Article

Dynamical analysis of discretized Logistic model with Caputo-Fabrizio fractional derivative

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Abstract
In this paper we consider a fractional order Logistic model with Caputo-Fabrizio fractional derivative. By applying two-step Adams-Bashforth scheme, we obtain a system of difference equations. By using the Schur-Cohn criterion, stability conditions of the positive equilibrium point of the discrete system are obtained. It is observed that the discrete system shows much richer dynamic behaviors than its fractional-order form such as Neimark-Sacker bifurcation and chaos. The direction and stability of the Neimark-Sacker bifurcation are determined by using the normal form and center manifold theory. In addition, the effect of fractional order parameter on the dynamical behavior of the system is investigated. Finally, numerical simulations are used to demonstrate the accuracy of analytical results.

Keywords Caputo-Fabrizio fractional derivative; two-step Adams-Basforth Method; Logistic differential equation; Neimark-Sacker bifurcation.

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1 Introduction

Fractional calculus is a generalization of integer order derivative and integral and is one of the most powerful mathematical tools used to model real-life problems in many areas of science, technology and engineering (Karaagac, 2019; Saad and Aguilar, 2018; Gomez-Aguilar et al., 2019). We can find many definitions of fractional order derivative in literature and the most important of these are Riemann-Liouville and Caputo fractional derivative. Although these definitions have successful applications in the literature, they have serious disadvantages that their kernel describing the memory effect had singularity. Because of this inconvenience Caputo and Fabrizio proposed a new definition of fractional derivative which is called Caputo Fabrizio fractional derivative (CF) (Caputo and Fabrizio, 2015). CF fractional derivative is a fractional derivative with a local and non-singular kernel (Karaagac, 2018), and is based on the exponential decay law (Saad and Aguilar, 2018). The CF fractional derivative has very attractive properties as it can describe heterogeneities and configurations of matter at different scales that cannot be command by known local theories and well known fractional derivatives (Atangana and Alqahtani, 2016). The calculation is much easier and resulting solutions can be expressed by the elemantary function for the fractional differential equation with CF fractional derivative sense (Abdulhameed et al., 2017). However CF fractional derivative takes on two different representations of temporal variable and spatial variable (Liu et al., 2017.). While in a structure that depends on the temporal variable the Laplace Transform is appropriate, in a structure depends on the spatial variable the Fourier Transform is more appropriate (Caputo and Fabrizio, 2015). Due to the above mentioned important advantages of this definition, CF fractional order derivative has many physical and biological applications in the literature (Abdulhameed et al., 2017; Atangana, 2016; Firoozjaee et al., 2018; Ullah et al., 2020; Khan et al., 2019; Ullah et al., 2018; Khan et al., 2018; Owolabi, 2019; Ghanbari and Gomez-Aguilar, 2018 and Noupoue et al., 2019). Abdulhameed et al. (2017) studied electro-magneto-hydrodynamic flow of the non-Newtonian behavior of biofluids, with heat transfer, through a cylindrical microchannel. Atangana (Atangana, 2016) applied the CF derivative to nonlinear Fisher reaction-diffusion model and given an algorithm in order to obtain numerical solutions of the model. A type of Fokker-Plank equation with CF derivative is considered in (Firoozjaee et al., 2018). In population dynamics, tuberculosis model (Ullah et al., 2020), hepatitis E model (Khan et al., 2019), hepatitis B virus model (Ullah et al., 2018), pine wilt disease model (Khan et al., 2018), love dynamics model (Owolabi, 2019), nutritient phytoplankton-zooplankton (Ghanbari and Gomez-Aguilar, 2018), logistic model (Noupoue et al., 2019) are studied via CF derivative. The Caputo fractional derivative of order α for a continuous function f is defined by

$$ {}_{a}^{C}D_{t}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha-n)} \int_{a}^{t} \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}} d\tau , \qquad (n-1<\alpha<n) \tag{1} $$

Let$\alpha \in (0,1)$ and $f \in H^1(a,b)$ where

$H^1(a,b) = \{f | f \in L^2(a,b) and\, f' \in L^2(a,b)\}$ and $L^2(a,b)$ is the space of square integrable functions on the interval(a,b). Then the Caputo-Fabrizio fractional derivative is defined as

$$ {}_{CF}D_{t}^{\alpha} f(t) = \frac{M(\alpha)}{1-\alpha} \int_{a}^{t} f'(\tau)\, exp\left[\frac{-\alpha(t-\tau)}{1-\alpha}\right] d\tau \tag{2} $$

where $M(\alpha)$ is a normalization function such that $M(0) = M(1) = 1.$

As with ordinary differential equations, many nonlinear fractional differential equations do not have an analytical solution and numerical approach is needed. There are several methods have been proposed for obtaining numerical solutions CF fractional differential equations such as Homotopy Analsysis method (Yepez-Martinez and Gomez-Aguilar, 2019), Finite difference approximation (Rangaig, 2018), Laplace transform method (Shaikh et al., 2019), MQ-RBF collocation method (Kazemi and Jafari, 2017), discretization scheme (Atangana and Gomez-Aguilar, 2017) and Adams- Bashforth scheme (Owolabi and Atangana, 2017). One of the suitable numerical methods for solving nonlinear equations is the Adams-Bashfort method (Koca, 2018). This method is developed with classical differentiation using the basic theorem of calculus and by taking between two times containing t_n and t_{n+1} (Atangana and Owolabi, 2017). Atangana extended this method to solve fractional differential equations with CF derivative and Atangana Baleanu fractional derivative.

