Article

Stability analysis of a system of second order rational difference equations

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Abstract
In this paper we consider a system of second order rational difference equations. We mainly discuss the boundedness and persistence, existence of fixed point, and uniqueness of positive fixed point, local and global behavior of positive fixed point and rate of convergence of every positive solution of the system under discussion. It will be shown that the system under discussion exhibits some special dynamics such as same mathematical condition for existence of fixed point and its global stability. Finally, some numerical examples are provided for verification of theoretical results.

Keywords system of rational difference equations of order two; boundedness and persistence; existence of fixed point; linearized stability; global stability analysis; rate of convergence.

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1 Introduction

Discrete dynamical system has a great worth in the field of applied mathematics. Each dynamical system $x_{n+1} = f(x_n, x_{n-1}, \cdots, x_{n-k})$ is equivalent to a difference equation and conversly. It is fairly gratifying and challenging to explore the dynamical proerties of nonlinear difference equations, more generally when we are dealing with difference equations of the rational types (Khan et al., 2014). Recently, many researchers have explored the dynamics of the constant solutions of nonlinear difference equations. In engineering, difference equations arise in switch engineering, ordinal signal processing and electrical systems. Periodic solutions of difference equations have been examined by many scholars, and many techniques have been introduced and applied for the existence and qualitative analysis of the constant solution (Liu, 2010; Din, 2013; Khan et al., 2019). There has remained an unlimited interest in the learning of the global behavioral analysis of difference equations and the boundedness and the periodic behavior of nonlinear difference equations (Din and Khan, 2014). Generally, linear difference equations are easy to solve. But here the main topic of attention is the solution of nonlinear difference equations. Hence we considered a non linear system of rational difference equations and explore its qualitative properties (Din and Khan, 2014). In nonlinear analysis we came across many types of recognized population models and many other well known forms of difference equations. Some well-known discrete systems are discrete-time population models, discrete-time chemical models, rational difference equations and exponential difference equations (Ahmed, 1993; Liu, 2010; Din, 2013; Din and Donchev, 2013; Din and Elsayed, 2014).

Aloqeili (2006) explored the stability and the semi-cycle nature of the solution of the following difference equation of rational form:

$$x_{n+1}=\frac{x_{n-1}}{a-x_{n-1}x_n},$$

where $n=0,1,2,\cdots$. In addition $a>0$.

Cinar (2004) provided the positive solution of the discrete-time mathematical equation:

$$x_{n+1}=\frac{x_{n-1}}{1+x_{n-1}x_n}.$$

Papaschinopoulos and Schinas (2012) studied the system of two nonlinear difference equations which is given as follows:

$$x_{n+1}=A+\frac{y_n}{x_{n-p}},$$
$$y_{n+1}=A+\frac{x_n}{y_{n-q}}.$$

where $n=0,1,2,\cdots$., and $p,\ q$ are positive real numbers.

Gibbons et al. (2000) investigated the qualitative behavior of the following second-order rational difference equation:

$$x_{n+1}=\frac{\alpha+\beta x_{n-1}}{\gamma+x_n}.$$

Papaschinopoulos et al. (2012) examined the asymptotic nature of the positive solutions of the following three systems of difference equations of exponential kind:

$$x_{n+1}=\frac{\alpha+\beta e^{-y_n}}{\gamma+y_{n-1}},y_{n+1}=\frac{\delta+\varepsilon e^{-x_n}}{\zeta+x_{n-1}},$$
$$x_{n+1}=\frac{\alpha+\beta e^{-y_n}}{\gamma+x_{n-1}},y_{n+1}=\frac{\delta+\varepsilon e^{-x_n}}{\zeta+y_{n-1}},$$

$$x_{n+1}=\frac{\alpha+\beta e^{-x_n}}{\gamma+y_{n-1}},y_{n+1}=\frac{\delta+\varepsilon e^{-y_n}}{\zeta+x_{n-1}}.$$

Din et al. (2014) studied the global dynamics of the second order competitive system of difference equations of rational type:

$$x_{n+1}=\frac{\alpha_1+\beta_1 x_{n-1}}{a_1+b_1 y_n},$$
$$y_{n+1}=\frac{\alpha_2+\beta_2 y_{n-1}}{a_2+b_2 x_n}, \qquad (1)$$

where $\alpha_i,\ \beta_i,\ a_i,\ b_i$ for $i\in\{1,2\}$ are positive parameters and conditions at initial points $x_0,\ x_{-1},\ y_0,\ y_{-1}$ are positive and real.

