Article

Modeling and analysis of the effects of gaseous pollutants and particulate matters on human health with control mechanisms

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Abstract
Human health is affected by various types of gaseous pollutants (CO, SO_2, NO_x, etc.) and particulate matters ($PM_{2.5}$, PM_5, PM_{10}) discharged from several sources such as vehicular traffic, small scale production and construction industries etc. in a city. These particulate matters are also discharged into atmosphere of the city by household emissions, causing respiratory diseases and innumerable deaths of the people. Therefore, in this paper, a nonlinear mathematical model is proposed and analyzed to study the effects of gaseous pollutants and particulate matters on human health in a city with control mechanisms. In the modeling process, six dependent variables are considered, namely, the density of human population, the cumulative density of various pollution emitting sources, the cumulative concentration of gaseous pollutants, the cumulative concentration of particulate matters, the cumulative concentration of suitable aerosols sprayed in the atmosphere to neutralize gaseous pollutants and the number density of water drops (amount of water) to wash out particulate matters from the atmosphere. In the modeling process, it is assumed that the human population density is governed by a logistic model, the growth rate of which decreases due to increased cumulative concentration of gaseous pollutants and particulate matters. It is further assumed that the growth rate of cumulative density of various sources discharging pollutants in the atmosphere is proportional to human population density. The cumulative concentration of aerosol sprayed in the atmosphere is considered to be proportional to the cumulative concentration of gaseous pollutants. Similarly the number density of water drops is assumed to be proportional to the cumulative concentration of particulate matters in the atmosphere. The model is proposed in the form of nonlinear ordinary differential equations which are analyzed by using the stability theory. The model analysis shows that in the absence of any control mechanism, the equilibrium level of population density is lower than that when the control mechanisms are applied. The numerical simulation of the model confirms the analytical findings. This study implies that human health is adversely affected without the control mechanism and the death rate of population increases due to various pollutants emitting sources such as vehicular traffic.

Keywords mathematical model; industry; vehicular traffic; gaseous pollutants; particulate matter; external species.

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1 Introduction

One of the main concerns, the world over, is the detrimental health effects on human population in a city due to increasing various pollutants emitting sources such as vehicular traffic, various types of small scale production and construction industries, etc. According to the report of WHO, every year nearly 4.2 million people die due to increasing traffic air pollution globally. More than 80% of these people are living in those urban areas where air pollution level is much higher than specified limits prescribed by the WHO. The main effect of the increased level of air pollution in a city in the form of gaseous pollutants and particulate matters is severe level of respiratory diseases in population leading to many deaths (Cohen et al., 2017; Dandona et al., 2019). Particulate matters having diameter less than 10μ (PM_{10}) are recognized as the pointer of the level of air pollution posing huge health problems in humans related to lungs (Varghas and Teran, 2012). The small particulate matters have much more detrimental effects on human health as compared to the large ones, because it is easier for these to enter into human respiratory system causing serious breathing problems (cardiovascular and respiratory) as reported in several studies across the world (Brunekreef and Holgate, 2002; Lee et al., 2013; Lepeule et al., 2012; Zang and Batterman, 2013; Lelived et al., 2015; Kheirbek et al., 2016; Munzel et al., 2018; Anenberg et al., 2019; Tong et al., 2020). Nearly 1.8% of the total global population die each year only due to cardiovascular problem which is much higher in those countries where the level of traffic air pollution is much higher (Giannadaki et al., 2014). The capital city, Beijing, of China is faced by increasing vehicular traffic pollution due to which public health is under threat (Tang et al., 2016). Children, because of their growing state and higher breathing rates, are more susceptible to lung diseases caused by gaseous pollutants and particulate matters (Nunes et al., 2016).

In view of the above discussion, it is clear that gaseous pollutants and particulate matters, caused by various pollutants emitting sources, damage human health seriously. Therefore, it is pertinent to apply some removal mechanism to remove these gaseous pollutants and particulate matters from the atmosphere. Water scavenging is one such process for the removal of these gaseous pollutants and particulate matters from the atmosphere. In this regard, Hales (1972) developed a theory for scavenging of gases from the atmosphere using rain. Bariie (1978) developed a mathematical model to calculate the redistribution and washout of sulphur dioxide by heavy rain. Shukla et al. (2008a) developed a mathematical model for the removal of gaseous pollutants and particulate matters by raindrops from the atmosphere of a city. Thus, introduction of some external species such as suitable aerosols and water spraying could be used to remove these gaseous pollutants and particulate matters from the atmosphere. In this regard, various investigations have been conducted (Nikulshina et al., 2007; Shukla et al., 2008b; Shukla et al., 2012; Wu and Lan, 2012; Sundar et al., 2013). A technique in which water has been sprayed from a mechanical device to remove gaseous pollutants and particulate matters showed high removal efficiency (Tsai et al., 2003).