Let a non-linear fractional differential equation with Caputo-Fabrizio sense defined by ${}_{CF}D_0^{\alpha}(t) = g\big(t, u(t)\big), \quad u(0) = u_0$ (3)

The numerical solution of (3) is based on the Adams-Bashforth method given as follows (Noupoue et al., 2019).

$$u_{n+1}=u_n+\left(\frac{1-\alpha}{M(\alpha)}+\frac{3\alpha h}{2M(\alpha)}\right)g(t_n,u_n)+\left(\frac{1-\alpha}{M(\alpha)}+\frac{\alpha h}{2M(\alpha)}\right)g(t_{n-1},u_{n-1}). \quad (4)$$

Consider fractional logistic differential equation (FLDE) with CF fractional derivative as

$${}_{CF}D_t^{\alpha}(t)=rN(t)\left(1-\frac{N(t)}{K}\right), \quad (5)$$

where $t>0,\ r>0,\alpha$ is the fractional order derivative with $0<\alpha\leq 1$. The existence and uniqueness of the solution of the model (5) are analyzed in (Noupoue et al., 2019). Furthermore, several numerical approaches such as generalized Eulers method, power series expansion method and Caputo-Fabrizio method are applied to model (5) and the results obtained from these methods are compared with the classical solution. Applying the Adams-Bashforth method to fractional logistic differential equation (5) in CF sense, the approximate solutions of FLDE obtained as (Noupoue et al., 2019).

$$\begin{aligned}N_{CF}(t_{n+1})=N(t_n)&+\left(\frac{1-\alpha}{M(\alpha)}+\frac{3\alpha h}{2M(\alpha)}\right)rN(t_n)\left(1-\frac{N(t_n)}{K}\right)\\&+\left(\frac{1-\alpha}{M(\alpha)}+\frac{\alpha h}{2M(\alpha)}\right)rN(t_{n-1})\left(1-\frac{N(t_{n-1})}{K}\right).\end{aligned} \quad (6)$$

In this article we deal with stability and Neimark-Sacker bifurcation analysis for the discretized model (6) obtained from the Caputo Fabrizio logistic differential equations (5).

2 Stability Analysis

Using the change of variables $N(t_n)=X_1(n)$ and $N(t_{n-1})=X_2(n)$ we obtain system of di□erence equations as follows

$$\begin{cases}X_1(n+1)=X_1(n)+\left(\frac{1-\alpha}{M(\alpha)}+\frac{3\alpha h}{2M(\alpha)}\right)rX_1(n)\left(1-\frac{X_1(n)}{K}\right)+\left(\frac{1-\alpha}{M(\alpha)}+\frac{\alpha h}{2M(\alpha)}\right)rX_2(n)\left(1-\frac{X_2(n)}{K}\right)\\ X_2(n+1)=X_1(n)\end{cases} \quad (7)$$

where $\alpha\in(0,1]$, $K>0$ and $M(\alpha)>0$. We note that system (6) and system (7) have the same equilibrium points that is (*0,0*) and (*K,K*). Now we deal with the stability analysis of these equilibrium points of system (7). Let's take $M(\alpha)=M$.

Theorem 2.1

(a) The equilibrium point (*0,0*) of the system (7) is unstable for $0<\alpha<1$.

(b) The equilibrium point (*0,0*) of the system (7) is local asympotically stable for $\alpha>1$, $h<\frac{\alpha-1}{\alpha}$ and

$$r<\frac{2M}{-2+2\alpha-h\alpha}. \quad (8)$$

Proof.

Let $X(n+1)=JX(n)$ is linearized system of (7) about (*0,0*). So the Jacobian matrix J can be calculated as

$$J(0,0)=\begin{pmatrix}\frac{2M+(2+(-2+3h)\alpha)r}{2M} & \frac{(2+(-2+h)\alpha)r}{2M}\\ 1 & 0\end{pmatrix},$$