For applications and elementary analysis of rational difference equations, we can see Kulenovic and Ladas (2002), Grove and Ladas (2004), and Sedaghat (2003). Some applications of difference equations in mathematical ecology can be seen also in past studies (Ahmed, 1993; Liu, 2010; Din, 2013; Din and Donchev, 2013; Din and Elsayed, 2014; Din et al., 2019).

Motivated by the study of Din et al. (2014), we have considered the following form of system (1) for further analysis of second order system of rational difference equations:

$$x_{n+1}=f(y_n,y_{n-1})=\frac{\alpha_1+\beta_1 y_{n-1}}{a_1+b_1 y_n},$$
$$y_{n+1}=g(x_n,x_{n-1})=\frac{\alpha_2+\beta_2 x_{n-1}}{a_2+b_2 x_n} \qquad (2)$$

where f, g are continuous functions and the initial conditions $x_i,\ y_i$ for $i\in\{-1,0\}$ are positive and real. The next part of manuscript is related to the qualitative analysis of system (2).

2 Persistence and Boundedness

Theorem 2.1 Assuming that $\beta_1 < a_1$ and $\beta_2 < a_2$ then every positive solution (x_n, y_n) of the system (2) is bounded. Furthermore, it persists.

Proof . Let us consider a positive solution (x_n, y_n) of the the system (2). Then, we have

$$x_{n+1}\le A_1+B_1 y_{n-1} \qquad (3)$$

$$y_{n+1}\le A_2+B_2 x_{n-1} \qquad (4)$$

$n=0,1,2,3,\ldots\ldots..$

where, A_i and B_i are constant values with $A_i=\frac{\alpha_i}{a_i}$ and $B_i=\frac{\beta_i}{b_i}$ for $i\in 1,2$.

Next we consider the following system of linear difference equations:

$$S_{n+1}=A_1+B_1 S_{n-1},$$
$$T_{n+1}=A_2+B_2 T_{n-1} \qquad (5)$$
$$n=0,1,2,\ldots$$

Then, by some critical computation one can get the following form of solutions of mathematical system (5):

$$S_n=\frac{A_1}{1-B_1}+r_1 B_1^{\frac{n}{2}}+r_2(-\sqrt{B_1})^n,$$
$$T_n=\frac{A_2}{1-B_2}+r_3 B_2^{\frac{n}{2}}+r_4(-\sqrt{B_2})^n. \qquad (6)$$
$$n=1,2,..$$

where r_1,r_2, r_3 *and* r_4 are constant and depends on initial values S_i and T_i for $i \in \{-1,0\}$. Assume that $\beta_1 < a_1$ and $\beta_2 < a_2$. Then, both sequences $\{S_n\}$ and $\{T_n\}$ of solutions of system (6) are bounded. Aditionally, we assume that S_{-1}=x_{-1}, S_0=x_0, and T_{-1}=y_{-1}, T_0=y_0 then eventually we have

$$x_n \leq \frac{\alpha_1}{a_1-\beta_1} = R_1,$$
$$y_n \leq \frac{\alpha_2}{a_2-\beta_2} = R_2. \quad (7)$$
$$n = 0,1,2,..$$

Moreover, from system (2) and (7) one can see that

$$x_{n+1} \geq \frac{\alpha_1}{a_1+b_1y_n} = \frac{\alpha_1(\alpha_2-\beta_2)}{a_1(\alpha_2-\beta_2)+b_1\alpha_2} = V_1,$$
$$y_{n+1} \geq \frac{\alpha_2}{a_2+b_2x_n} = \frac{\alpha_2(\alpha_1-\beta_1)}{a_2(\alpha_1-\beta_1)+b_2\alpha_1} = V_2. \quad (8)$$

Clearly, by joining system (7) and (8) by mathematical inequalities one can get the following true mathematical condition:

$$V_1 \leq x_n \leq R_1,$$
$$V_2 \leq y_n \leq R_2. \quad (9)$$
$$n = 1,2.3,\ldots.$$

Hence, the required goal is acchieved .

Lemma 2.1 Let $\{(x_n, y_n)\}$ be a positive solution of (2) .Then, $[V_1, R_1] \times [V_2, R_2]$ be an invariant set concerning to the mathematical system (2).

Proof. The proof of this theorem is exactly followed by the method of mathematical induction.

Next we discuss the stability analysis of mathematical system (2).

3 Stability Analysis

The following theorem gives us the necessary and sufficient conditions for existence of the unique positive fixed point of system (2).

