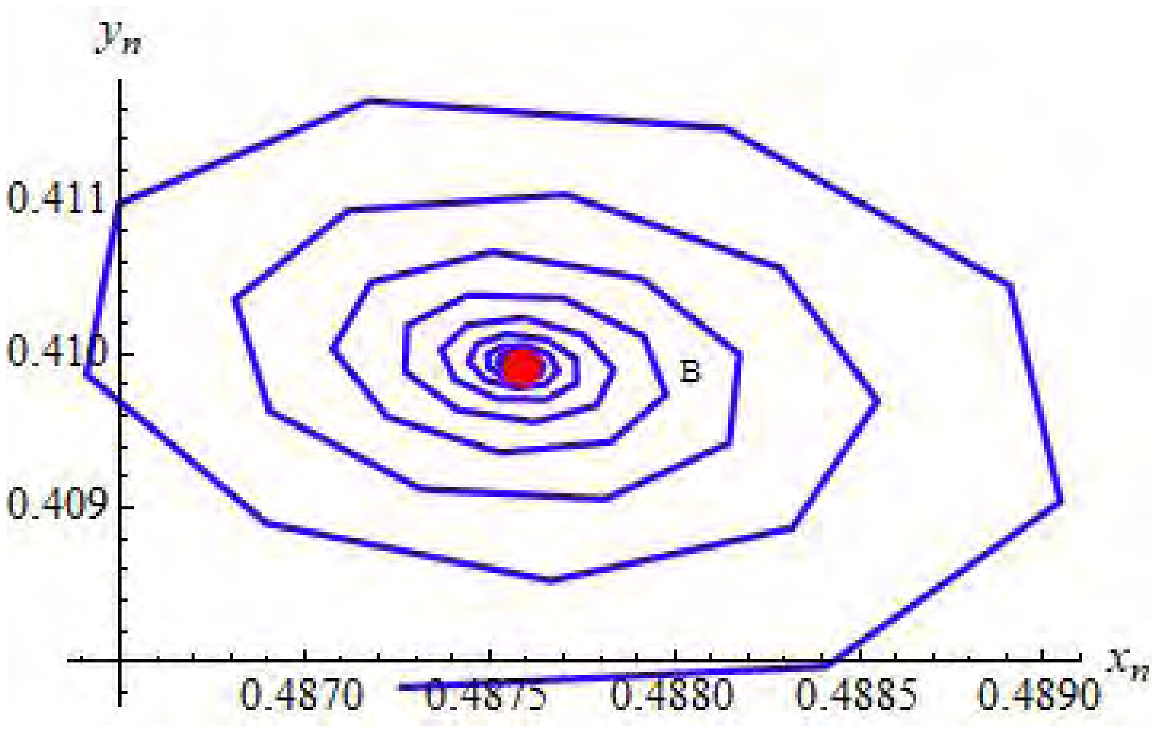
(D) $m = 2.43$ with (0.04,0.09)



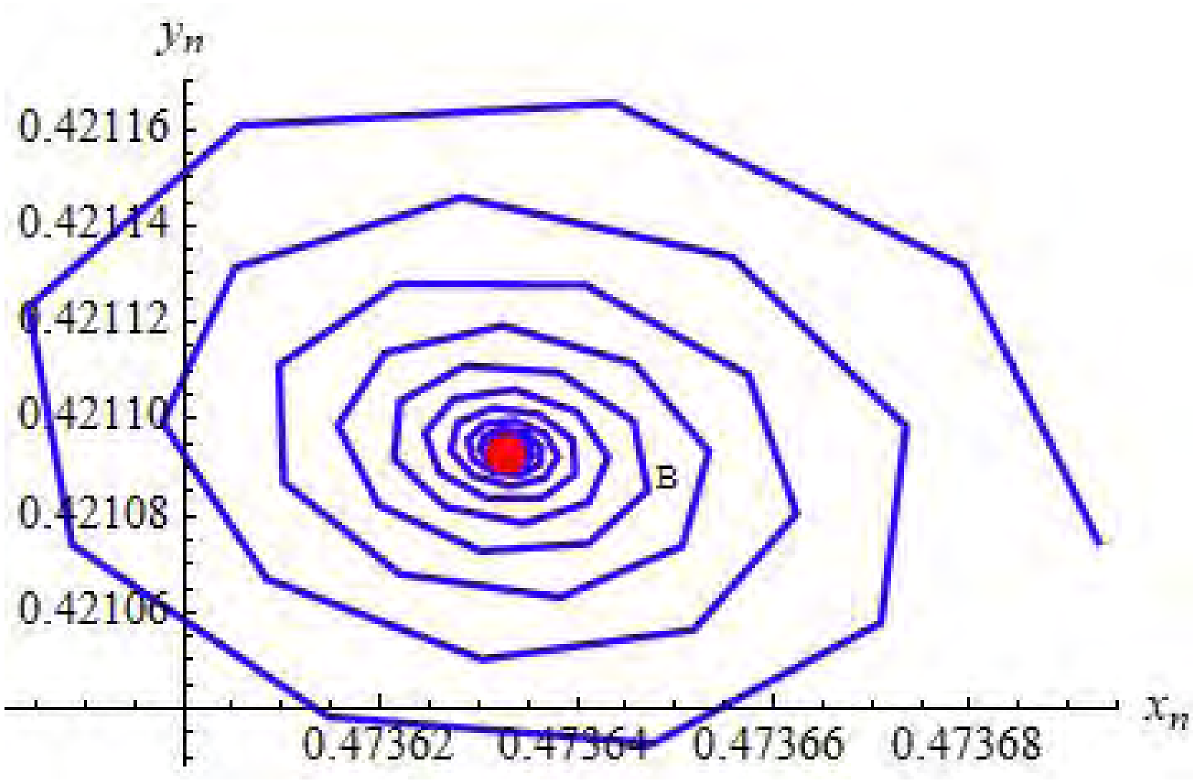
(E) $m = 2.47$ with (0.4,0.09)



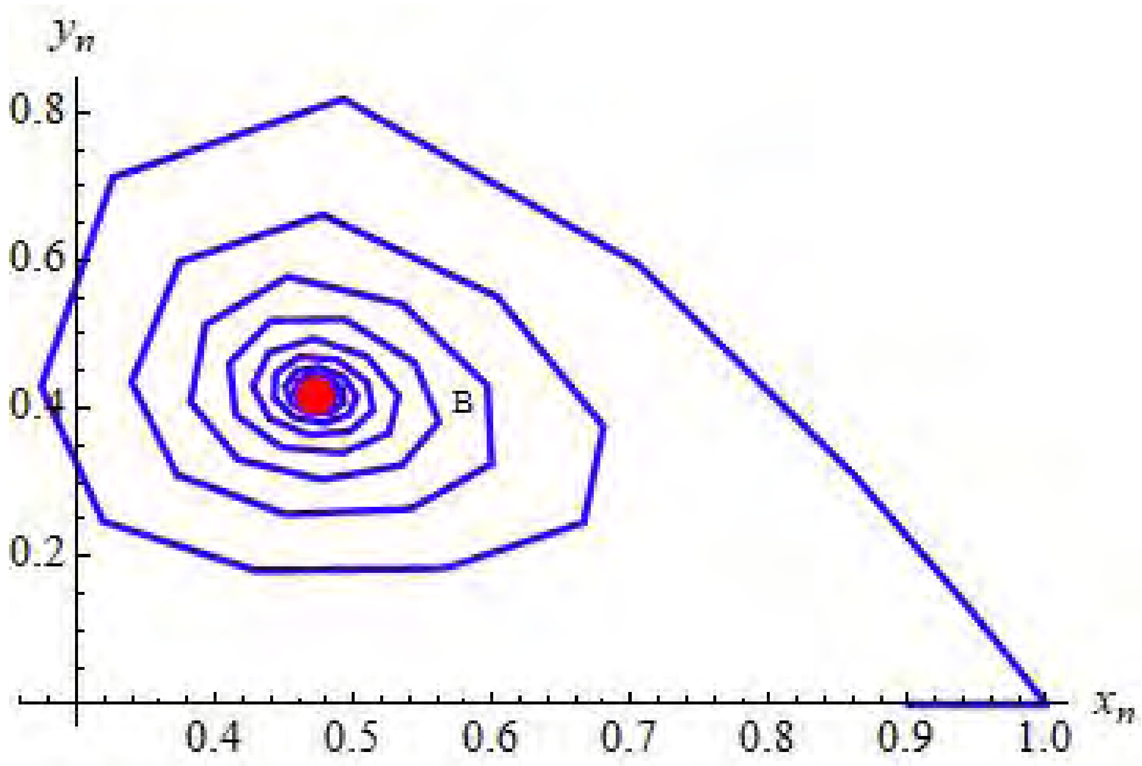
(F) $m = 2.479$ with (0.4,0.39)



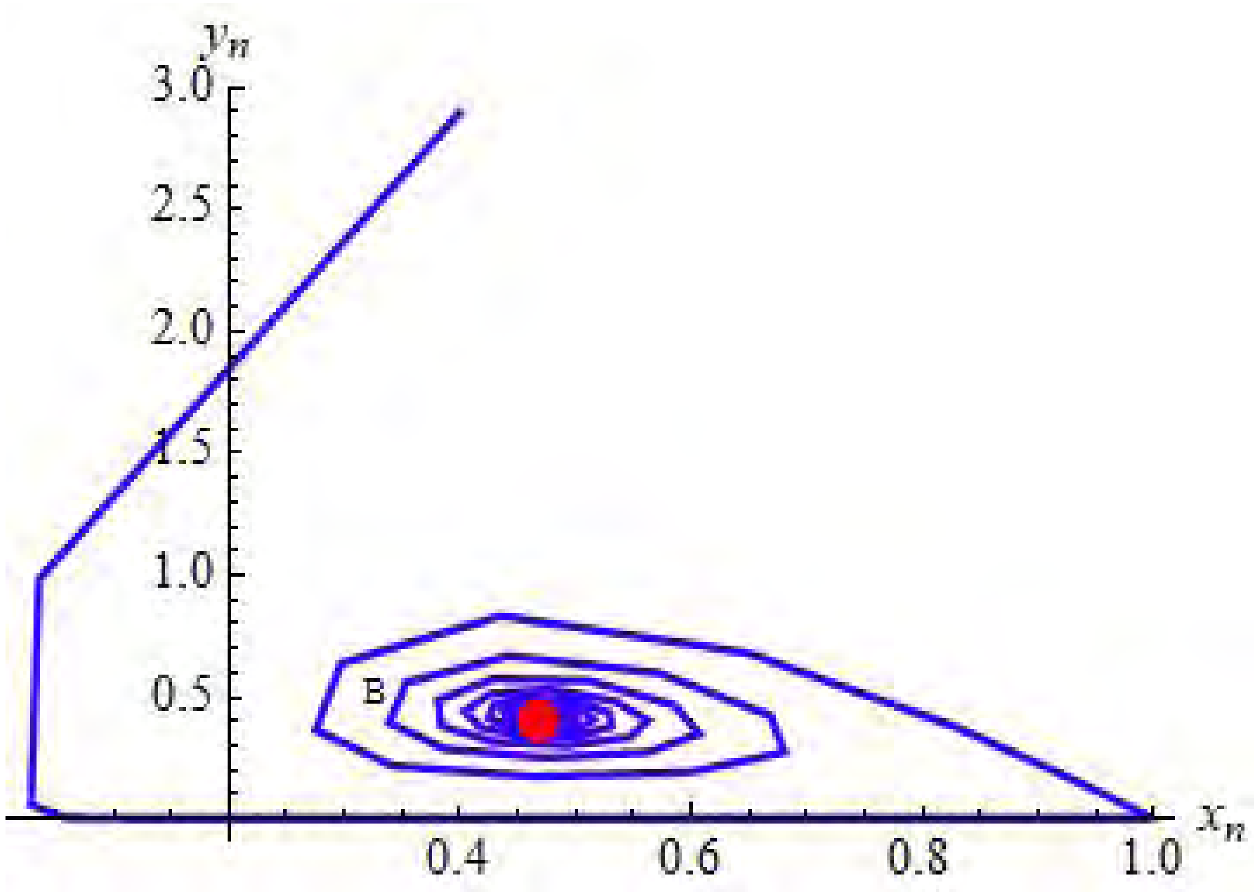
(G) $m = 2.5$ with (0.54,0.9)