In view of the above, in this paper, we have proposed a nonlinear mathematical model to study the effect of gaseous pollutants and particulate matters (without and with control mechanisms) on human health in a city.

2 Mathematical Model

Let us consider that the atmosphere of a city is affected by gaseous pollutants and particulate matters in a city causing various health problems to its citizens. Let C_1 be the cumulative concentration of gaseous pollutants and C_2 be the cumulative concentration of particulate matters caused by various pollutants emitting sources. Let N be the density of human population modelled logistically with decreased growth rate due to particulate matters and gaseous pollutants. Let S be the cumulative density of various pollutants emitting sources, the growth rate of which is assumed to be proportional to human population density N. Let Q_1 be the constant rate of emission of gaseous pollutants and Q_2 be the constant rate of emission of particulate matters discharged into the atmosphere of the city caused by small scale production industries, construction of buildings, roads etc. It is considered that the cumulative concentration of particulate matters increases in the city due to discharge of household sources also assumed to be proportional to the human density N. Let C_{d1} be the cumulative density of some suitable aerosol introduced in the atmosphere to neutralize gaseous pollutants and C_{d2} be the number density of water drops, sprayed in the atmosphere to remove particulate matters.

The proposed model, in terms of nonlinear differential equations, is proposed as follows

$$\frac{dN}{dt} = r\left(N - \frac{N^2}{K}\right) - r_1 N - d_1 N C_1 - d_2 N C_2 \tag{1}$$

$$\frac{dS}{dt} = \lambda N - \lambda_0 S \tag{2}$$

$$\frac{dC_1}{dt} = Q_1 + \delta_1 S - \delta_{10} C_1 - \nu_1 C_1 N - \mu_{11} C_1 C_{d1} \tag{3}$$

$$\frac{dC_2}{dt} = Q_2 + \delta_2 S + \delta_{21} N - \delta_{20} C_2 - \nu_2 C_2 N - \mu_{22} C_2 C_{d2} \tag{4}$$

$$\frac{dC_{d1}}{dt} = \mu_1 C_1 - \mu_{10} C_{d1} - \mu_{11} C_1 C_{d1} \tag{5}$$

$$\frac{dC_{d2}}{dt} = \mu_2 C_2 - \mu_{20} C_{d2} - \mu_{22} C_2 C_{d2} \tag{6}$$

$$N(0) > 0,\ S(0) \ge 0,\ C_1(0) \ge 0,\ C_2(0) \ge 0,\ C_{d1}(0) \ge 0,\ C_{d2}(0) \ge 0$$

In eq (1), r is the intrinsic growth rate of human population density N with carrying capacity K and r_1 is its death rate coefficient due to some natural factors. Since the human health is adversely affected due to the uptake of gaseous pollutants and particulate matters during breathing, it is assumed that the density of human population decreases due to these pollutants with rate coefficients d_1 and d_2 respectively. The growth rate of cumulative density of pollutants emitting sources density, S, in equation (2), is assumed to be proportional to the density of human population, where $\lambda > 0$ is assumed to be its growth rate coefficient and $\lambda_0 > 0$ is the depletion rate coefficient due to inefficient working of sources .

The cumulative concentration of gaseous pollutants C_1 is modelled in equation (3), where Q_1 represents the constant emission rate of gaseous pollutants in the atmosphere such as from construction of buildings, roads etc. The constant $\delta_{10} > 0$ represents depletion rate coefficient of gaseous pollutants due to some natural factors. Since the gaseous pollutants are uptaken directly during breathing by the human population with density N, the cumulative concentration C_1 decreases with depletion rate coefficient $\nu_1 > 0$ (proportional to $C_1 N$) and then subsequently death rate of N increases with rate d_1 (in proportion to NC_1). Similarly, the cumulative concentration of particulate matters with density C_2 is modelled in equation (4), where δ_{21} is the rate of increase of particulates due to household discharges.

It is assumed further that the growth rate of the cumulative density of aerosol (C_{d1}) is directly proportional to the cumulative concentration C_1 of gaseous pollutants in the atmosphere (see equation (5)) with a growth rate $\mu_1 > 0$, depletion rate $\mu_{10} > 0$ where $\mu_{11} > 0$ is the depletion rate due to interaction of C_1 and C_{d1} i.e. $C_1 C_{d1}$ (see equation (3) also) in this equation. Similarly, the number density of water drops C_{d2} is modelled in equation (6), where $\mu_2 > 0$ is the growth rate coefficient, $\mu_{20} > 0$ is the depletion rate coefficient and $\mu_{22} > 0$ is the depletion rate due to interaction of C_2 and C_{d2} i.e. C_2C_{d2} (see equation (4) also).