Theorem 3.1 Assme that

$$\frac{\alpha_1}{a_1} < \frac{\beta_1}{b_1}, \frac{\alpha_2}{a_2} > \frac{\beta_2}{b_2} \text{ or } \frac{\alpha_1}{a_1} > \frac{\beta_1}{b_1}, \frac{\alpha_2}{a_2} < \frac{\beta_2}{b_2} \quad (10)$$

Then there exist the unique positive fixed point $(\overline{x}, \overline{y})$ of mathematical system (2). Additionally, each solution

(x_n, y_n) of system (2) converges to $(\overline{x}, \overline{y})$as asymptotically.

Proof . Consider the following system

$$\hat{f}(u,v) = \frac{\alpha_1+\beta_1 v}{a_1+b_1 u},$$
$$\hat{g}(z,w) = \frac{\alpha_2+\beta_2 w}{a_2+b_2 z}, \tag{11}$$

where $z, w \in [V_1, R_1] = U_1$ and $u, v \in [V_2, R_2] = U_2$. Moreover, it is clearly seen that $\hat{f}(u,v) \in U_1,$ and $\hat{g}(z,w) \in U_2$ as $\hat{f}: U_2 \times U_2 \to U_1$ and $\hat{g}: U_1 \times U_1 \to U_2$.

Assume that (x_n, y_n) is positive solution of original system (2) then formerly we have $x_n \in U_1$ and $y_n \in U_2$.

Now assume that m_2, m_1, h_2, h_1 are positive numbers satisfying (11) such that

$$m_2 = \hat{f}(m_1, m_1) = \frac{\alpha_1+\beta_1 m_1}{a_1+b_1 m_1},$$
$$m_1 = \hat{f}(m_2, m_2) = \frac{\alpha_1+\beta_1 m_2}{a_1+b_1 m_2} \tag{12}$$

$$h_2 = \hat{g}(h_1, h_1) = \frac{\alpha_2+\beta_2 h_1}{a_2+b_2 h_1},$$
$$h_1 = \hat{g}(h_2, h_2) = \frac{\alpha_2+\beta_2 h_2}{a_2+b_2 h_2}. \tag{13}$$

Let

$$\tilde{F}(x) = \frac{\alpha_1+\beta_1 \underbrace{f}(x)}{a_1+b_1 \underbrace{f}((x)} - x = \frac{\beta_1}{b_1} + (\frac{\alpha_1 b_1 - a_1\beta_1}{b_1})(\frac{1}{a_1+b_1 f(x)}) - x \tag{14}$$

where

$$\underbrace{f}((x) = \frac{\alpha_2+\beta_2 x}{a_2+b_2 x} = \frac{\beta_2}{b_2} + \left(\frac{\alpha_2 b_2 - a_2\beta_2}{b_2}\right)\left(\frac{1}{a_2+b_2 x}\right) \tag{15}$$

where $\tilde{F}: U_1 \to U_1$ is into.

Now one can see that:

$$\hat{F}'(x) = -\underbrace{f}'(x)\frac{(\alpha_1 b_1 - a_1\beta_1)}{(a_1+b_1 f(x))^2} - 1, \tag{16}$$

where

$$\underbrace{f}'(x) = -\frac{(\alpha_2 b_2 - a_2\beta_2)}{(a_2+b_2 x)^2}. \tag{17}$$

By using equation (17) in (16) one can acquire

$$\hat{F}'(x) = \frac{(\alpha_1 b_1 - a_1\beta_1)}{(a_1+b_1 f(\bar{x}))^2} \times \frac{(\alpha_2 b_2 - a_2\beta_2)}{(a_2+b_2\bar{x})^2} - 1. \tag{18}$$

Hence, under the condition defined in equation (11) one can have :

$\hat{F}'(x) < 0$

(19)

(19) indicates that $\hat{F}(x) = 0$ takes exceptional unique positive solution in U_1. Additionally, from equation (12) we realize that m_2 and m_1 are satisfying the equation $F(x) = 0$ which demonstrate that $m_1 = m_2$. Therefore from equation (13) it is clear that $h_2 = h_1$. Moreover, by using a result from Din and Khan (2017) it follow that system of equation (2) has a unique positive equilibrium $(\overline{x}, \overline{y})$ and every positive solution of system equation (2) tends to the unique positive equilibrium $(\overline{x}, \overline{y})$ as n→ ∞. This completes the proof of the theorem.

The very next theorem is related to the linearized stability of system (2) about $(\overline{x}, \overline{y})$.