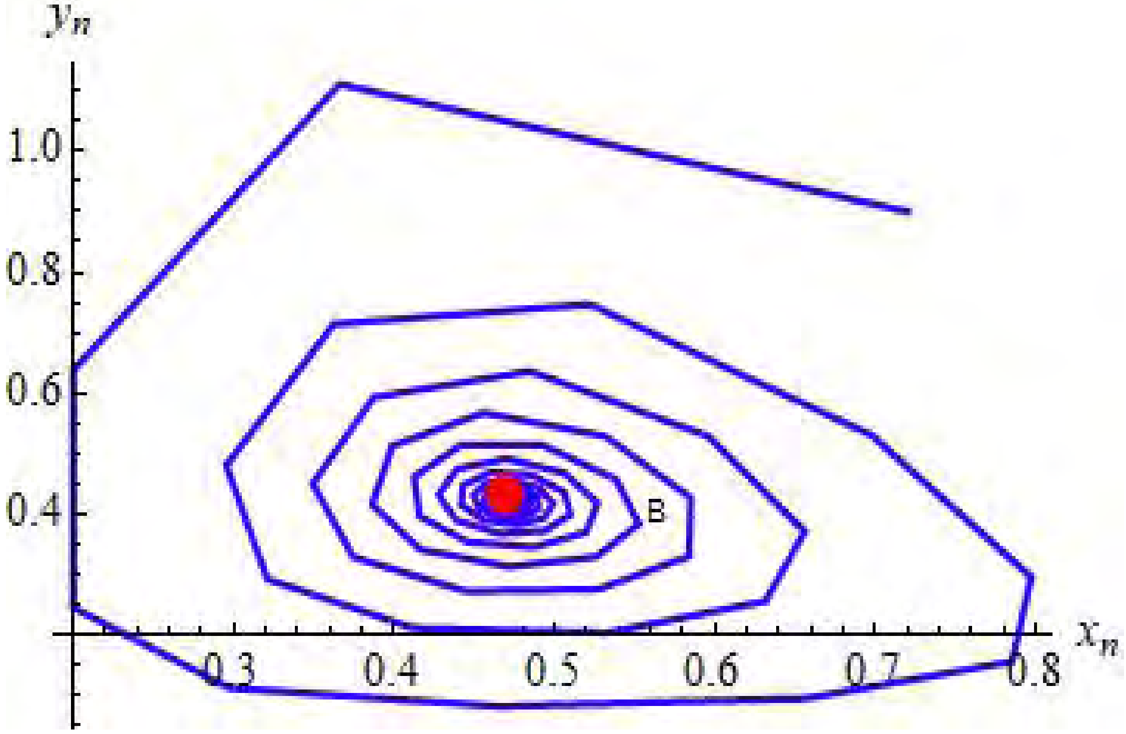
(H) $m = 2.587$ with (0.54,0.09)



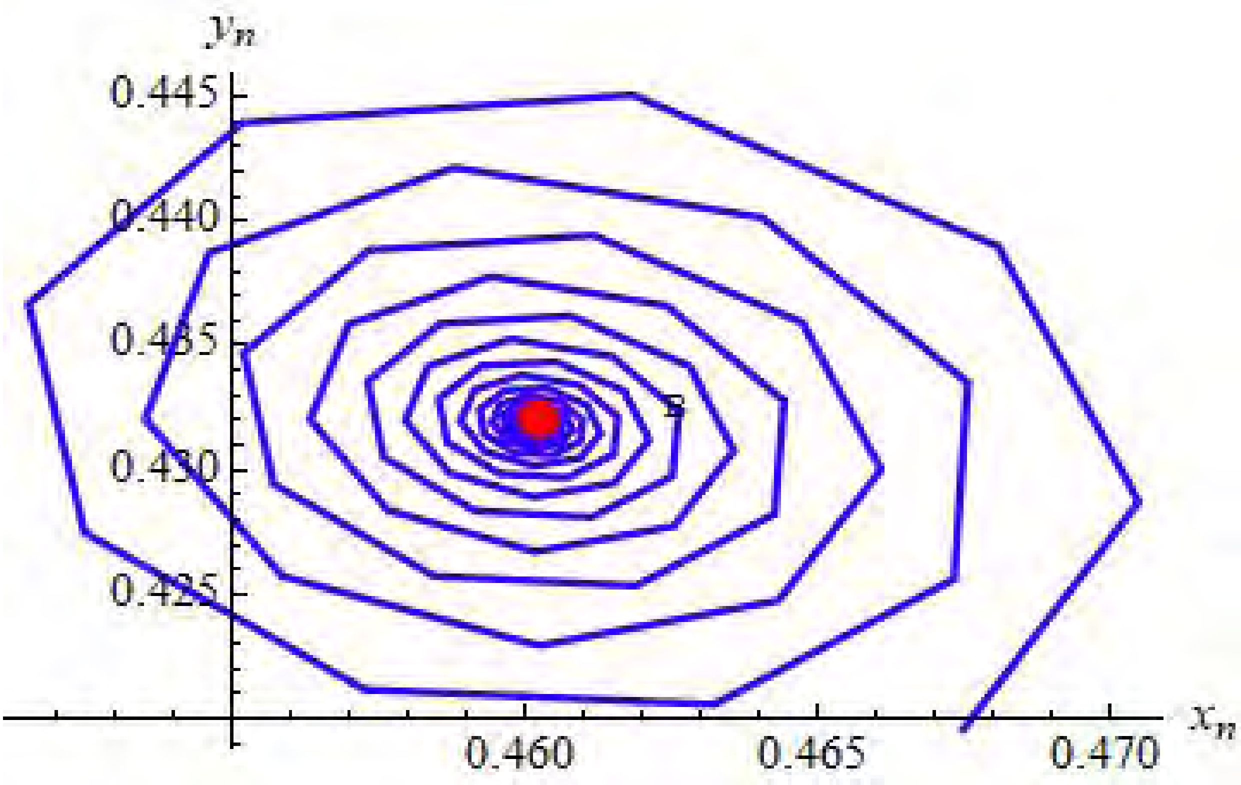
(I) $m = 2.587987$ with (0.9,0.000009)



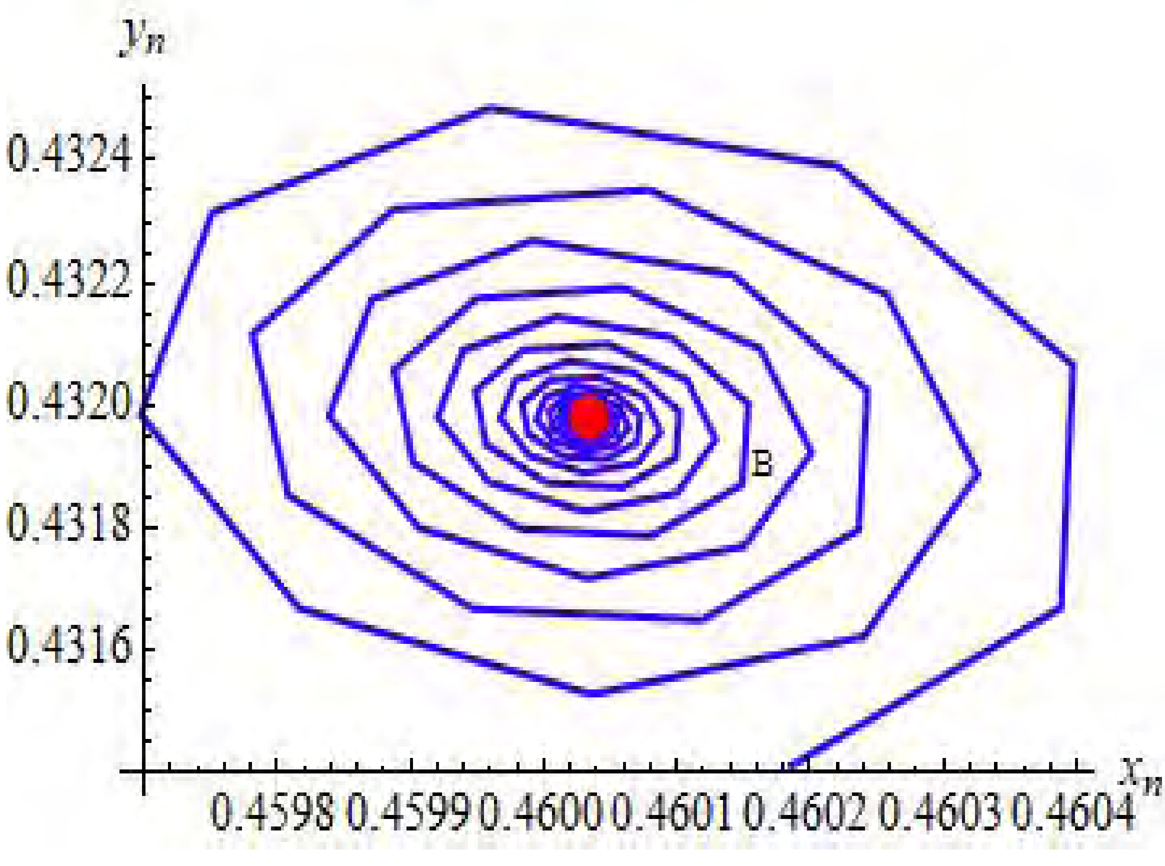
(J) $m = 2.6$ with (0.4,2.9)



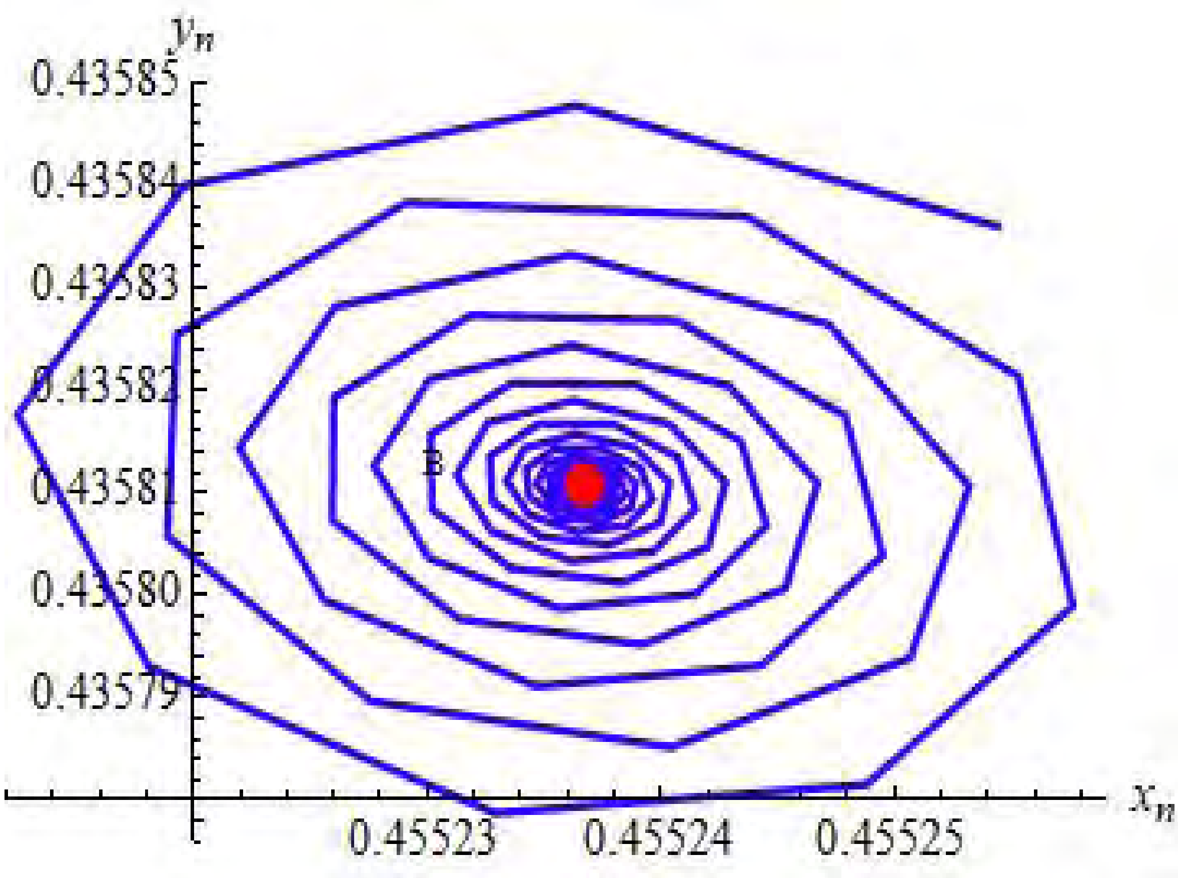
(K) $m = 2.6$ with (0.72,0.9)