which gives the characteristic equation

$p(\lambda)=\lambda^2+p_{10}\lambda+p_{00}=0$,

where

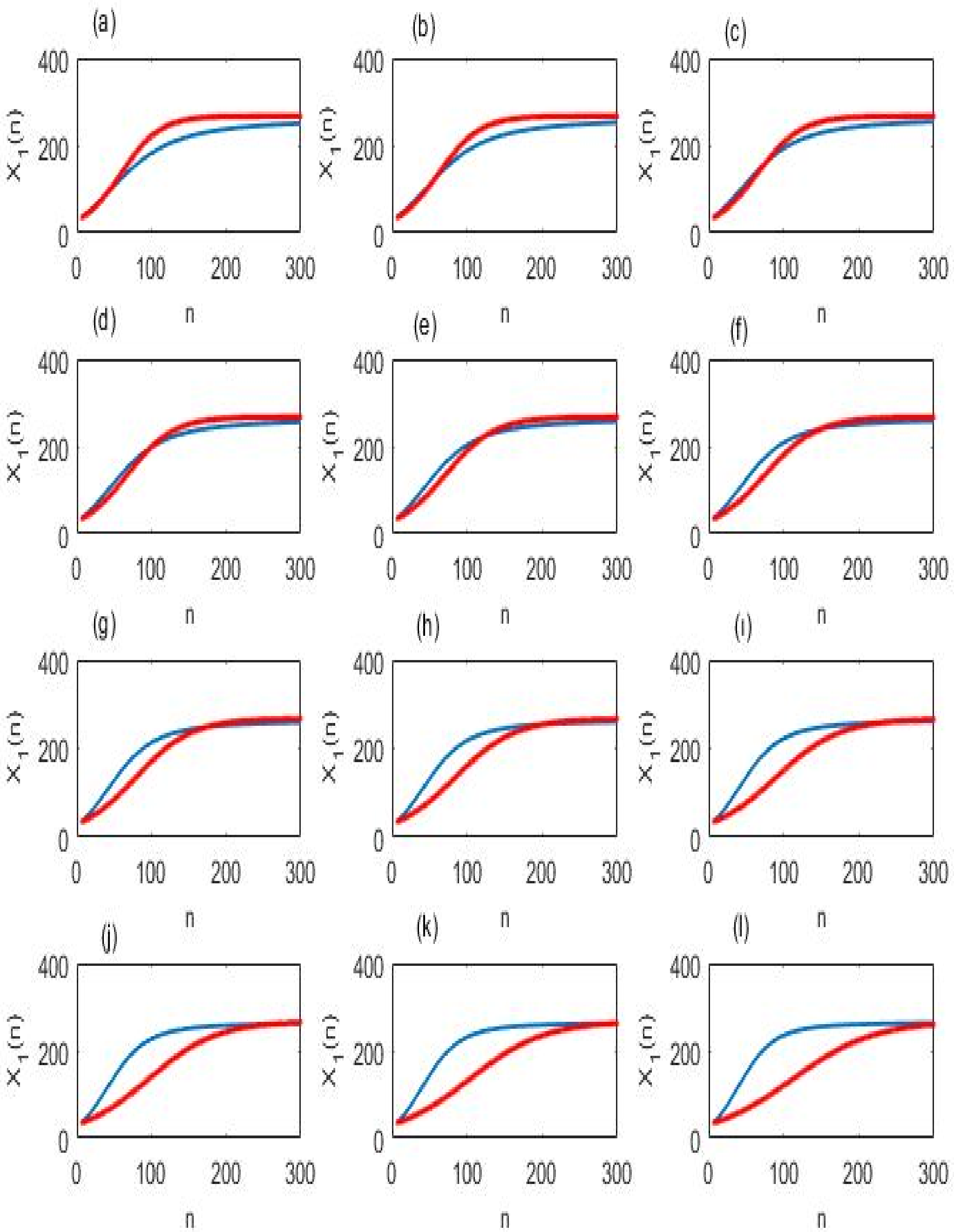

Fig. 1 Numerical solution of fractional logistic equation with Caputo and Caputo-Fabrizio fractional derivative represented by blue and red curve respectively $\alpha = 0.77$ (a), $\alpha = 0.78$ (b), $\alpha = 0.79$ (c), $\alpha = 0.80$ (d), $\alpha = 0.81$ (e), $\alpha = 0.82$ (f), $\alpha = 0.83$ (g), $\alpha = 0.84$ (h), $\alpha = 0.85$ (ı), $\alpha = 0.86$ (j), $\alpha = 0.87$ (k), $\alpha = 0.88$ (l). Parameters values and initial conditionals are $K = 267.5301$, $h = 0.001, r = 0.07060, X_1(7) = 35$.

$$p_{10} = -\frac{2M + (2 + (-2 + 3h)\alpha)r}{2M}$$

and

$p_{00} = -\frac{(2+(-2+h)\alpha)r}{2M}$

To check asymptotically stable of equilibrium points, we use the following Jury conditions.

$i)\ 1+p_{10}+p_{00}>0,\ \ ii)\ 1-p_{10}+p_{00}>0,\ \ iii)\ 1-p_{00}>0$

The first Jury condition leads to

$$1+p_{10}+p_{00} = \frac{(-1+(1-h)\alpha)2r}{M}$$

and does not satisfy $1+p_{10}+p_{00}>0$ for $0<\alpha<1$. Therefore equilibrium point (*0,0*) is unstable for $0<\alpha<1$.

Now we assume that $\alpha>1$. In this situation we have $1+p_{10}+p_{00}>0$ under the condition $h<\frac{\alpha-1}{\alpha}$.

From (ii) we always have

$$1-p_{10}+p_{00} = 2+\frac{h\alpha r}{M}>0.$$

From the last condition (iii) we get

$$1-p_{00} = 1+\frac{(2+(-2+h)\alpha)r}{2M}>0,$$

for $r<\frac{2M}{-2+2\alpha-h\alpha}$, $h<\frac{\alpha-1}{\alpha}$ and $\alpha>1$. This completes the proof.