Theorem 3.2 The one and only positive equilibrium point $(\overline{x}, \overline{y})$ of the mathematical system (2) is locally asymptotically stable if the following conditions are true:

$$\beta_2 < a_1,\ \beta_1 < a_2,\ b_1 = b_2 \tag{20}$$

Proof. Assume that $(\overline{x}, \overline{y})$ is the one and only fixed point of the system (2), then individual can see that

$$\overline{x} = \frac{\alpha_1 + \beta_1 \overline{y}}{a_1 + b_1 \overline{y}},$$
$$\overline{y} = \frac{\alpha_2 + \beta_2 \overline{x}}{a_2 + b_2 \overline{x}}$$

(21)

Formally, by linearization of the system (2) about $(\overline{x}, \overline{y})$ and by using system (21), we get:

$$x_{n+1} = -\frac{b_1 \overline{x}}{a_1 + b_1 \overline{y}} y_n + \frac{\beta_1}{a_1 + b_1 \overline{y}} y_{n-1}, \tag{22}$$

$$y_{n+1} = -\frac{b_2 \overline{y}}{a_2 + b_2 \overline{x}} x_n + \frac{\beta_2}{a_2 + b_2 \overline{x}} x_{n-1}, \tag{23}$$

Futrhermore, equations (22) and (23) are jointly equivalent to the following mathematical matrix form:

$X_{n+1} = AX_n$.

where

$$A=\begin{pmatrix}0 & D_1 & 0 & D_2\\ D_3 & 0 & D_4 & 0\\ 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\end{pmatrix}, X_n=.\begin{pmatrix}x_n\\ y_n\\ x_{n-1}\\ y_{n-1}\end{pmatrix}$$

$$D_1=-\frac{b_1\overline{x}}{a_1+b_1\overline{y}}, D_2=\frac{\beta_1}{a_1+b_1\overline{y}}, D_3=-\frac{b_2\overline{y}}{a_2+b_2\overline{x}}, D_4=\frac{\beta_2}{a_2+b_2\overline{x}}.$$

The characteristic equation of the matrix A can be described mathematically as

$$\lambda^4-(D_1D_3)\lambda^2-(D_1D_4-D_2D_3)\lambda-D_2D_4=0 \qquad (24)$$

Furthermore, under condition (20) one has:

$$|D_1D_3|+|D_1D_4|+|D_2D_3|+|D_2D_4|=\left(\frac{-b_1\overline{x}}{a_1+b_1\overline{y}}\right)\times\left(\frac{-b_2\overline{y}}{a_2+b_2\overline{x}}\right)+\left(\frac{-b_1\overline{x}}{a_1+b_1\overline{y}}\right)\times\left(\frac{\beta_2}{a_2+b_2\overline{x}}\right)+\left(\frac{\beta_1}{a_1+b_1\overline{y}}\right)\times\left(\frac{-b_2\overline{y}}{a_2+b_2\overline{x}}\right)+$$
$$\left(\frac{\beta_1}{a_1+b_1\overline{y}}\right)\times\left(\frac{\beta_2}{a_2+b_2\overline{x}}\right)=\frac{b_1b_2\overline{x}\overline{y}+b_1\beta_2\overline{x}+b_2\beta_1\overline{y}+\beta_1\beta_2}{(a_1+b_1\overline{y})(a_2+b_2\overline{x})}\leq\frac{(\beta_2+b_2\overline{y})(\beta_1+b_1\overline{x})}{(a_1+b_1\overline{y})(a_2+b_2\overline{x})}=\left[\frac{(\beta_2+b_2\overline{y})}{(a_1+b_1\overline{y})}\right]\times\left[\frac{(\beta_1+b_1\overline{x})}{(a_2+b_2\overline{x})}\right]<1$$

(25)

Therefore, the sufficient condition for linearized stability of system (2) is satisfied (Din and Khan, 2017) which finalizes the proof of theorem.