(L) $m = 2.676$ with (0.2,0.9)

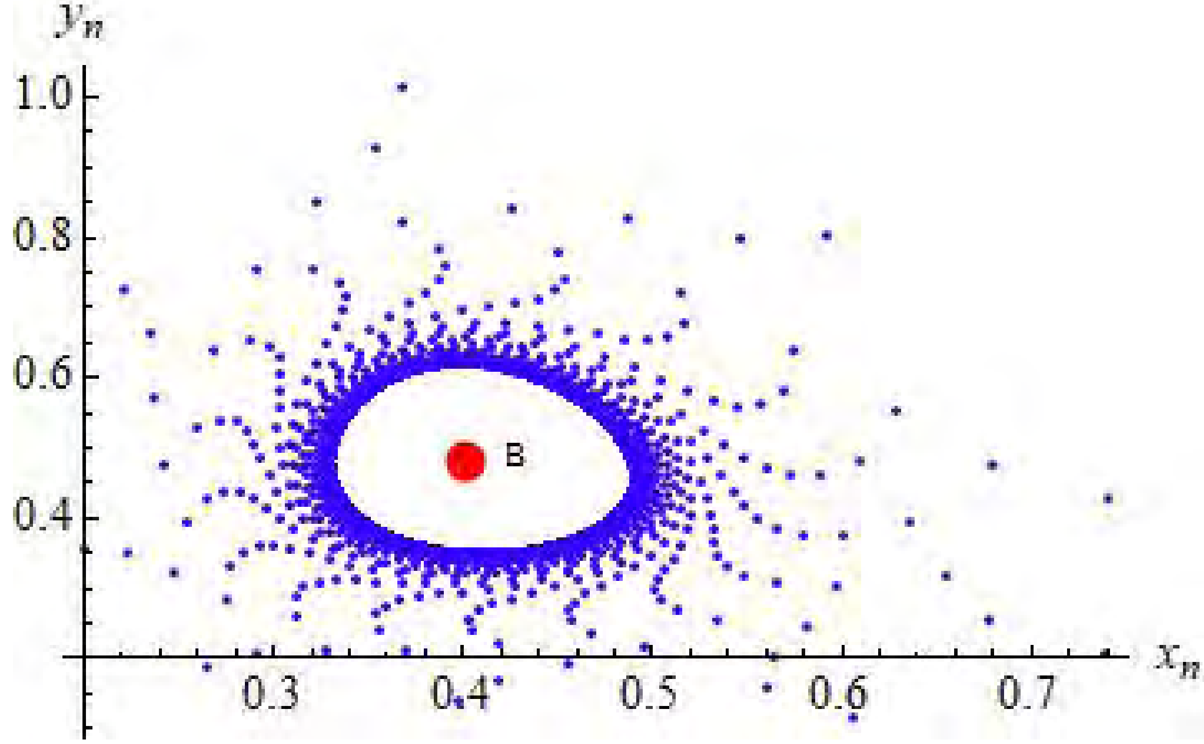


(M) $m = 2.676987$ with (0.2,0.09)

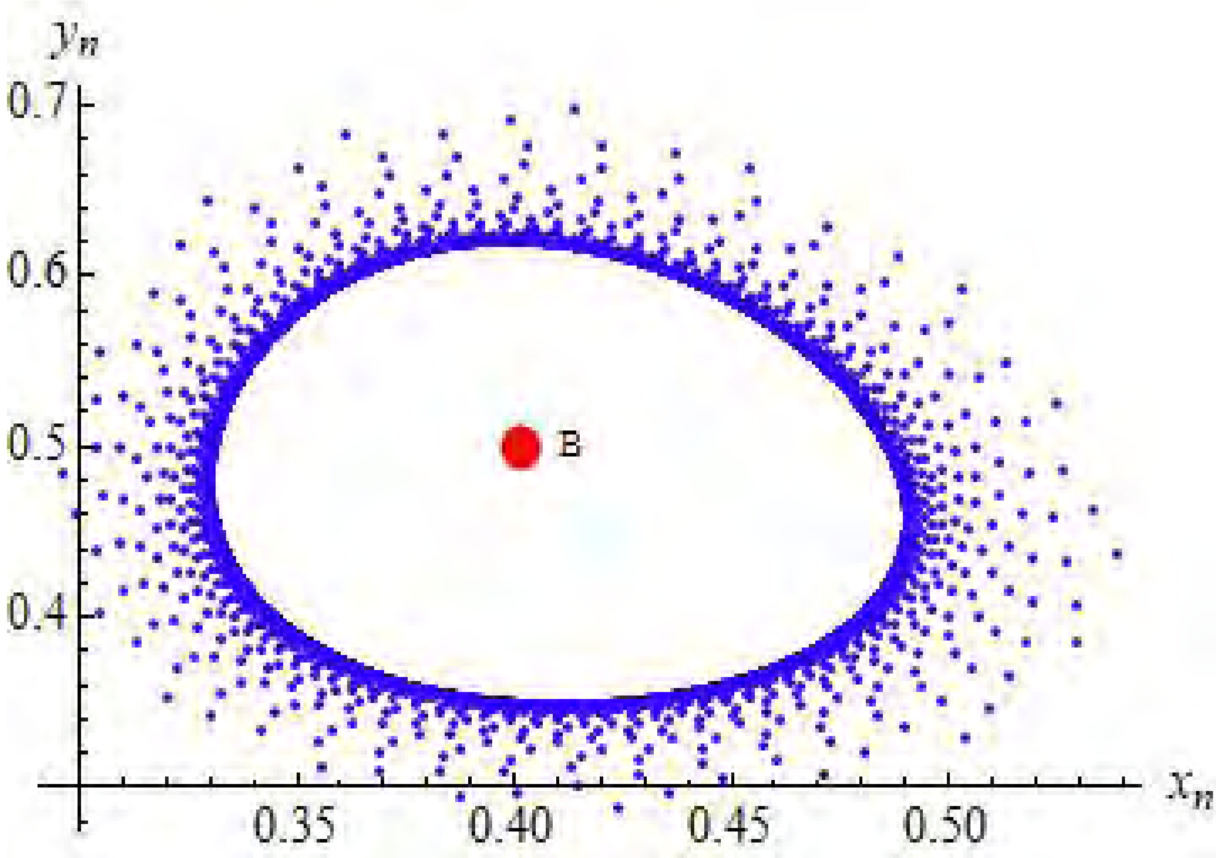


(N) $m = 2.71$ with (0.6,0.92)

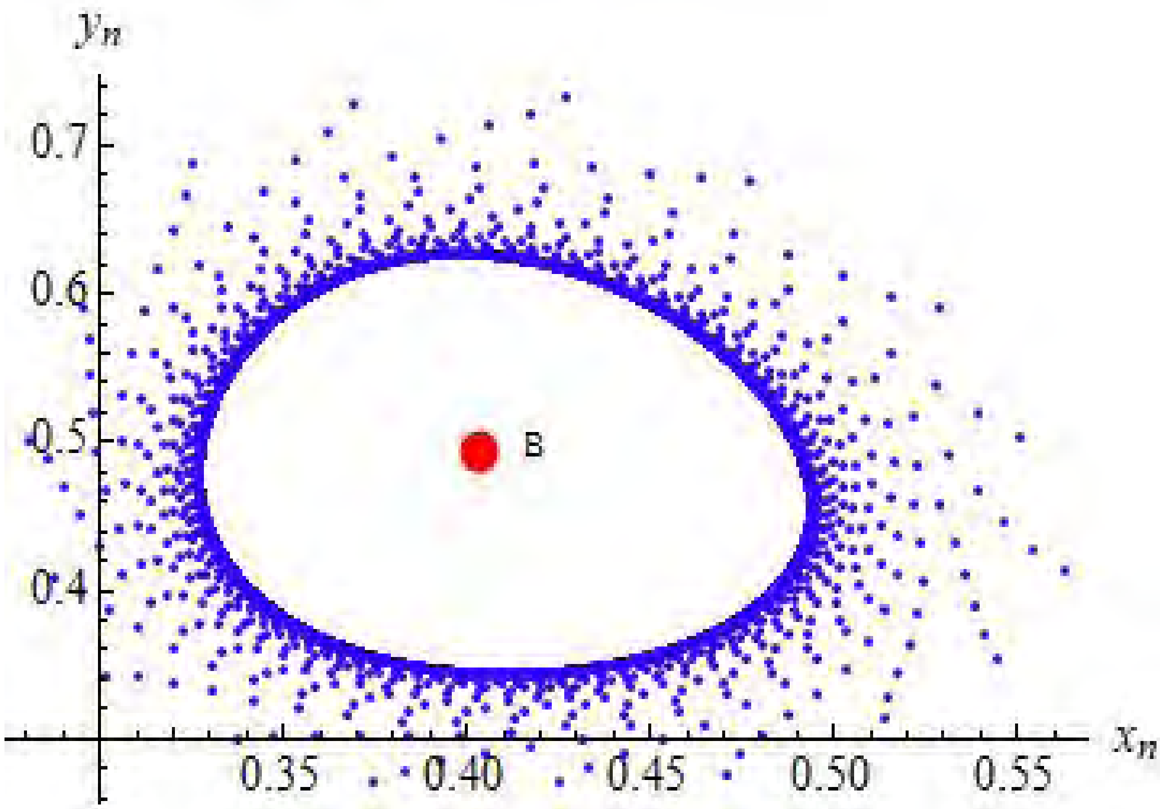
Fig. 1 Phase portraits for model (2).