Now in the following, we state the lemma without proof required for analyzing the model (1) - (6).

Lemma

The region of attraction of the model system (1) – (6) is given as follows,

$$\Omega = \{(N, S, C_1, C_2, C_{d1}, C_{d2}) \in R^6 : 0 \le N \le K, 0 \le S \le S_m, \frac{Q_1}{\delta_{10}} \le C_1 \le C_{1m},$$
$$\frac{Q_2}{\delta_{20}} \le C_2 \le C_{2m}, \frac{\mu_1 Q_1}{\mu_{10}\delta_{10}} \le C_{d1} \le C_{d1m}, \frac{\mu_2 Q_2}{\mu_{20}\delta_{20}} \le C_{d2} \le C_{d2m})\}$$

where, $S_m = \frac{\lambda}{\lambda_0}K$, $C_{1m} = \frac{Q_1 + \bar{\delta}_1 K}{\delta_{10}}$, $C_{2m} = \frac{Q_2 + \bar{\delta}_2 K}{\delta_{20}}$, $C_{d1m} = \frac{\mu_1}{\mu_{10}} C_{1m}$, $C_{d2m} = \frac{\mu_2}{\mu_{20}} C_{2m}$, $\bar{\delta}_1 = \frac{\delta_1 \lambda}{\lambda_0}$,

$\bar{\delta}_2 = \frac{\delta_2 \lambda}{\lambda_0} + \delta_{21}$, , $C_{d1\min} = \frac{\mu_1 Q_1}{\mu_{10}\delta_{10}}$, $C_{d2\min} = \frac{\mu_2 Q_2}{\mu_{20}\delta_{20}}$

Feasibility conditions: For feasibility of the model (1) - (6), we have the following conditions,

(i) $r - r_1 - d_1C_1 - d_2C_2 > 0$ (7)

(ii) $\frac{\bar{\delta}_1}{v_1} > C_1$ including maximum of C_1 (8)

(iii) $\frac{\bar{\delta}_2}{v_2} > C_2$ and $\delta_{21} - v_2C_2 > 0$ including maximum of C_2 (9)

(iv) $\frac{\mu_1}{\mu_{11}} > C_{d1}$ including maximum of C_{d1} (10)

(v) $\frac{\mu_2}{\mu_{22}} > C_{d2}$ including maximum of C_{d2} (11)

3 Equilibrium and Stability Analysis

In this section, the model is analyzed under the following two cases,

Case I: Without control mechanism (i.e. $C_{d1} = 0, C_{d2} = 0$)

Case II: With control mechanism using external species (i.e. $C_{d1} \ne 0$, $C_{d2} \ne 0$)

3.1 Case I: Analysis of model without control mechanism (i.e. $C_{d1} = 0, C_{d2} = 0$)

In this case, we study the effects of cumulative concentration of gaseous pollutants and particulate matters on human population density in the absence of control mechanism i.e. ($C_{d1} = 0, C_{d2} = 0$). The model system (1) – (6) in this case is reduced to the following form;

$$\frac{dN}{dt} = r\left(N - \frac{N^2}{K}\right) - r_1N - d_1NC_1 - d_2NC_2 \quad (12)$$

$$\frac{dS}{dt} = \lambda N - \lambda_0 S \quad (13)$$

$$\frac{dC_1}{dt} = Q_1 + \delta_1 S - \delta_{10}C_1 - v_1C_1N \quad (14)$$

$$\frac{dC_2}{dt} = Q_2 + \delta_2 S + \delta_{21} N - \delta_{20} C_2 - v_2 C_2 N \tag{15}$$

There are only two non-zero equilibria of the model (12) – (15) as given below,

(i) $E_0\left(0, 0, \frac{Q_1}{\delta_{10}}, \frac{Q_2}{\delta_{20}}\right)$ This implies that in the absence of density of human population and density of various pollutants emitting sources, the cumulative concentration of both pollutants will remain at their natural equilibrium levels $\frac{Q_1}{\delta_{10}}$ and $\frac{Q_2}{\delta_{20}}$ respectively.

(ii) $\tilde{E}(\tilde{N}, \tilde{S}, \tilde{C}_1, \tilde{C}_2)$. The existence of $\tilde{E}$ is proved as follows.