4 Convergence Rate of Solution

Let $\{(\boldsymbol{x_n},\boldsymbol{y_n})\}$ be any solution of system (2) and satisfying the following equations:

$$\lim_{n\to\infty}x_{n+1}=\overline{x},$$

and

$$\lim_{n\to\infty}y_{n+1}=\overline{y}.$$

where $\overline{x}\in U_1$ and $\overline{y}\in U_2$. To determine the error terms one has the following from system (2)

$$x_{n+1}-\overline{x}=\frac{\alpha_1+\beta_1y_{n-1}}{a_1+b_1y_n}-\overline{x}=\frac{\beta_1(y_{n-1}-\overline{y})}{a_1+b_1y_n}-\frac{b_1\overline{x}(y_n-\overline{y})}{a_1+b_1y_n},$$

$$y_{n+1}-\overline{y}=\frac{\alpha_2+\beta_2x_{n-1}}{a_2+b_2x_n}-\overline{y}=\frac{\beta_2(x_{n-1}-\overline{x})}{a_2+b_2x_n}-\frac{b_2\overline{y}(x_n-\overline{x})}{a_2+b_2x_n}.$$

Let $E_n^1=x_n-\overline{x}$ and $E_n^2=y_n-\overline{y}$ then one has

$$E_{n+1}^1=a_nE_n^2+b_nE_{n-1}^2,$$

$$E_{n+1}^2=c_nE_n^1+d_nE_{n-1}^1.$$

where

$$b_n=\frac{\beta_1}{a_1+b_1y_n},\ a_n=-\frac{b_1x_n}{a_1+b_1y_n},$$

$$c_n = -\frac{b_2 y_n}{a_2 + b_2 x_n}, \qquad d_n = \frac{\beta_2}{a_2 + b_2 x_n}$$

Moreover ,

$$\lim_{n\to\infty} a_n = -\frac{b_1 \bar{x}}{a_1 + b_1 \bar{y}} = D_1, \quad \lim_{n\to\infty} b_n = \frac{\beta_1}{a_1 + b_1 \bar{y}} = D_2,$$
$$\lim_{n\to\infty} c_n = -\frac{b_2 \bar{y}}{a_2 + b_2 \bar{x}} = D_3, \qquad \lim_{n\to\infty} d_n = \frac{\beta_2}{a_2 + b_2 \bar{x}} = D_4.$$

Now, the limiting system of error terms can be described as follows:

$$\begin{pmatrix} e_{n+1}^1 \\ e_{n+1}^2 \\ e_n^1 \\ e_n^2 \end{pmatrix} = \begin{pmatrix} 0 & D_1 & 0 & D_2 \\ D_3 & 0 & D_4 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} e_n^1 \\ e_n^2 \\ e_{n-1}^1 \\ e_{n-1}^2 \end{pmatrix}.$$

which is similar to linearized system of (2) about one and only fixed point $(\bar{x}, \bar{y})$. Finally, we get the following result by using a Proposition from Pituk (2002).

Theorem 4.1 (Pituk, 2002) Assume that (x_n, y_n) be any solution of system (2) such that $\lim_{n\to\infty} x_{n+1} = \bar{x}$ and $\lim_{n\to\infty} y_{n+1} = \bar{y}$, where $\bar{x} \in U_1$ and $\bar{y} \in U_2$. Then, the error vector $E_n = \begin{pmatrix} E_n^1 \\ E_n^2 \\ E_{n-1}^1 \\ E_{n-1}^2 \end{pmatrix}$ of each solution of (2) satisfies both of the following asymptotic relations:

$$\operatorname{Lim}_{n\to\infty} (||E_n||)^{\frac{1}{n}} = |\lambda_{1,2,3,4} F_J(\bar{x}, \bar{y})|, \quad \lim_{n\to\infty} \frac{||E_{n+1}||}{||E_n||} = |\lambda_{1,2,3,4} F_J(\bar{x}, \bar{y})|, \tag{26}$$

where $\lambda_{1,2,3,4} F_J(\bar{x}, \bar{y})$ are the characteristic values of Jacobian matrix $F_J(\bar{x}, \bar{y})$ of system (2) about one and only fixed point $(\bar{x}, \bar{y})$.

5 Numerical Simulation and Discussion

Example 5.1 Let $\alpha_1 = 1.8,\ \beta_1 = 1.5,\ a_1 = 0.06,\ b_1 = 0.002,\ \alpha_2 = 1.1,\ \beta_2 = 0.01,\ a_2 = 0.22,\ b_2 = 0.003$. Then, the system (2) has the following mathematical form :

$$x_{n+1} = \frac{1.8 + 1.5 y_{n-1}}{0.06 + 0.002 y_n},$$
$$y_{n+1} = \frac{1.1 + 0.01 x_{n-1}}{0.22 + 0.003 x_n}. \tag{27}$$

where the initial conditions are $x_0 = 114.4,\ x_{-1} = 114.3,\ y_{-1} = 3.94,\ y_0 = 3.95$. By using these values in system (2) we get the following unique positive fixed point $(\bar{x}, \bar{y}) = (114.4, 3.98)$ for the system (2). Moreover, Fig. 1 and Fig. 2 respectively represent the plots of x_n and y_n. Additionally, attracting nature of the system (2) can be seen from Fig. 3**.**