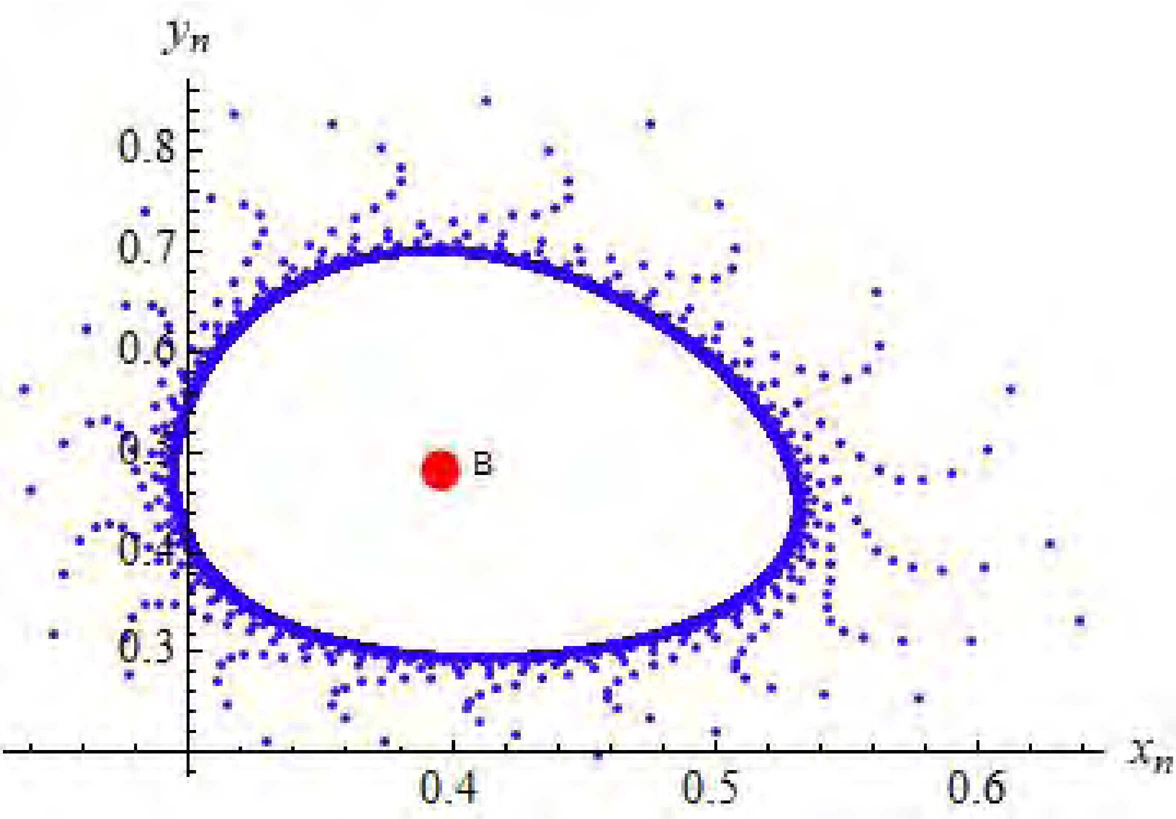
(A) $m = 3.1$ with (0.9987,0.02)



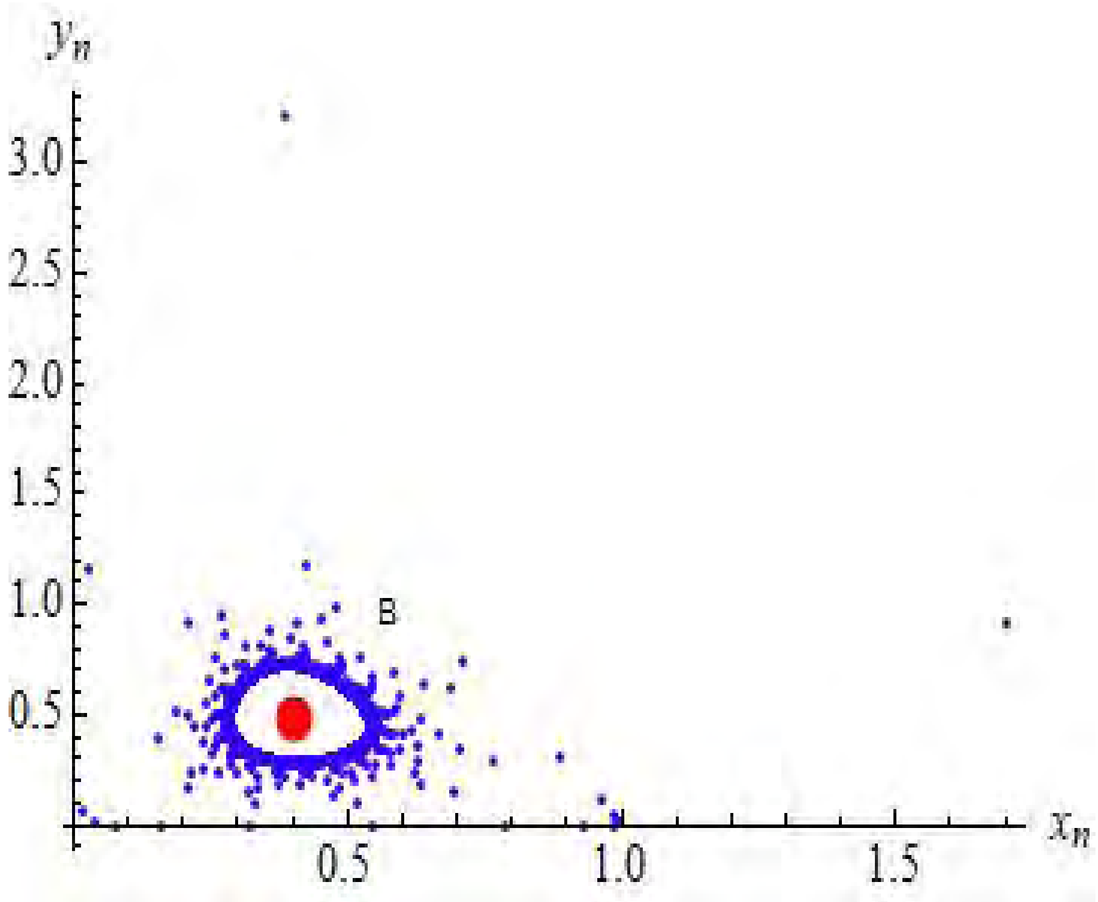
(B) $m = 3.1$ with (0.1,0.2)



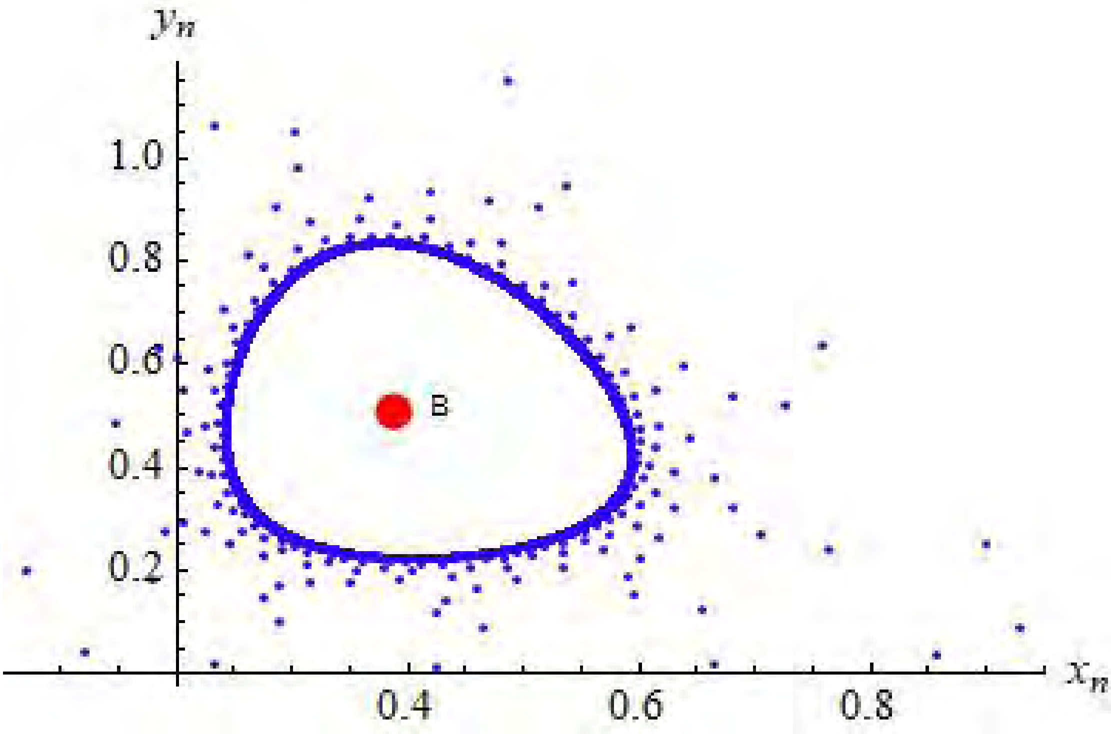
(C) $m = 3.102$ with (0.79,0.002)



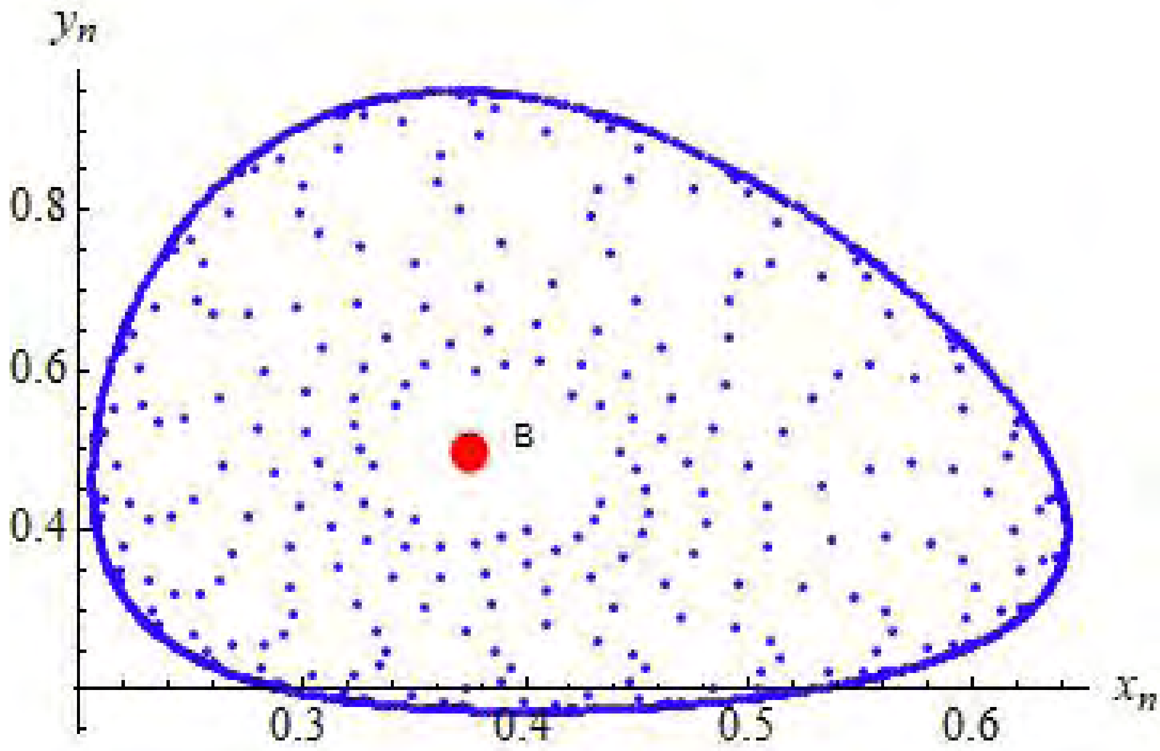
(D) $m = 3.13$ with $(0.97,1.92)$



(E) $m = 3.139$ with $(1.7,0.92)$

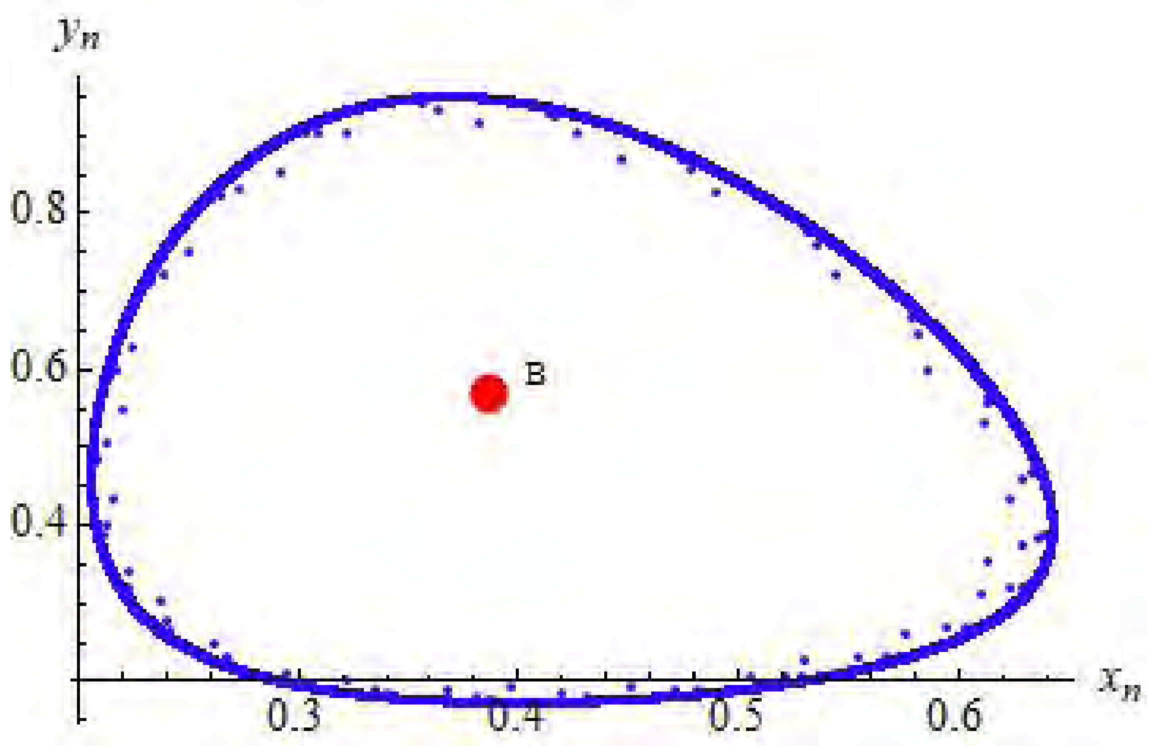


(F) $m = 3.2$ with (0.07,0.2)