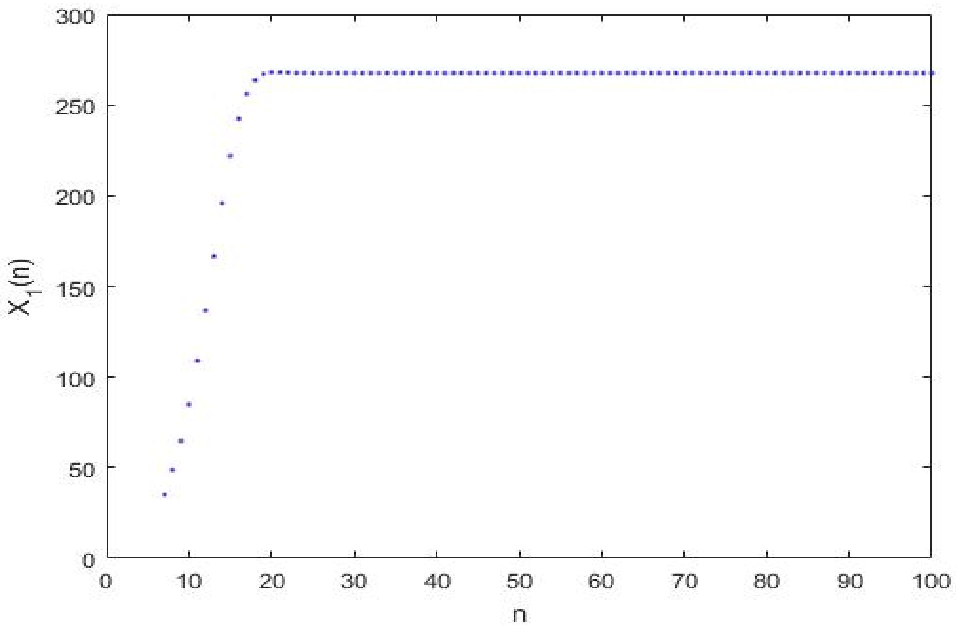

Fig. 2 Asymptotically stable equilibrium point (*K*,*K*) of the system (7) for $\alpha=0.8$, $K=267.5301$, $M=0.88715, h=0.001, r=0.07060, X_1(7)=35,\ X_2(7)=35$.

Theorem 2.2

The positive equilibrium point (*K*,*K*) of system (7) is local asymptotically stable if $0<\alpha\le 1$ and

$$r<\frac{2M}{2-2\alpha+h\alpha}. \tag{9}$$

Proof.

The Jacobian matrix J at the equilibrium point (K,K) is the form

$$J(K,K) = \begin{pmatrix} \frac{2M + (-2 + (2-3h)\alpha)r}{2M} & -\frac{(2 + (-2+h)\alpha)r}{2M} \\ 1 & 0 \end{pmatrix}$$

that has the characteristic equation

$$p(\lambda) = \lambda^2 + p_1\lambda + p_0 = 0 \qquad (10)$$

where

$$p_1 = \frac{-2M + (2 + (-2 + 3h)\alpha)r}{2M}$$

and

$$p_0 = \frac{(2 + (-2+h)\alpha)r}{2M}$$

From the Jury condition (i) we have

$$1 + p_1 + p_0 = \frac{2(1 + (-1+h)\alpha)r}{M} > 0,$$

for $0 < \alpha \leq 1$. Under the condition $r < \frac{2M}{h\alpha}$, we get

$$1 - p_1 + p_0 = 2 - \frac{h\alpha r}{M} > 0.$$

The last Jury condition (iii) leads to

$$1 - p_0 = 1 - \frac{(2 + (-2+h)\alpha)r}{2M} > 0,$$

for $r < \frac{2M}{2+(-2+h)\alpha}$ and $\alpha < 1$. We note that

$$r < \frac{2M}{2 + (-2+h)\alpha} < \frac{2M}{h\alpha},$$

for $\alpha < 1$. This completes the proof.

3 Neimark-Sacker Bifurcation Analysis

In this section we study Neimark-Sacker bifurcation around the positive equilibrium point (K,K) by using the bifurcation theory in (Kuznetsov, 1998; He and Li, 2014; Kartal, 2017, 2018). We choose the parameter r as a bifurcation parameter.

The eigenvalues of characteristic equation (10) are

$$\lambda_{1,2}(r) = \frac{-p_1(r) \pm \sqrt{p_1^2(r) - 4p_0(r)}}{2}$$
$$= \frac{2M + (-2 + (2-3h)\alpha)r \pm \sqrt{-8M(2 + (-2+h)\alpha)r + (2M + (-2 + (2-3h)\alpha)r)^2}}{4M}.$$

For the Neimark-Sacker bifurcation these eigenvalues must be complex and so we require $p_1^2 - 4p_0 < 0$ that leads to

$$-8M(2 + (-2+h)\alpha)r + (2M + (-2 + (2-3h)\alpha)r)^2 < 0.$$

Let

$$\bar{r} = \frac{2M}{2 - 2\alpha + h\alpha},$$

then we have $p_0(\bar{r}) = 1$. Now the eigenvalues are

$$\lambda, \bar{\lambda} = -\frac{p_1}{2} \pm \frac{i}{2}\sqrt{4p_0^2 - p_1^2}$$
$$= -h\alpha \pm 2i\sqrt{(1-\alpha)(1+(-1+h)\alpha)}$$

and $|\lambda(\bar{r})| = 1$. For $\alpha \neq \frac{2}{2-h}$, the transversality condition leads to

$$\left.\frac{d|\lambda(r)|}{dr}\right|_{r=\bar{r}} = \frac{2+(-2+h)\alpha}{4M} \neq 0 \qquad (11)$$

In addition from the non-resonance condition, we have $p_1(\bar{r}) = \frac{2h\alpha}{2+(-2+h)\alpha} \neq 0,1$ under the condition $\alpha \neq \frac{2}{2+h}$. This means that $\lambda^k(\bar{r}) \neq 1$ for $k = 1,2,3,4$.