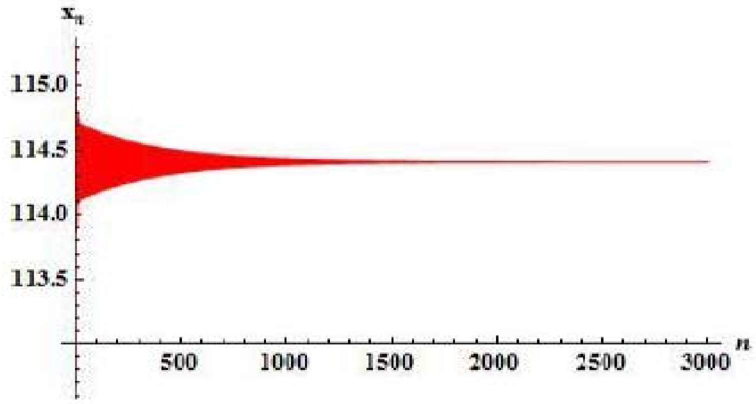

Fig. 1 Plot of x_n for system (2).

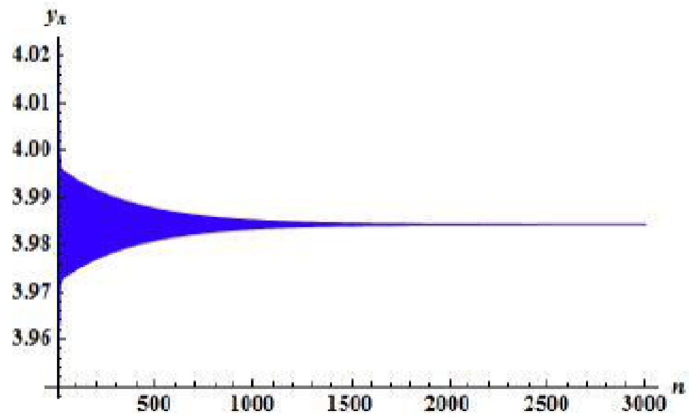

Fig. 2 Plot of y_n for system (2).

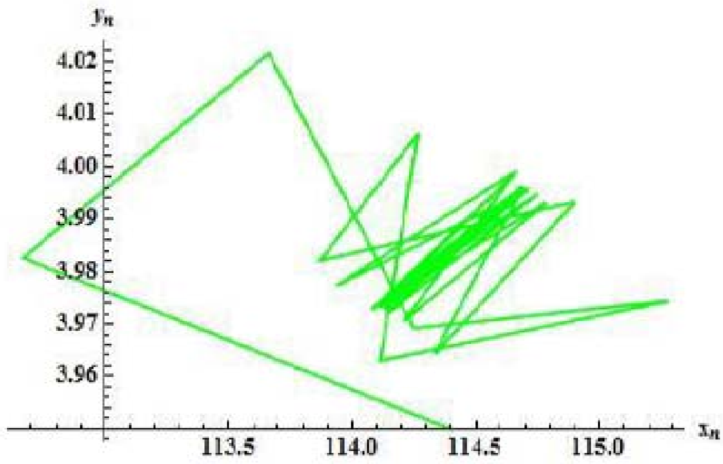

Fig. 3 An attractor for system (2).

Example 5.2 Let $\alpha_1 = 3.91,\ \beta_1 = 2.6,\ a_1 = 1.3,\ b_1 = 1.29,\ \alpha_2 = 0.81,\ \beta_2 = 2.1,\ a_2 = 2.7,\ b_2 = 3.7$.

Then, the system (2) has the following mathematical form:

$$x_{n+1} = \frac{3.91+2.6y_{n-1}}{1.3+1.29y_n},$$
$$y_{n+1} = \frac{0.81+2.1x_{n-1}}{2.7+3.7x_n}, \qquad (28)$$

where, the initial conditions are $x_0 = 2.669,\ x_{-1} = 2.668,\ y_{-1} = 0.5101,\ y_0 = 0.5102$. By using these values in system (2) we get the following unique positive fixed point $(\bar{x},\bar{y}) = (2.668,0.5101)$ for the system (2). Moreover, Fig. 4 and Fig. 5 respectively represent the plots of x_n and y_n. Additionally, attracting nature of the system (2) can be seen from Fig. 6.