3.1.1 Existence of the equilibrium $\tilde{E}$

The equilibrium values of different variables are given by the following algebraic equations,

$$r\left(1 - \frac{N}{K}\right) - r_1 - d_1 C_1 - d_2 C_2 = 0 \tag{16}$$

$$\lambda N - \lambda_0 S = 0 \tag{17}$$

$$Q_1 + \delta_1 S - \delta_{10} C_1 - v_1 C_1 N = 0 \tag{18}$$

$$Q_2 + \delta_2 S + \delta_{21} N - \delta_{20} C_2 - v_2 C_2 N = 0 \tag{19}$$

From equations (21) - (24) we get,

$$N = \frac{K}{r}\left(r - r_1 - d_1 C_1 - d_2 C_2\right) \tag{20}$$

$$S = \frac{\lambda}{\lambda_0} N \tag{21}$$

$$C_1 = \frac{Q_1 + \bar{\delta}_1 N}{\delta_{10} + v_1 N} \tag{22}$$

$$C_2 = \frac{Q_2 + \bar{\delta}_2 N}{\delta_{20} + v_2 N} \tag{23}$$

Using equations (21) – (23) in equation (16) we can define $F(N)$ as follows,

$$F(N) = r\left(1 - \frac{N}{K}\right) - r_1 - d_1\left(\frac{Q_1 + \bar{\delta}_1 N}{\delta_{10} + v_1 N}\right) - d_2\left(\frac{Q_2 + \bar{\delta}_2 N}{\delta_{20} + v_2 N}\right) = 0 \tag{24}$$

From equation (24), we note that,

(i) $F(0) = r - r_1 - \frac{d_1 Q_1}{\delta_{10}} - \frac{d_2 Q_2}{\delta_{20}} > 0$, due to feasibility condition (7)

(ii) $F(K) = -r_1 - d_1\left(\frac{Q_1 + \bar{\delta}_2 K}{\delta_{10} + v_1 K}\right) - d_2\left(\frac{Q_2 + \bar{\delta}_2 K}{\delta_{20} + v_2 K}\right) < 0$

(iii) $F'(N) = -\frac{r}{K} - \frac{\nu_1 \delta_{10} d_1}{(\delta_{10}+\nu_1 N)^2}\left(\frac{\bar{\delta}_1}{\nu_1} - \frac{Q_1}{\delta_{10}}\right) - \frac{\nu_2 \delta_{20} d_2}{(\delta_{20}+\nu_2 N)^2}\left(\frac{\bar{\delta}_2}{\nu_2} - \frac{Q_2}{\delta_{20}}\right) < 0$

(see feasibility condition (8)and (9))

This implies that $F(N) = 0$ has a unique positive root (say $N = \tilde{N}$) in $(0 \le N \le K)$. Using $N = \tilde{N}$, the values of $\tilde{S}$, $\tilde{C}_1$ and $\tilde{C}_2$ can be obtained from equations (21), (22) and (23) respectively.

3.1.2 Stability analysis

To study the local stability behavior of equilibrium E_0, we compute the Jacobian matrix $M(E)$ for the model system (16) – (19) as follows,

$$M(E) = \begin{bmatrix} r - \frac{2rN}{K} - r_1 - d_1 C_1 - d_2 C_2 & 0 & -d_1 N & -d_2 N \\ \lambda & -\lambda_0 & 0 & 0 \\ -\nu_1 C_1 & \delta_1 & -(\delta_{10}+\nu_1 N) & 0 \\ -\nu_2 C_2 + \delta_{21} & \delta_2 & 0 & -(\delta_{20}+\nu_2 N) \end{bmatrix}$$

From the above matrix, we note that the equilibrium E_0 is unstable as one eigenvalue $r - r_1 - d_1\tilde{C}_1 - d_2\tilde{C}_2$ of $M(E_0)$ is positive, in view of the conditions (7).

Theorem 3.1.1 The equilibrium $\tilde{E}$ is locally asymptotically stable provided the following conditions are satisfied,

$$\frac{1}{6}\frac{r}{K}(\delta_{10}+\nu_1\tilde{N}) - d_1\nu_1\tilde{C}_1 > 0 \quad (25)$$

$$\frac{1}{6}\frac{r}{K}(\delta_{20}+\nu_2\tilde{N}) + d_2(\delta_{21} - \nu_2\tilde{C}_2) > 0 \quad (26)$$

$$\frac{8}{27}\frac{r}{K}(\delta_{10}+\nu_1\tilde{N})\nu_1\tilde{C}_1 - d_1\bar{\delta}_1^2 > 0 \quad (27)$$

(See Appendix A for proof).