Let $x_1 = X_1 - K, x_2 = X_2 - K$, $A(r) = J(K,K)$. Now we transform the equilibrium point (K,K) of system (7) into the origin, then system (7) becomes

$$\binom{x_1}{x_2} \to A(r)\binom{x_1}{x_2} + \binom{F_1(x_1,x_2,r)}{F_2(x_1,x_2,r)} \qquad (12)$$

where

$$F_1 = \frac{(-2+(2-3h)\alpha)r}{2KM}x_1^2 - \frac{(2+(-2+h)\alpha)r}{2KM}x_2^2 + O(|x|^3)$$
$$F_2 = 0. \qquad (13)$$

Now we calculate multilinear function:

$$B_1(x,y) = \sum_{j,k=1}^{2} \left.\frac{\partial^2 F_1(\psi,r)}{\partial\psi_j \partial\psi_k}\right|_{\psi=0} x_j y_k$$
$$= \frac{(-2+(2-3h)\alpha)r}{KM}x_1 y_1 - \frac{(2+(-2+h)\alpha)r}{KM}x_2 y_2$$
$$B_2(x,y) = \sum_{j,k=1}^{2} \left.\frac{\partial^2 F_2(\psi,r)}{\partial\psi_j \partial\psi_k}\right|_{\psi=0} x_j y_k$$
$$= 0 \qquad (14)$$
$$C_1(x,y,u) = \sum_{j,k,l=1}^{2} \left.\frac{\partial^3 F_1(\psi,r)}{\partial\psi_j \partial\psi_k \partial\psi_l}\right|_{\psi=0} x_j y_k u_l$$
$$= 0$$
$$C_2(x,y,u) = \sum_{j,k,l=1}^{2} \left.\frac{\partial^3 F_1(\psi,r)}{\partial\psi_j \partial\psi_k \partial\psi_l}\right|_{\psi=0} x_j y_k u_l$$
$$= 0$$

Let $q \epsilon C^2$ be an eigenvector of $A(\bar{r})$corresponding to the eigenvalues $\lambda(\bar{r})$ such that, $A(\bar{r})q = e^{i\theta_0}q$ and let $p \epsilon C^2$ be an eigenvector of the transposed matrix $A^T(\bar{r})$ corresponding to its eigenvalue $\overline{\lambda(\bar{r})}$ such

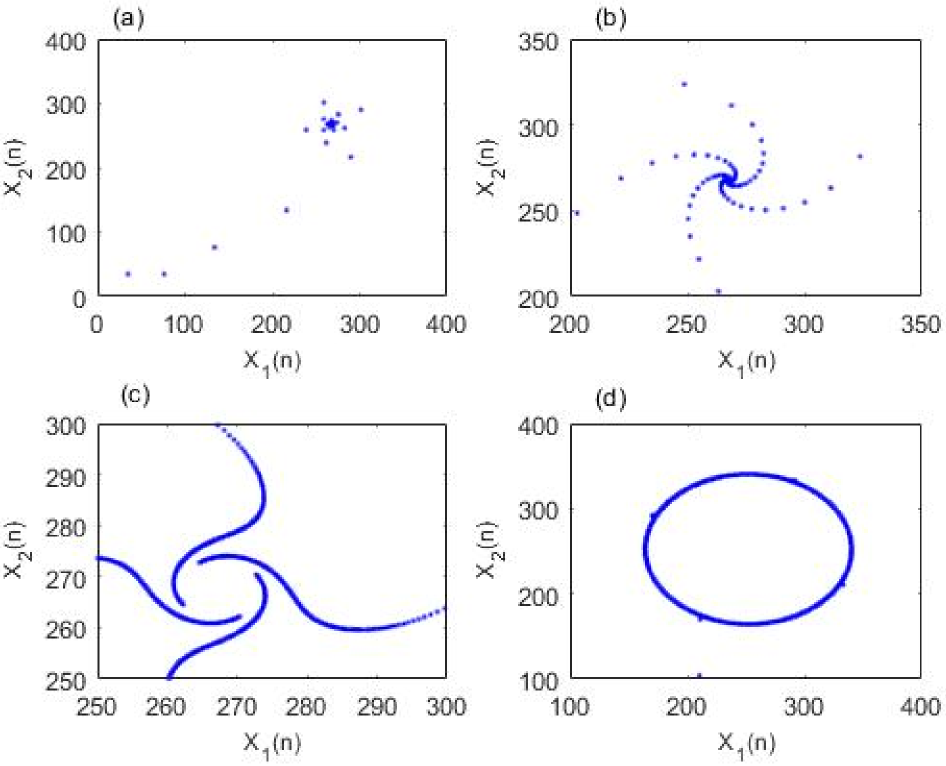

Fig. 3 Phase portraits of the system (7) for values r for $r = 3$ (a), $r = 4$ (b), $r = 4.42695$ (c), $r = 5$ (d). The other parameter values and initial conditions as the same as in Fig. 1.

that $A^T(\bar{r})p = e^{-i\theta_0}p$.

As a result of necessary calculation we obtain

$$q \sim (-h\alpha + 2i\sqrt{(1-\alpha)(1+(-1+h)\alpha)}, (2+(-2+h)\alpha))^T \tag{15}$$

and

$$p \sim (h\alpha + 2i\sqrt{(1-\alpha)(1+(-1+h)\alpha)}, (2+(-2+h)\alpha))^T \tag{16}$$

In order to normalize p with respect to q , we choose $p = m(a+ib, c)$ where $a = h\alpha,\ b = 2\sqrt{(1-\alpha)(1+(-1+h)\alpha)},\ \ c = 2+(-2+h)\alpha$ and $m = \frac{1}{-a^2-2iab+b^2+c^2}$. Now the normalized vectors are

$$q = (-a+ib, c)^T \tag{17}$$

and

$$p = \left(\frac{a+ib}{-a^2-2iab+b^2+c^2}, \frac{c}{-a^2-2iab+b^2+c^2}\right)^T . \tag{18}$$

Now it can be easily seen that $\langle p,q \rangle = 1$, where $\langle .,. \rangle$ means the standart scalar product $C^2: \langle p,q \rangle = \overline{p_1}q_1 + \overline{p_2}q_2$.