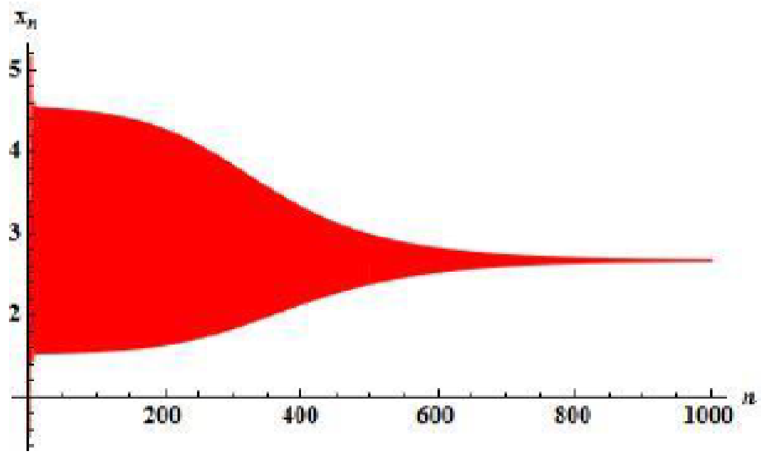

Fig. 4 Plot of x_n for system (2).

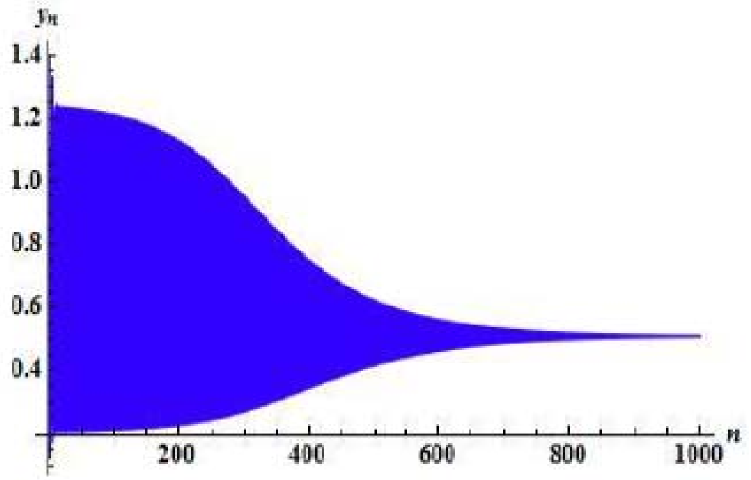

Fig. 5 Plot of x_n for system (2).

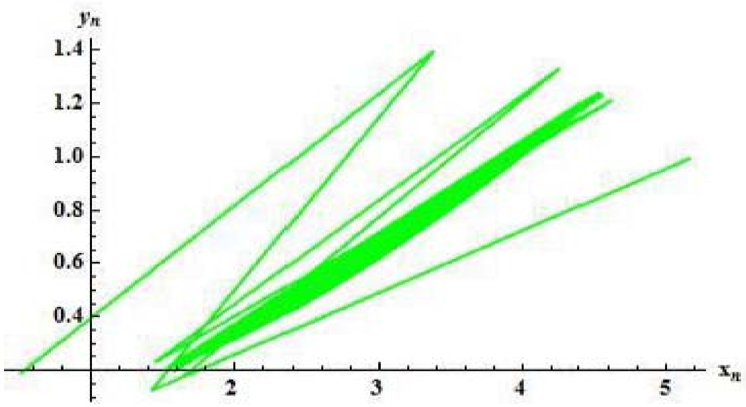

Fig. 6 An attractor for system (2).

Example 5.3 Let $\alpha_1 = 0.5,\ \beta_1 = 12,\ a_1 = 13,\ b_1 = 0.2,\ \alpha_2 = 0.1,\ \beta_2 = 17,\ a_2 = 17.5,\ b_2 = 0.3$. Then, the system (2) has the following mathematical form:

$$x_{n+1} = \frac{0.5+12x_{n-1}}{13+0.2y_n},$$
$$y_{n+1} = \frac{0.1+17y_{n-1}}{17.5+0.3x_n}, \qquad (29)$$

where, the initial conditions are $x_0 = 0.46,\ x_{-1} = 0.5,\ y_{-1} = 0.11,\ y_0 = 0.14$. By using these values in system (2) we get the following unique positive fixed point $(\bar{x}, \bar{y}) = (0.484974, 0.154921)$ for the system (2). Moreover, Fig. 7 and Fig. 8 respectively represent the plots of x_n and y_n. Additionally, attracting nature of the system (2) can be seen from Fig. 9.