Theorem 3.1.2

The equilibrium $\tilde{E}$ is nonlinearly asymptotically stable provided the following conditions are satisfied in Ω,

$$\frac{1}{6}\frac{r}{K}\delta_{10} - d_1\nu_1\tilde{C}_1 > 0 \quad (28)$$

$$\frac{1}{6}\frac{r}{K}\delta_{20} + d_2(\delta_{21} - \nu_2\tilde{C}_2) > 0 \quad (29)$$

$$\frac{8}{27}\frac{r}{K}\delta_{10}\nu_1\tilde{C}_1 - d_1\bar{\delta}_1^2 > 0 \quad (30)$$

(See Appendix B for proof)

These stability conditions imply that all the four variables reach to their equilibrium values under above conditions. Further, if $\delta_1, \delta_2, \nu_1$ and ν_2 approach to zero then the possibility of satisfying the conditions (25)

– (30) is more plausible. Hence, these parameters have destabilizing effect on the system.

3.2 Case II: Analysis of model with control mechanism (i.e. $C_{d1} \neq 0$, $C_{d2} \neq 0$**)**

In this case, we study the effect of cumulative concentrations of gaseous pollutants and particulate matters on human population using control mechanisms in the form of spraying of suitable aerosols and water drops in the atmosphere i.e. ($C_{d1} \neq 0$, $C_{d2} \neq 0$) by considering the full model system (1) - (6).

There are again two equilibrium points of the model system (1) - (6) as given below,

(i) $\overline{E} = (0,0,\overline{C}_1,\overline{C}_2,\overline{C}_{d1},\overline{C}_{d2})$

where $$\overline{C}_1 = \frac{-(\mu_{10}\delta_{10} - Q_1\mu_{11}) + \sqrt{(\mu_{10}\delta_{10} - Q_1\mu_{11})^2 + 4\mu_{10}\mu_{11}Q_1(\mu_1 + \delta_{10})}}{2\mu_{11}(\mu_1 + \delta_{10})}$$

$$\overline{C}_2 = \frac{-(\mu_{20}\delta_{20} - Q_2\mu_{22}) + \sqrt{(\mu_{20}\delta_{20} - Q_2\mu_{22})^2 + 4\mu_{20}\mu_{22}Q_2(\mu_2 + \delta_{20})}}{2\mu_{20}\mu_{22}(\mu_2 + \delta_{20})}$$

$$\overline{C}_{d1} = \frac{\mu_1\overline{C}_1}{\mu_{10} + \mu_{11}\overline{C}_1} \text{ and } \overline{C}_{d1} = \frac{\mu_2\overline{C}_2}{\mu_{20} + \mu_{22}\overline{C}_2}$$

(ii) $E^* = (N^*, S^*, C_1^*, C_2^*, C_{d1}^*, C_{d2}^*)$

3.2.1 Existence of equilibrium E^*

The equilibrium values of different variables in E^* are given by the following algebraic equations,

$$r\left(1 - \frac{N}{K}\right) - r_1 - d_1C_1 - d_2C_2 = 0 \tag{31}$$

$$\lambda N - \lambda_0 S = 0 \tag{32}$$

$$Q_1 + \delta_1 S - \delta_{10}C_1 - \nu_1 C_1 N - \mu_{11}C_1C_{d1} = 0 \tag{33}$$

$$Q_2 + \delta_2 S + \delta_{21}N - \delta_{20}C_2 - \nu_2 C_2 N - \mu_{22}C_2C_{d2} = 0 \tag{34}$$

$$\mu_1 C_1 - \mu_{10}C_{d1} - \mu_{11}C_1C_{d1} = 0 \tag{35}$$

$$\mu_2 C_2 - \mu_{20}C_{d2} - \mu_{22}C_2C_{d2} = 0 \tag{36}$$

From equation (34), (35) and (37) we get

$$(\mu_{11}\mu_1 + \delta_{10}\mu_{11} + \nu_1\mu_{11}N)C_1^2 - (Q_1\mu_{11} + \overline{\delta}_1\mu_{11}N - \delta_{10}\mu_{10} - \nu_1\mu_{10}N)C_1$$
$$-(Q_1\mu_{10} + \overline{\delta}_1\mu_{10}N) = 0 \tag{37}$$