Now we form

$x = zq + \overline{zq}$

In this way system (12) can be transformed for sufficiently small$|r|$into following form

$z \to \lambda(r)z + g(z,\bar{z},r),$
where $\lambda(r)$ can be written as $\lambda(r) = (1+\phi(r))e^{i\theta(r)}$ $(\phi(r)\ is\ smooth\ function\ with\ \phi(\bar{r})=0)$ and g is a complex valued smooth function of $z, \bar{z}$ and r whose Taylor expression with respect to $(z,\bar{z})$ are

$$g(z,\bar{z},r) = \sum_{k+j\geq 2} \frac{1}{k!\,j!} g_{kj}(r)\, z^k z^{-j}, \qquad k,j = 0,1,2,\ldots$$

where

$$g_{20} = \langle p, B(q,q)\rangle$$
$$= -\frac{2(a-ib)(2a(a-ib)^2 + (a-ib)^2c + c^3)}{c(-(a-ib)^2 + c^2)K}$$
$$g_{11} = \langle p, B(q,\bar{q})\rangle$$
$$= -\frac{2(a-ib)(2\alpha(a^2+b^2) + (a^2+b^2)c + c^3)}{c(-(a-ib)^2 + c^2)K}$$
$$g_{02} = \langle p, B(q,\bar{q})\rangle$$
$$= -\frac{2(a-ib)(2a(a+ib)^2 + (a+ib)^2c + c^3)}{c(-(a-ib)^2 + c^2)K}$$
$$g_{21} = \langle p, C(q,q,\bar{q})\rangle$$
$$= 0$$

Now the coefficient $k(\bar{r})$ which determines the direction of the appearence of the invariant curve in a generic system exhibiting Neimark-Sacker bifurcation can be computed via

$$k(\bar{r}) = Re\left(\frac{e^{-i\theta(\bar{r})}g_{21}}{2}\right) - Re\left(\frac{\left(1-2e^{i\theta(\bar{r})}\right)e^{-2i\theta(\bar{r})}}{2(1-e^{i\theta(\bar{r})})} g_{20}g_{11}\right) - \frac{1}{2}|g_{11}|^2 - \frac{1}{4}|g_{02}|^2. \qquad (19)$$

By using the above arguments and the theorems in (Kuznetsov, 1998; He and Li, 2014; Kartal, 2017, 2018), we have the following result.

Theorem 3.1

Let (*K*,*K*) is the possitive equilibrium point of the system (7). If $p_1^2 - 4p_0 < 0, \alpha \neq \frac{2}{2-h}$, $\alpha \neq \frac{2}{2+h}$ and $k(\bar{r}) < 0$ (respectively $k(\bar{r}) > 0$), then the Neimark-Sacker of the system at $r = \bar{r}$ is supercritical (respectively subcritical) and closed invariant curve bifurcation from (*K*,*K*) for $r = \bar{r}$, which is asymptotically stable (respectively unstable).

From the conditions of Theorem 2.2, the stable region of the system (7) is obtained as $r \leq 4.42695$ for the parameter values $\alpha = 0.8$, $K = 267.5301$, $h = 0.001$ and $M = 0.88715$. For $r = 0.07060$, which falls in this stable region, the positive equilibrium point $(267.5301, 267.5301)$ of the system (7) is local asymptotically stable (Fig. 2). For $r = \bar{r} = 4.42695$ we have complex conjugate eigenvalues $|\lambda_{1,2}| = -0.00199601 - 0.999998i$ with moduls $|\lambda_{1,2}(\bar{r})| = 1$. In addition, it can be easily seen that transversality condition $\frac{d|\lambda_{1,2}(r)|}{dr}\Big|_{r=\bar{r}} = 0.112945 \neq 0$ and non-resonance condition $\lambda^k(\bar{r}) \neq 1$ for $\alpha \neq 1.0005$, $\alpha \neq 0.9995$ are satisfied. So the Neimark-Sacker bifurcation occurs around the positive equilibrium point (Fig.3, Fig. 4). On the other hand, the Taylor coefficients can be calculated as $g_{20} = 6.00451\text{x}10^{-6} - 5.96866\text{x}10^{-6}i$, $g_{11} = 3.13112\text{x}10^{-21} + 0.00300228\text{x}10^{-6}i$, $g_{02} = -6.00451\text{x}10^{-6} - 5.96866\text{x}10^{-6}i$ and $g_{21} = 0$ with $\theta = 1.52729$. Therefore the critical real point is obtained as$k(\bar{r}) = -4.4889\text{x}10^{-6}$ that is Neimark-Sacker bifurcation is supercritical. In addition, we also compute Lyapunov exponents corresponding to Fig. 4 to confirm further complexity of the dynamical behaviours. Fig. 5 demostrates the existence of the chaotic regions and period orbits in the parametric space.