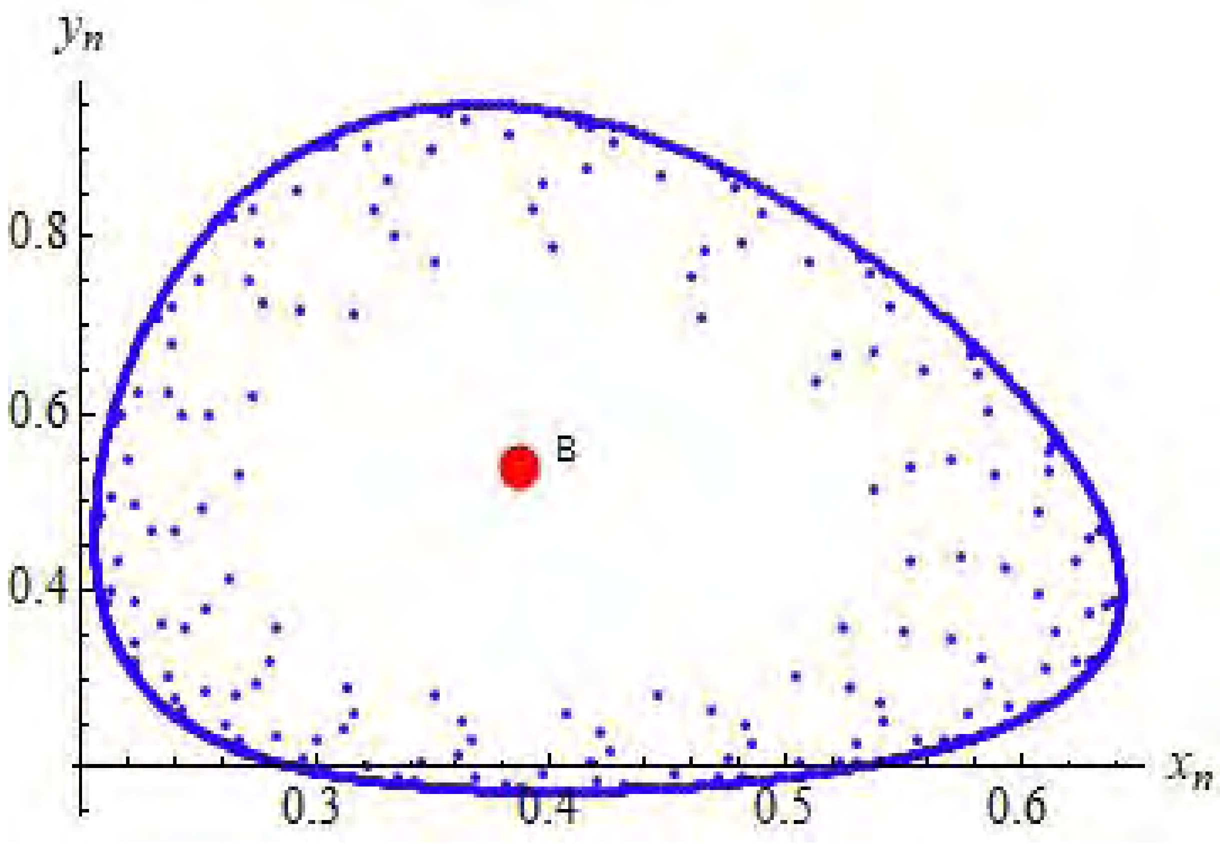


(G)

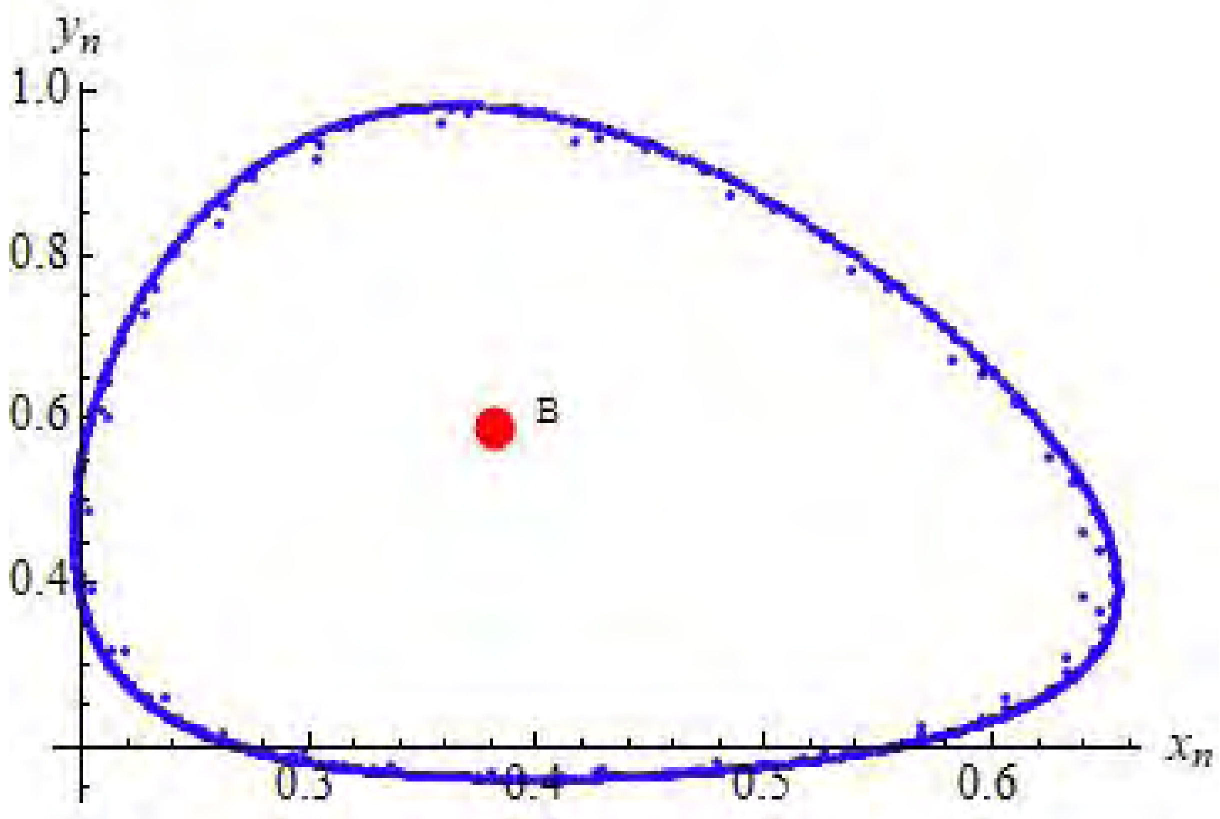
(G) $m = 3.276$ with (0.4,0.4)