From equation (39) we get

$$C_1 = f(N) > 0 \tag{38}$$

where

$$f(N)=\frac{\mu_{11}\left(Q_1+\bar{\delta}_1 N\right)-\mu_{10}(\delta_{10}+\nu_1 N)+\sqrt{\begin{bmatrix}\left\{\mu_{11}\left(Q_1+\bar{\delta}_1 N\right)-\mu_{10}(\delta_{10}+\nu_1 N)\right\}^2+\\ 4\mu_{11}(\delta_{10}+\mu_1+\nu_1 N)\left(Q_1\mu_{10}+\mu_{10}\bar{\delta}_1 N\right)\end{bmatrix}}}{2\mu_{11}(\delta_{10}+\mu_1+\nu_1 N)}$$

(39)

Similarly from equation (34), (36) and (38) we get,

$$(\mu_{22}\mu_2+\delta_{20}\mu_{22}+\nu_2\mu_{22}N)C_2^2-(Q_2\mu_{22}+\bar{\delta}_2\mu_{22}N-\delta_{20}\mu_{20}-\nu_2\mu_{20}N)C_2$$

$$-(Q_2\mu_{20}+\bar{\delta}_2\mu_{20}N)=0 \quad (40)$$

From equation (42) we get,

$$C_2=g(N)>0 \quad (41)$$

where

$$g(N)=\frac{\mu_{22}\left(Q_2+\bar{\delta}_2 N\right)-\mu_{20}(\delta_{20}+\nu_2 N)+\sqrt{\begin{bmatrix}\left\{\mu_{22}\left(Q_2+\bar{\delta}_2 N\right)-\mu_{20}(\delta_{20}+\nu_2 N)\right\}^2+\\ 4\mu_{22}(\delta_{20}+\mu_2+\nu_2 N)\left(Q_2\mu_{20}+\mu_{20}\bar{\delta}_2 N\right)\end{bmatrix}}}{2\mu_{22}(\delta_{20}+\mu_2+\nu_2 N)}>0$$

(42)

From equations (39) and (42), on differentiating with respect to N, we get
$f'(N)>0$ and $g'(N)>0$ (43)
Using equations (40), (43) in equation (33) we define $G(N)$ as follows,

$$G(N)=r\left(1-\frac{N}{K}\right)-r_1-d_1 f(N)-d_2 g(N)=0 \quad (44)$$

From equation (46), in view of conditions (7) – (11) and (45), we note that (using feasibility condition) (7) – (11)

(i) $G(0)=r-r_1-d_1 f(0)-d_2 g(0)>0$ (45)

(ii) $G(K)=-r_1-d_1 f(K)-d_2 g(K)<0$ (46)

(iii) $G'(N)=\frac{-r}{K}-r_1-d_1 f'(N)-d_2 g'(N)<0$ (47)

This implies that $G(N)=0$ has a unique positive root (say $N=N^*$) in $0\le N\le K$. Using $N=N^*$ the values of $S^*, C_1^*, C_2^*, C_{d1}^*$ and C_{d2}^* can be obtained from equations (31) - (36) respectively.

3.2.2 Stability analysis

The following Jacobin matrix $J(E)$ for the model system (1)-(6) is computed to study the local stability behavior of equilibrium $\bar{E}$,

$$J(E)=\begin{bmatrix} \begin{pmatrix} r-\dfrac{2rN}{K}-r_1 \\ -d_1C-d_2C_2 \end{pmatrix} & 0 & -d_1N & -d_2N & 0 & 0 \\ \lambda & -\lambda_0 & 0 & 0 & 0 & 0 \\ -\nu_1C_1 & \delta_1 & -\begin{pmatrix} \delta_{10}+\mu_{11}C_{d1} \\ +\nu_1N \end{pmatrix} & 0 & -\mu_{11}C_1 & 0 \\ -\nu_2C_2+\delta_{21} & \delta_2 & 0 & -\begin{pmatrix} \delta_{20}+\mu_{22}C_{d2} \\ +\nu_2N \end{pmatrix} & 0 & -\mu_{22}C_2 \\ 0 & 0 & \begin{pmatrix} \mu_1 \\ -\mu_{11}C_{d1} \end{pmatrix} & 0 & -\begin{pmatrix} \mu_{10} \\ +\mu_{11}C_1 \end{pmatrix} & 0 \\ 0 & 0 & 0 & \begin{pmatrix} \mu_2 \\ -\mu_{22}C_{d2} \end{pmatrix} & 0 & -\begin{pmatrix} \mu_{20} \\ +\mu_{22}C_2 \end{pmatrix} \end{bmatrix}$$

From the above matrix, we note that the equilibrium $\overline{E}(0,0,\overline{C}_1,\overline{C}_2,\overline{C}_{d1},\overline{C}_{d2})$ is unstable as one of the eigenvalues $r-r_1-d_1\overline{C}_1-d_2\overline{C}_2$ of $J(\overline{E})$ is positive.