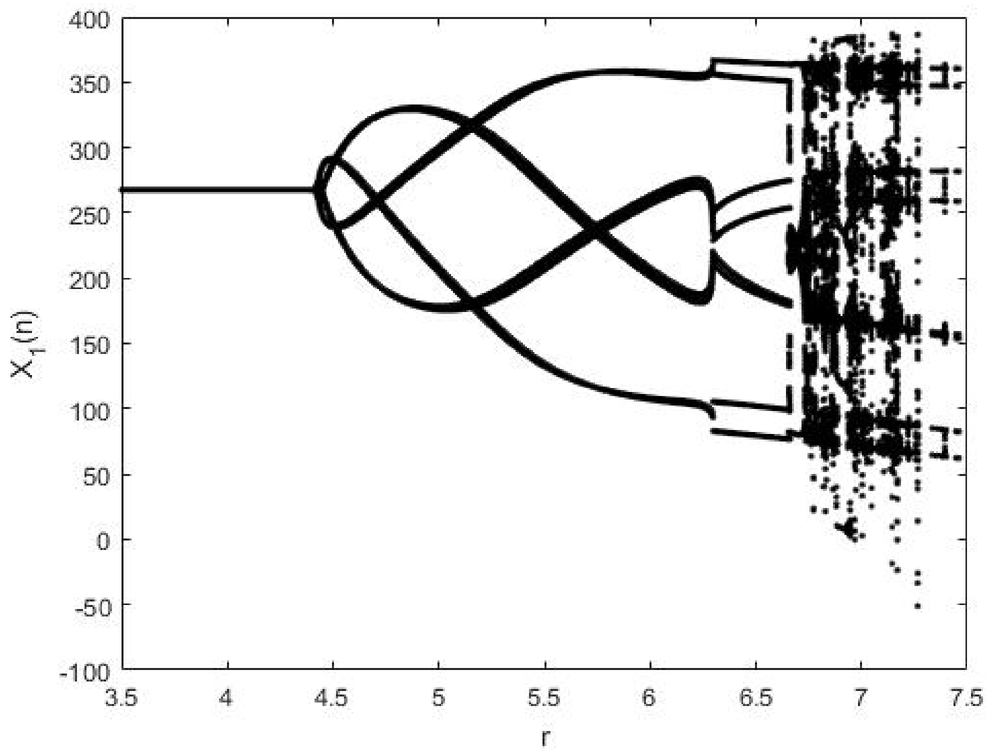

Fig. 4 Bifurcation diagram for $(r-X_1)$ plane. The other parameter values and initial conditions as the same as in Fig. 1.

4 Results and Discussion

In this study, by applying two-step Adams-Bashforth scheme to the Caputo-Fabrizio fractional logistic equation, we obtain the system of difference equations (7). The parameter values are taken from the data resulting from the experimental observation of the annual growth rate of the helianthus plant and these data can be found in (Noupoue et al., 2019; Reed and Holland, 1919). The height of the plants measured at a constant spacing time of 7 days given in centimeters. Stability analysis show that the positive equilibrium point (*K*,*K*) of the discrete system (7) is local asymptotically stable under the some algebraic conditions depending on the parameter r (Fig. 2). When we compare solution of Caputo and Caputo-Fabrizio fractional logistic equations we find that both fractional differential equations give closer results in the interval $\alpha \epsilon [0.8-0.82]$. Considering this harmony we take the fractional order parameter α as 0.80. As α exceeds this value and approaches to 1 the difference between the solutions of the two equations increases. We also deal with the bifurcation analysis of the discrete system and show that system (7) undergoes a Neimark-Sacker bifurcation at the critical parameter value $r = 4.42695$ that leads to stable limit cycle around the positive equilibrium point (Fig. 3 and Fig. 4).Maximum Lyapunov exponents show that the discrete system exhibit chaotic dynamic according to changing parameter r (Fig. 5). From Fig. 5, it is observed that some Lyapunov exponents are bigger than 0, some are smaller than 0, so there exists stable fixed point or stable quasi-periodic windows in the chaotic region. In addition we choose normalization function M as $M = 1-\alpha+\frac{\alpha}{\Gamma(\alpha)}$ which satisfies $M(0) = M(1) = 1$ (Bastos, 2018). The phase portrait of the system for increasing value of fractional order parameter α is given in Fig. 6. This figure demonstrates the process of how a smooth invariant circle appears and disappears from the fixed point. The critical Neimark-Sacker bifurcation according to the fractional order parameter α can be determined as $\bar{\alpha} = 0.598995$ (Fig. 6). From this figure, we observe that stable behavior of system destabilize for increasing the fractional order parameter α.

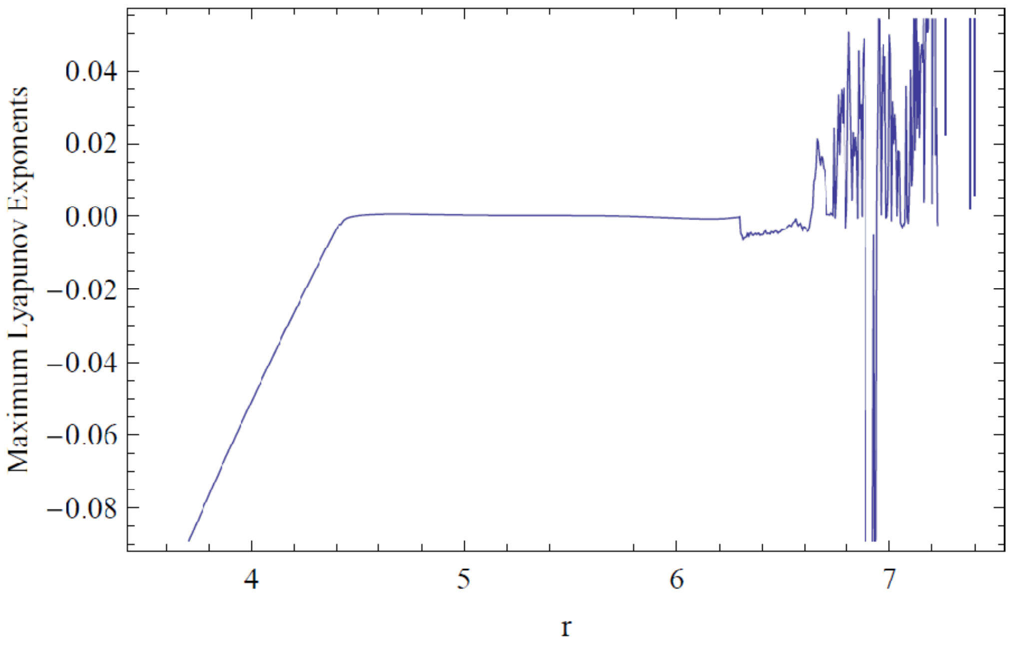

Fig. 5 Maximum Lyapunov exponents corresponding to Fig. 4.