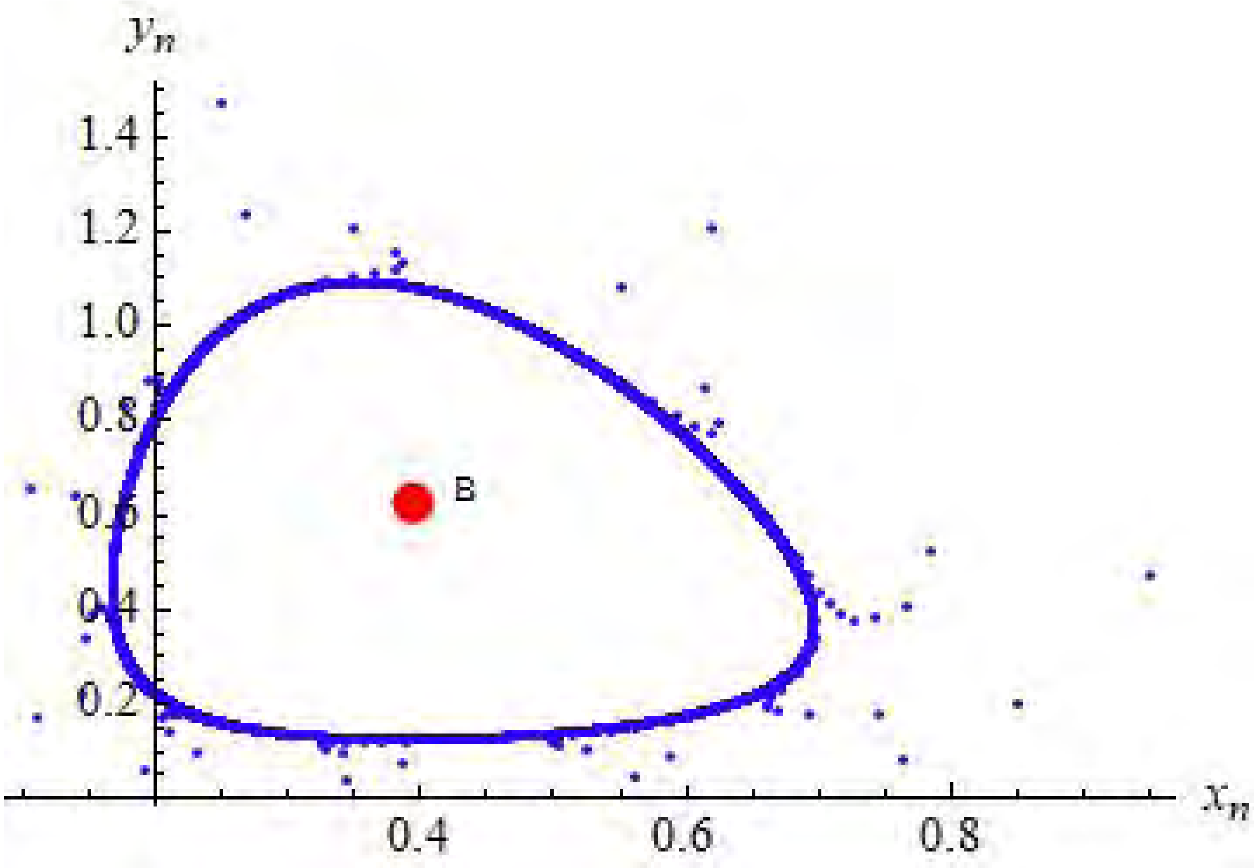
(H) $m = 3.277$ with $(0.5, 0.4)$



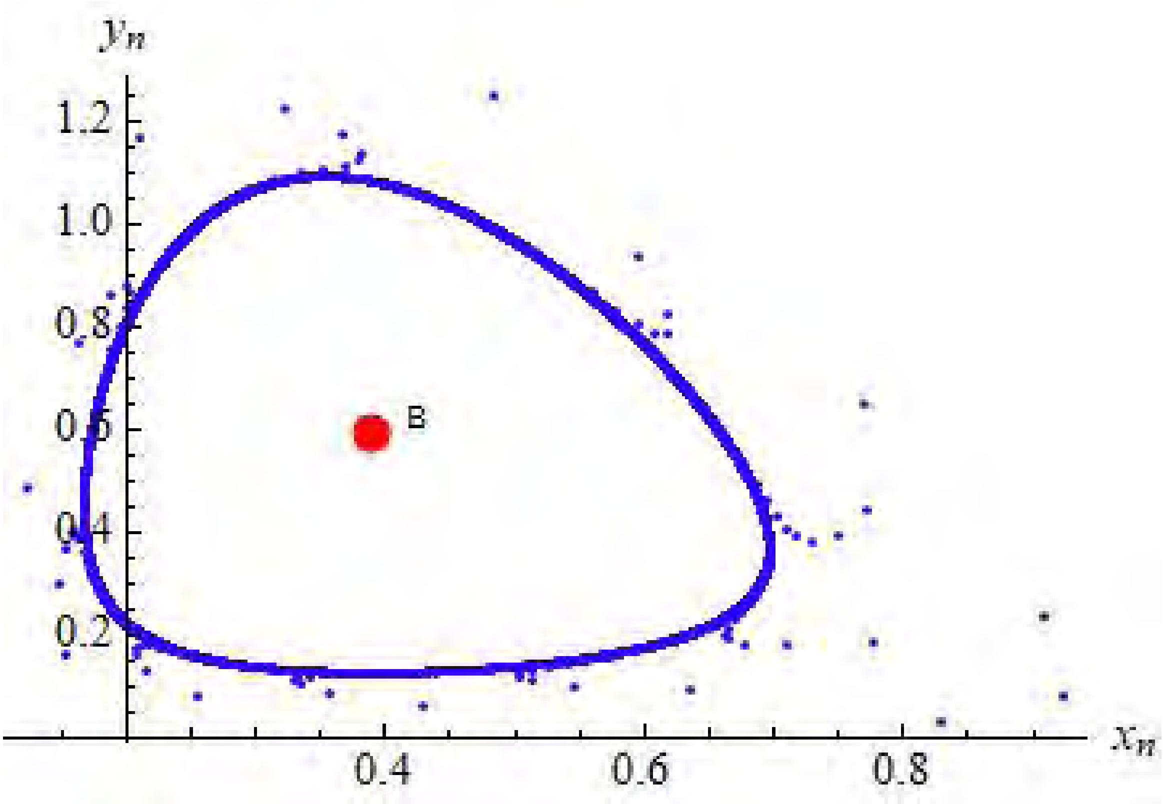
(I) $m = 3.2769$ with $(0.5,0.4)$



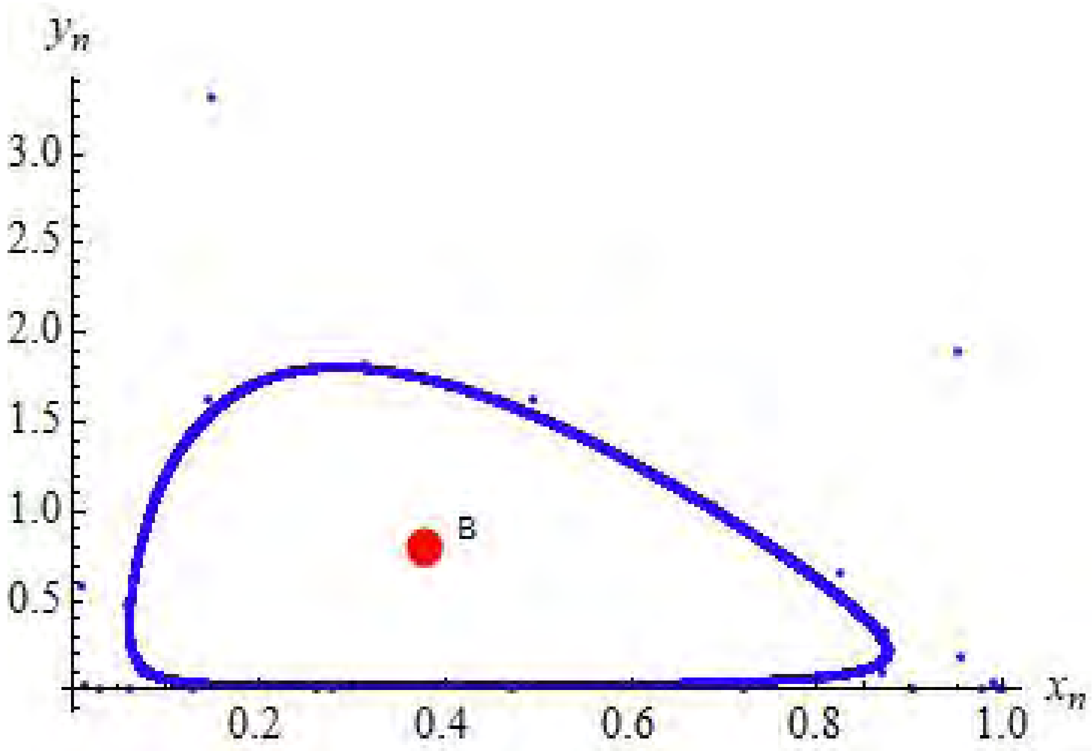
(J) $m = 3.3$ with $(0.5, 0.7)$



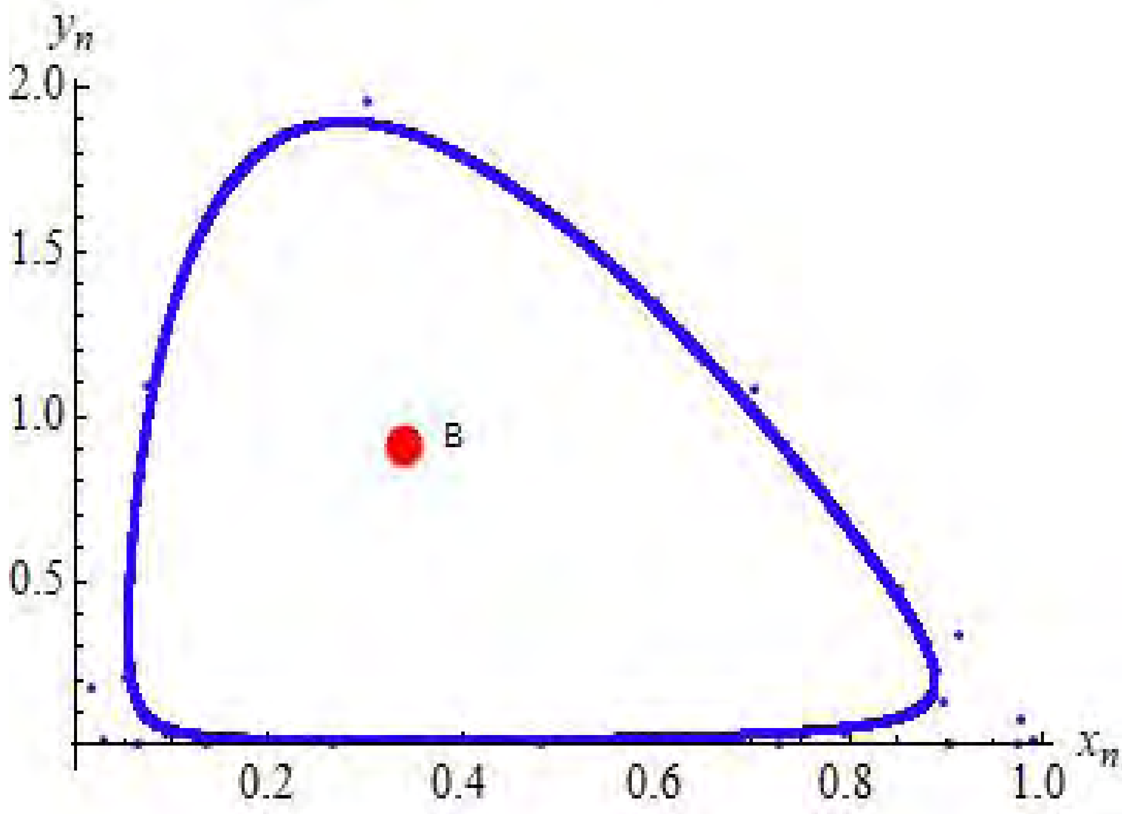
(K) $m = 3.387$ with $(0.95, 0.47)$



(L) $m = 3.38798$ with $(0.95, 0.89)$

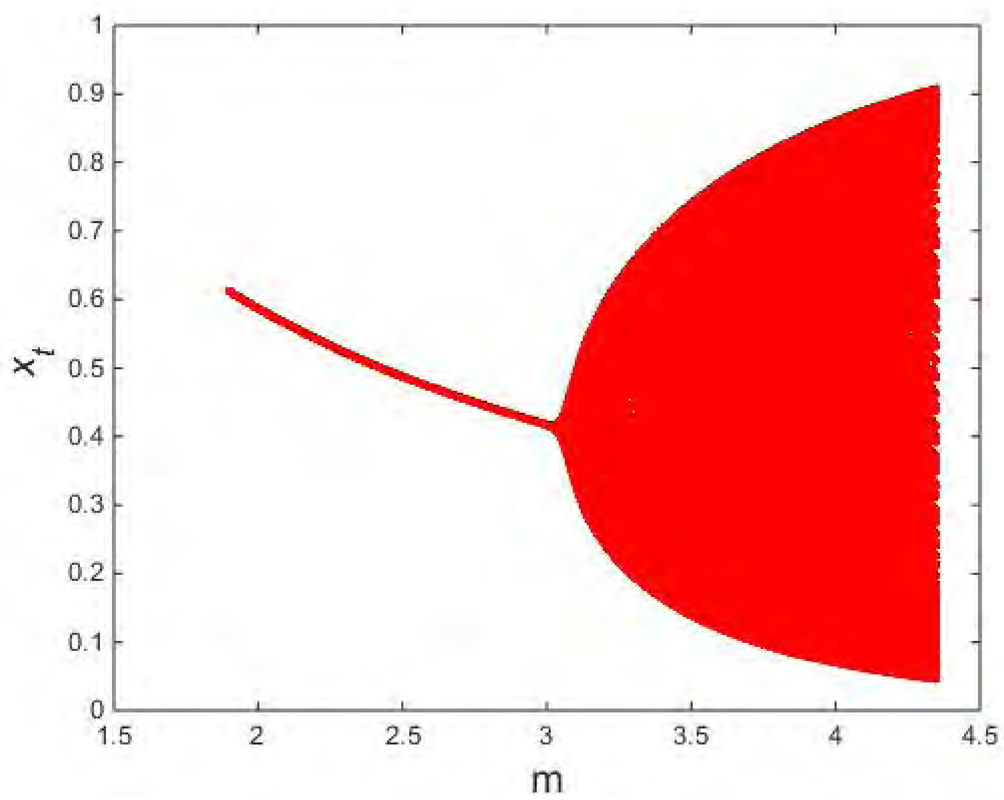


(M) $m = 4.1$ with (0.95,1.9)