Theorem 3.2.1 The equilibrium E^* is locally asymptotically stable provided the following conditions are satisfied,

$$\frac{1}{6}\frac{r}{K}(\delta_{10}+\nu_1N^*+\mu_{11}C_{d1}^*)-d_1\nu_1C_1^*>0 \tag{48}$$

$$\frac{1}{6}\frac{r}{K}(\delta_{20}+\nu_2N^*+\mu_{22}C_{d2}^*)+d_2(\delta_{21}-\nu_2C_2^*)>0 \tag{49}$$

$$\frac{8}{27}\frac{r}{K}(\delta_{10}+\nu_1N^*+\mu_{11}C_{d1}^*)\nu_1C_1^*-d_1\overline{\delta}_1^2>0 \tag{50}$$

(See Appendix C for proof)

Theorem 3.2.2 The equilibrium E^* is nonlinearly asymptotically stable provided the following conditions are satisfied in Ω,

$$S_1=\frac{1}{6}\frac{r}{K}(\delta_{10}+\mu_{11}C_{d1\min})-d_1\nu_1C_1^*>0 \tag{51}$$

$$S_2=\frac{1}{6}\frac{r}{K}(\delta_{20}+\mu_{22}C_{d2\min})+d_2(\delta_{21}-\nu_2C_2^*)>0 \tag{52}$$

$$S_3=\frac{8}{27}\frac{r}{K}(\delta_{10}+\mu_{11}C_{d1\min})\nu_1C_1-d_1\overline{\delta}_1^2>0 \tag{53}$$

(See Appendix D for proof)

The above stability conditions imply that if δ_1, δ_2, ν_1 and ν_2 approach to zero then the possibility of satisfying the conditions (48) – (53) is more feasible. Hence, these parameters have destabilizing effect on the model system. These conditions are more favourable than the case without control mechanisms.

4 Numerical Simulations

We have performed here some numerical simulations to study the local and nonlinear stability behavior of equilibria and feasibility of the model system (1) – (6) using MAPLE 18 by choosing the following set of values of parameters,

$Q_1 = 4$, $Q_2 = 2$, $r = 0.5$, $K = 10000$, $r_1 = 0.001$, $d_1 = 0.000004$, $d_2 = 0.000006$, $\lambda = 0.1$,

$\lambda_0 = 0.15$, $\delta_1 = 0.025$, $\delta_2 = 0.008$, $\delta_{10} = 0.3$, $\delta_{20} = 0.35$, $\nu_1 = 0.000004$, $\nu_2 = 0.000006$

$\mu_1 = 0.1$, $\mu_2 = 0.1$, $\mu_{10} = 0.05$, $\mu_{20} = 0.04$, $\mu_{11} = 0.0002$, $\mu_{22} = 0.0004$, $\delta_{21} = 0$

The equilibrium values of different variables in $E^*(N^*, S^*, C_1^*, C_2^*, C_{d1}^*, C_{d2}^*)$ corresponding to above data are given as below,

$N^* = 9932.080$, $S^* = 6621.386$, $C_1^* = 421.222$, $C_2^* = 118.515$, $C_{d1}^* = 313.772$, $C_{d2}^* = 135.591$

The eigenvalues of the Jacobean matrix corresponding to $E^*(N^*, S^*, C_1^*, C_2^*, C_{d1}^*, C_{d2}^*)$ for the model system (1)-(6) are obtained as $-0.5037, -0.3854, -0.1537, -0.1445, -0.4538, -0.0932$.

Since all the eigenvalues are negative, as a result, the internal equilibrium $E^*(N^*, S^*, C_1^*, C_2^*, C_{d1}^*, C_{d2}^*)$ is locally asymptotically stable.

Using the above set of parameter values, the nonlinear stability behavior of $E^*(N^*, S^*, C_1^*, C_2^*, C_{d1}^*, C_{d2}^*)$ is shown in the Fig. 1. It shows that the solution trajectories starting at any point within the region of attraction approach to equilibrium point $E^*(N^*, S^*, C_1^*, C_2^*, C_{d1}^*, C_{d2}^*)$. In Fig. 2, the variation of cumulative concentration of gaseous pollutants C_1 with time t is plotted for different values of δ_1 (the growth rate coefficient of gaseous pollutants) with control mechanism using some suitable external specie ($\mu_1 \neq 0$) and without control mechanism ($\mu_1 = 0$). It is observed from the figure that as δ_1 increases, the cumulative concentration of gaseous pollutants increases in the atmosphere but it decreases significantly when external species is introduced as control mechanism. In Fig. 3, the variation of cumulative concentration of particulate matters C_2 with time t is plotted for different values of δ_2 (the growth rate coefficient of particulate matters) with control mechanism using some other suitable external species ($\mu_2 \neq 0$) and without control mechanism ($\mu_2 = 0$). It is observed from the figure that as δ_2 increases, the cumulative concentration of particulate matters increases in the atmosphere but it decreases significantly when this external species is introduced as control mechanism.