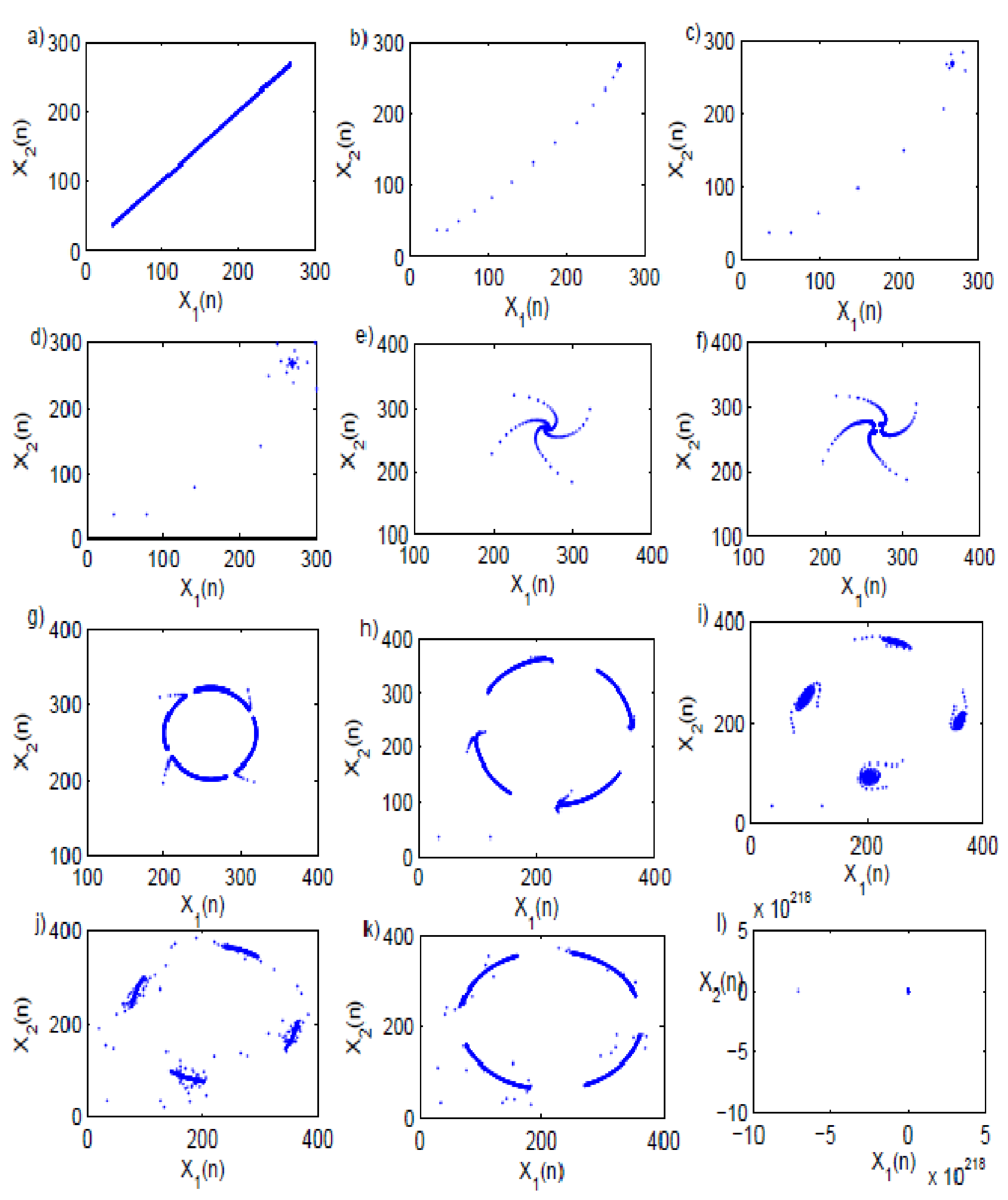

Fig. 6 Phase portrais for values of α and normalization function M for $\alpha = 1$, $M = 1$ (a), $\alpha = 0.9$, $M = 0.942201$ (b), $\alpha = 0.8$, $M = 0.88715$ (c), $\alpha = 0.7$, $M = 0.839268$ (d), $\alpha = 0.61$, $M = 0.805903$ (e), $\alpha = 0.598995$, $M = 0.80261$ (f), $\alpha = 0.58$, $M = 0.797326$ (g), $\alpha = 0.45$, $M = 0.778643$ (h), $\alpha = 0.44$, $M = 0.778558$ (i), $\alpha = 0.4$, $M = 0.78033$ (j), $\alpha = 0.35$, $M = 0.787463$ (k), $\alpha = 0.3, M = 0.800281$ (l). The other parameter values and initial conditions as the same as in Fig. 1.

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