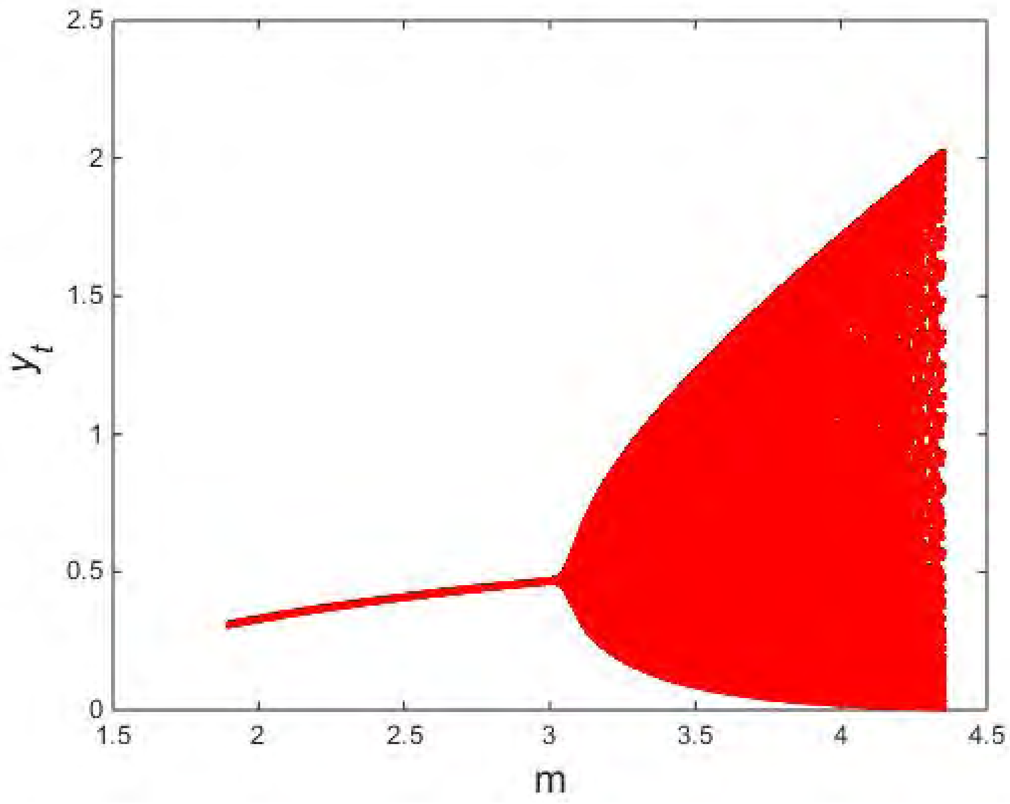


(N) $m = 4.2$ with $(0.05, 1.9)$

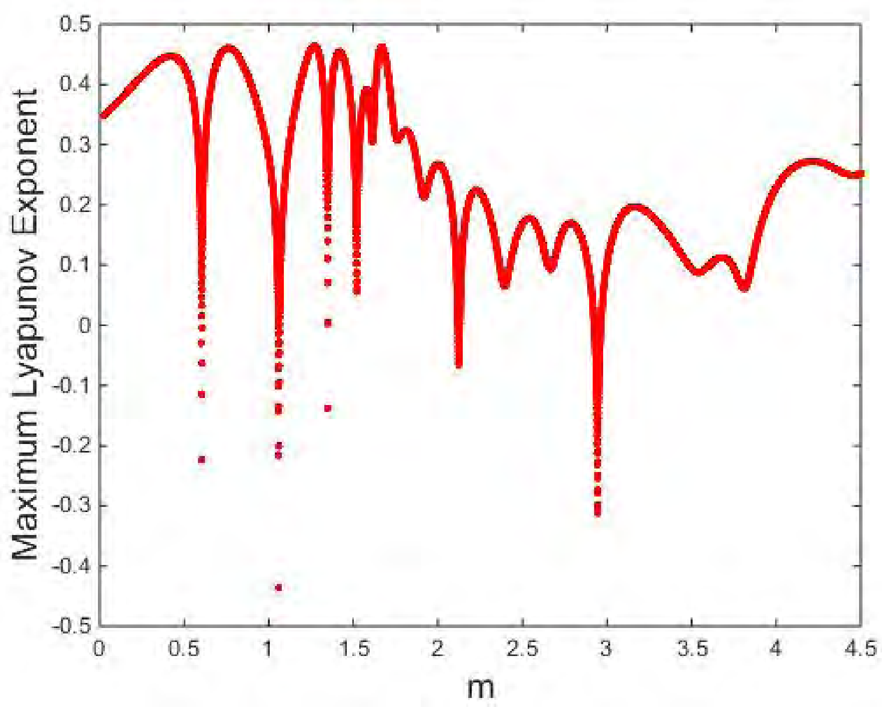
Fig. 2 Phase portraits for model (2).



(A)

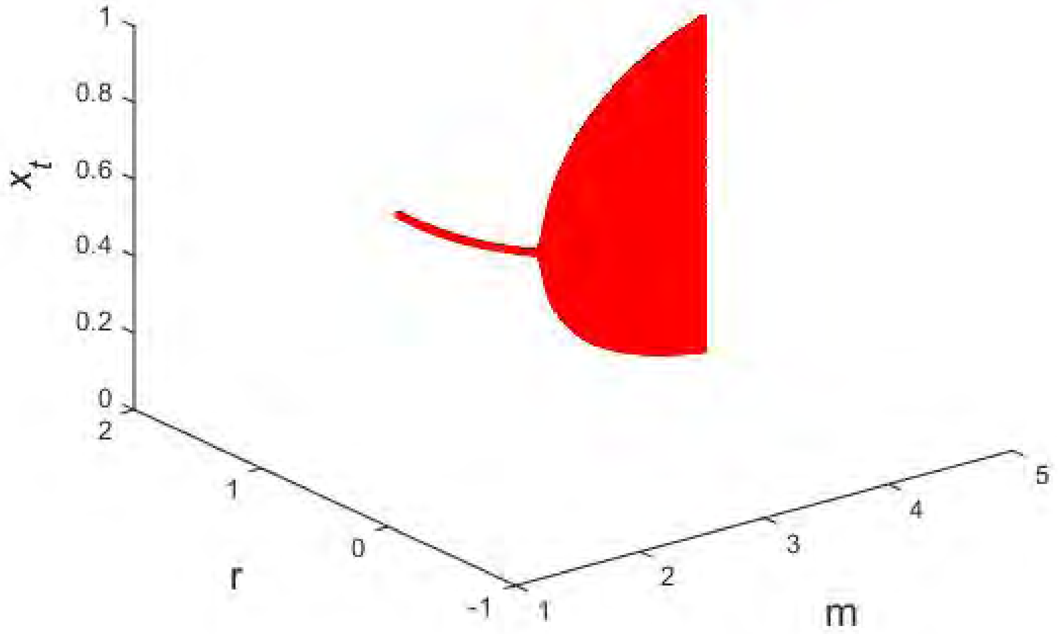


(B)

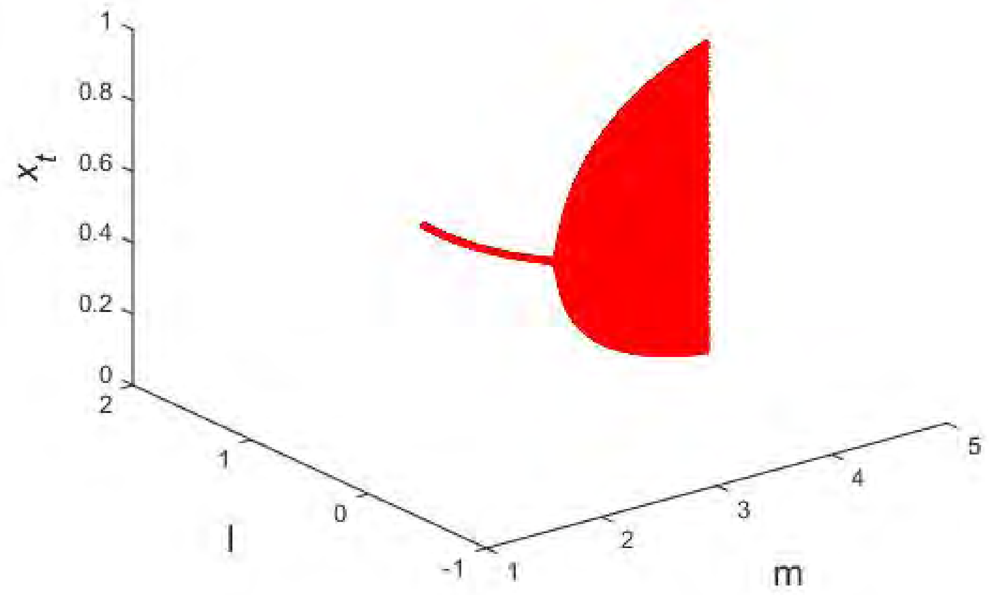


(C)

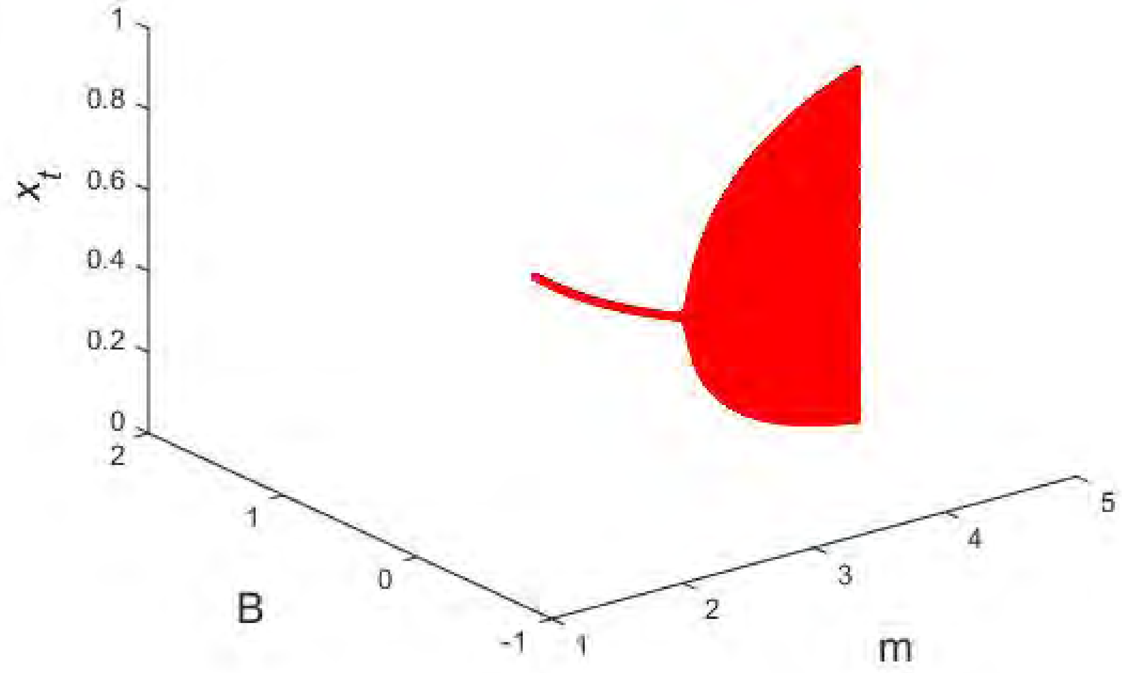
Fig. 3 Bifurcation diagram and their corresponding Maximum Lyapunov Exponent of the model (2) about $B(\theta, r(1-\theta))$. (A-B)Bifurcation diagram of (2) if $1.9 \leq m \leq 4.35$ and $(0.4,0.09)$. (C) Maximum Lyapunov Exponent corresponding to (A-B).