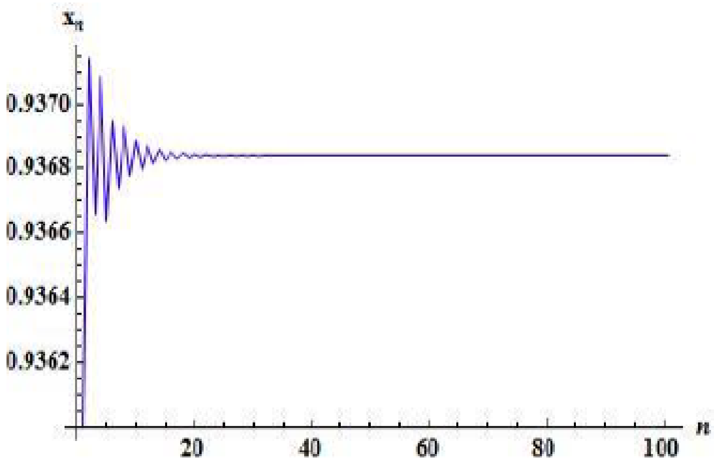

Fig. 7 Plot of x_n for system (2).

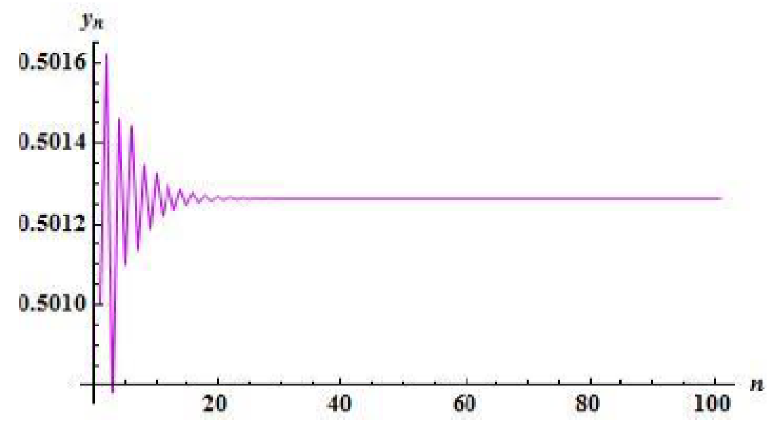

Fig. 8 Plot of y_n for system (2).

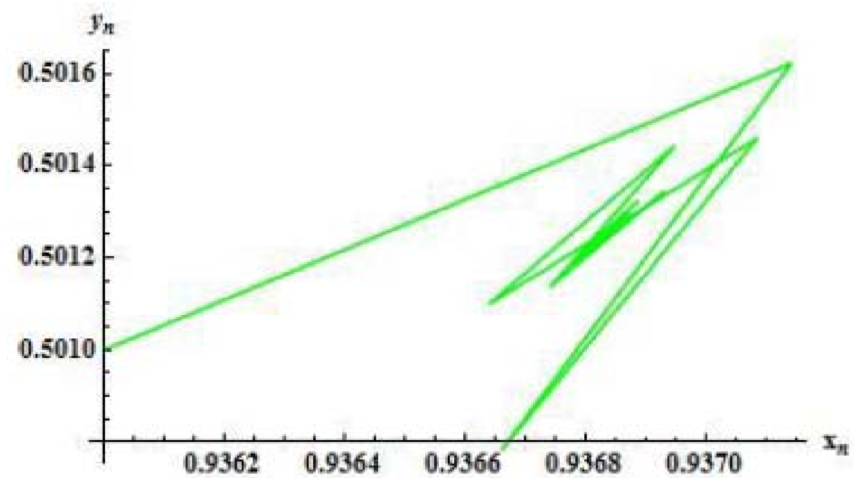

Fig. 9 An attractor for system (2).

6 Conclusion

In literature, many articles are related to qualitative study of competitive system of second order rational difference equations (Garic et al., 2009). It is very interesting to study the problems related to the dynamical study of competitive systems in higher dimension. This article is related to qualitative study of system of second-order rational difference equations. We have investigated the boundedness and persistence of every positive solution of system (2). Under certain parametric conditions the existence of one and only positive fixed point is proved. Moreover, we have shown that unique positive equilibrium point of system (2) is locally as well as globally asymptotically stable. Furthermore, we have explored the rate at which each solution of (2) convergence to the unique positive fixed point of (2). Finally, some numerical examples are provided to support our theoretical discussion.

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