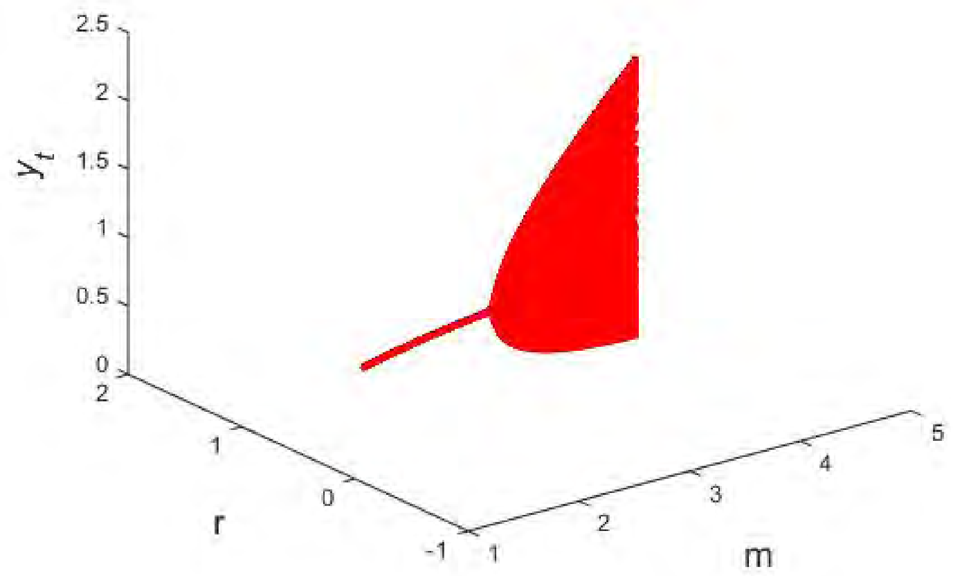
(A)



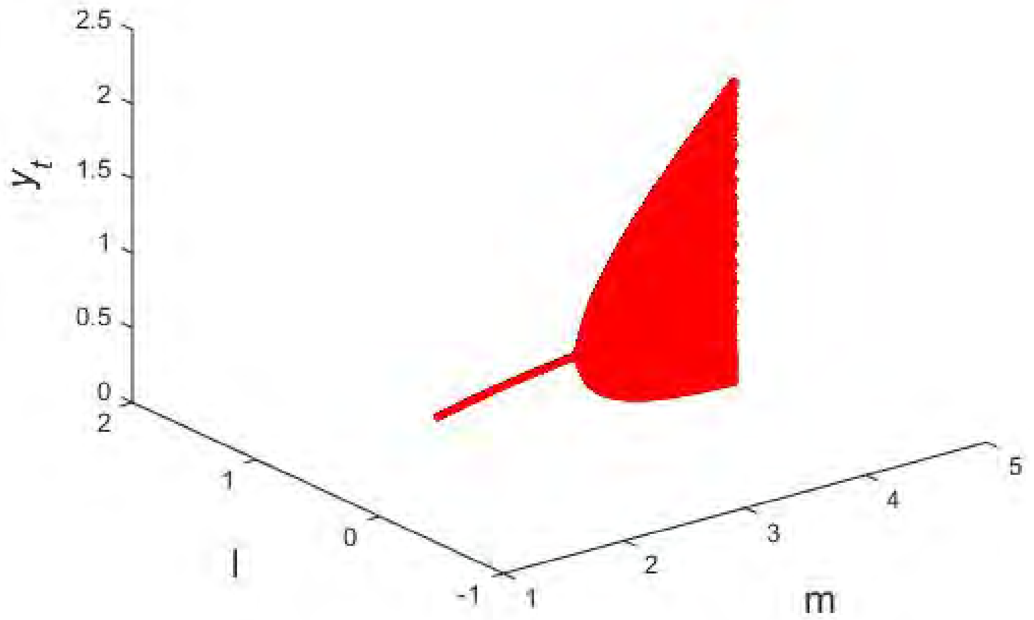
(B)



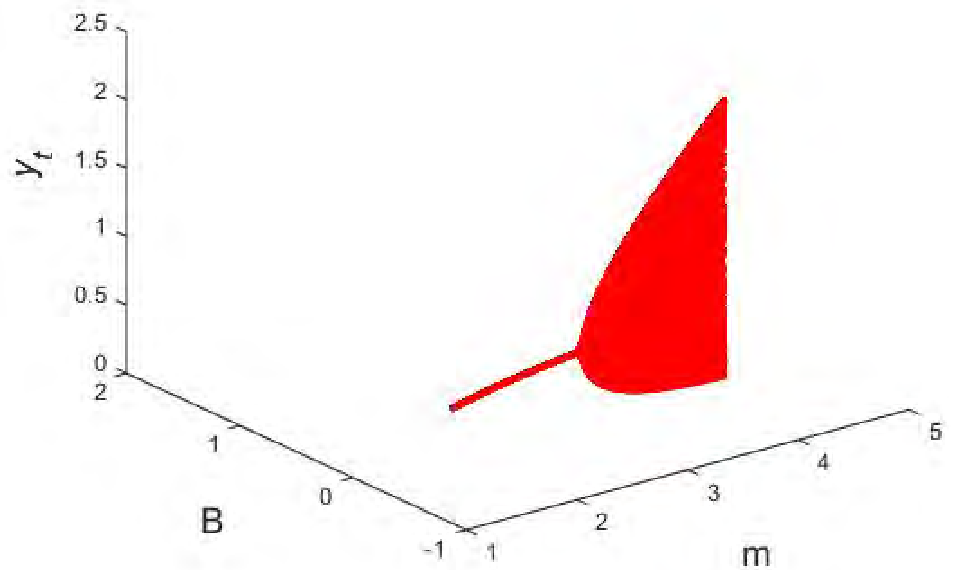
(C)



(D)



(E)



(F)

Fig. 4 Bifurcation diagrams in $3D$ of the model (2).

6 Conclusion

This work is about local dynamical properties and supercritical N-S bifurcations of a Beddington model with Allee effect in $\mathbb{R}_+^2$. We have studiedlocal dynamical properties along withtopological classification about $O(0,0)$, $A(1,0)$ and $B(\theta, r(1-\theta))$ of model (2), and conclusions are presented in Table 1. We have exploredthat about$A(1,0)$, parasitoidgoes extinction while host population undergoes a flip bifurcation to chaos when parameters are in theset: $F_{A(1,0)} = \{(r,m): r = 2, r, m > 0\}$. Further, we also explored that about $B(\theta, r(1-\theta)$, model (2) undergoes N-S bifurcation, if $(r,m,\theta) \in N_{B(\theta,r(1-\theta))}$, i.e., $N_{B(\theta,r(1-\theta))} = \left\{(r,m,\theta): \Delta < 0 \text{ and } m = \frac{r(1-\theta)(1-r\theta)}{\theta\left(r(1-2\theta)+r\theta(1-r+r\theta)e^{-r(1-\theta)}\right)}, r, m > 0, 0 < \theta < 1\right\}$. Finally theoretical results are verified numerically.

Table 1 Equilibria with corresponding behavior of model (2).

E.P	Corresponding behavior
$O(0,0)$	Saddle but never sink; source and non-hyperbolic.
$A(1,0)$	Sink if $0 < r < 2$; never source; saddle if $r > 2$; non-hyperbolic if $r = 2$.
$B(\theta, r(1-\theta))$	sink if $m < min\left\{\frac{r^2(1-\theta)}{2-r+r\theta+(-2+r-r\theta+r^2\theta-r^2\theta^2)e^{-r(1-\theta)}}, \frac{r(1-\theta)(2-r\theta)}{\theta\left(2+r-3r\theta+(-2+r+r\theta-r^2\theta+r^2\theta^2)e^{-r(1-\theta)}\right)}\right\}$, and $m > \frac{r(1-\theta)(1-r\theta)}{\theta\left(2+r-2r\theta+(-2+r\theta-r^2\theta+r^2\theta^2)e^{-r(1-\theta)}\right)}$; Repeller if $\frac{r(1-\theta)(2-r\theta)}{\theta(2+r-3r\theta+(-2+r+r\theta-r^2\theta+r^2\theta^2)e^{-r(1-\theta)})} < m < \frac{r(1-\theta)(1-r\theta)}{\theta(2+r-2r\theta+(-2+r\theta-r^2\theta+r^2\theta^2)e^{-r(1-\theta)})}$; Saddle if $\left(1-r\theta+\frac{-r+r\theta+m\theta-m\theta(1-r+r\theta)e^{-r(1-\theta)}}{m\theta(1-e^{-r(1-\theta)})}\right)^2 + 4\left(\frac{(1-r\theta)\left(-r+r\theta+m\theta-m\theta(1-r+r\theta)e^{-r(1-\theta)}\right)}{m\theta\left(1-e^{-r(1-\theta)}\right)} + r(1-\theta)\right) > 0$, and $0 < m < \frac{r(1-\theta)(2-r\theta)}{\theta\left(2+r-3r\theta+(-2+r+r\theta-r^2\theta+r^2\theta^2)e^{-r(1-\theta)}\right)}$; Non-hyperbolic if $m = \frac{r(1-\theta)(2-r\theta)}{\theta\left(2+r-3r\theta+(-2+r+r\theta-r^2\theta+r^2\theta^2)e^{-r(1-\theta)}\right)}$, or $m = \frac{r(1-\theta)(1-r\theta)}{\theta\left(2+r-2r\theta+(-2+r\theta-r^2\theta+r^2\theta^2)e^{-r(1-\theta)}\right)}$, and $m \geq \frac{r(1-\theta)}{\theta\left(2-r\theta+(-2-r+2r\theta)e^{-r(1-\theta)}\right)}$;

locally asymptotically node if

$$0 < m < \frac{r(1-\theta)(2-r\theta)}{\theta\left(4+r-3r\theta+(-4+r+r\theta-r^2\theta+r^2\theta^2)e^{-r(1-\theta)}\right)};$$

unstable node if

$$m > \frac{r(1-\theta)(2-r\theta)}{\theta\left(4+r-3r\theta+(-4+r+r\theta-r^2\theta+r^2\theta^2)e^{-r(1-\theta)}\right)};$$

non-hyperbolic if

$$m = \frac{r(1-\theta)(2-r\theta)}{\theta\left(4+r-3r\theta+(-4+r+r\theta-r^2\theta+r^2\theta^2)e^{-r(1-\theta)}\right)};$$

locally asymptotically focus if

$$0 < m < \frac{r(1-\theta)(1-r\theta)}{\theta\left(r(1-2\theta)+r\theta(1-r+r\theta)e^{-r(1-\theta)}\right)};$$

unstable focus if

$$m > \frac{r(1-\theta)(1-r\theta)}{\theta\left(r(1-2\theta)+r\theta(1-r+r\theta)e^{-r(1-\theta)}\right)};$$

Non-hyperbolic if

$$m = \frac{r(1-\theta)(1-r\theta)}{\theta\left(r(1-2\theta)+r\theta(1-r+r\theta)e^{-r(1-\theta)}\right)}.$$

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References

Beddington JR, Free CA, Lawton JH. 1975. Dynamic complexity in predator-prey models framed in difference equations. Nature, 225: 58-60

Cao H, Yue Z, Zhou Y. 2013. The stability and bifurcation analysis of a discrete Holling-Tanner model. Advances in Difference Equations, 2013(1): 330

Chen Q, Teng Z, Hu Z. 2013. Bifurcation and control for a discrete-time prey–predator model with Holling-IV functional response. International Journal of Applied Mathematics and Computer Science, 23(2): 247-261

Guckenheimer J, Holmes P. 1983. Nonlinear Oscillations, Dynamical Systems, and Bifurcation of Vector Fields. Springer, New York, USA

Hu Z, Teng Z, Zhang L. 2011.Stability and bifurcation analysis of a discrete predator–prey model with nonmonotonic functional response. Nonlinear Analysis: Real World Applications, 12(4): 2356-2377

Khan AQ, Ma J, Xiao D. 2017. Global dynamics and bifurcation analysis of a host–parasitoid model with strong Allee effect. Journal of biological dynamics, 11(1): 121-146

Kulenovic MRS, Ladas G. 2003. Dynamics of Second-Order Rational Difference Equations with Open Problems and Conjectures. Chapman & Hall/CRC Press, Boca Raton, USA

Kuznetsov YA. 2004. Elements of Applied Bifurcation Theory (Vol. 112 of Applied Mathematical Sciences). Springer, New York, USA

Ufuktepe U, Kapçak S, Akman, O. 2015. Stability analysis of the Beddington model with Allee effect. Applied Mathematics and Information Sciences, 9(2): 603-608