In Fig. 4, the variation of cumulative concentration of gaseous pollutants C_1 with time t is plotted for different values of λ, the growth rate coefficient of pollutants emitting sources due to human population with control mechanism ($\mu_1 \neq 0$) and without control mechanism ($\mu_1 = 0$). It is found that as the growth rate of pollutants emitting sources due to human population increases, the cumulative concentration of gaseous pollutants also increases in the atmosphere but it decreases significantly when first external species is introduced as control mechanism. In Fig. 5, the variation of cumulative concentration of particulate matters C_2 with time t is plotted for different values of λ, the growth rate coefficient of pollutants emitting sources density due to human population with control mechanism using the second external species ($\mu_2 \neq 0$) and without control mechanism ($\mu_2 = 0$). It is found that as the growth rate of the density of traffic and industry due to human population increases, the cumulative concentration of particulate matters increases in the atmosphere but it decreases significantly when second external species is introduced as control mechanism. Fig. 6 depicts the variation of human population density N with time t for different values of δ_1 (the growth rate coefficient of gaseous pollutants) with $(\mu_1 \neq 0)$ and without control mechanism $(\mu_1 = 0)$. It is observed that as δ_1 increases, human population density decreases but it increases when the first external species is introduced. This implies that the excess of gaseous pollutants in the atmosphere adversely affects the human health. Fig. 7 shows the variation of human population density N with time t for different values of δ_2 (the growth rate coefficient of particulate matters) with $(\mu_2 \neq 0)$ and without control mechanism $(\mu_2 = 0)$. It is observed that as δ_2 increases, the human population density decreases but it increases when second external species is introduced. This implies that the excess of particulate matters in the atmosphere also adversely affects the human health. In Figs 8 and 9, the variation of human population density N with time t is plotted for different values of λ, the growth rate coefficient of pollutants emitting sources density due to human population density with and without control mechanism applied to control gaseous pollutants and particulate matters respectively. It is noted from this figure that, as the growth rate of pollutants emitting sources density increases, the human population density decreases due to emission of gaseous pollutants and particulate matters as discussed before. In Fig. 10, the cumulative concentration of gaseous pollutants C_1 with time t for different values of μ_1 is plotted. It is found that cumulative concentration of gaseous pollutants due to pollutants emitting sources density decreases but is much higher when $\mu_1 = 0$. In Fig. 11, the cumulative concentration of particulate matters C_2 with time t for different values of μ_2 is plotted. It is seen that cumulative concentration of particulate matters due to pollutants emitting sources density decreases but is much higher when $\mu_2 = 0$. The variation of population density N with time t for different values of μ_1 and μ_2 is explicitly shown in Figs 12 and 13 respectively. It is easily seen that as μ_1 and μ_2 increases, the human population density increases and its equilibrium level is much lower when $\mu_1 = 0$ and $\mu_2 = 0$. In Figs 14 and

15, we have shown the variation of nonlinear stability conditions S_1 and S_2 (as given in equations (51) and (52)), respectively, with respect to crucial parameters. It is clear from the Fig. 14 that S_1 remains positive for $\nu_1 < 0.002256$ and it is negative for $\nu_1 > 0.002256$. This implies that the stability condition is satisfied for $0 < \nu_1 < 0.002256$ and for higher values of ν_1 it will not be satisfied indicating that ν_1 has destabilizing effect on the model system. From Fig. 15, it is noted that S_2 remains positive for $\nu_2 < 0.061524$ and is negative for $\nu_2 > 0.061524$ implying that the stability condition is satisfied for $0 < \nu_2 < 0.061524$ for and for higher values of ν_2 it will not be satisfied indicating that ν_2 has destabilizing effect on the model system. In Fig. 16, we have shown the variation of nonlinear stability conditions S_3 (as given in equation (53)) respectively, with respect to crucial parameters to study the effect of these parameters on stability conditions. It is clear from the Fig. 16 that S_3 remains positive for $\delta_1 < 0.064670$ and it is negative for $\delta_1 > 0.064670$. This implies that the stability condition is satisfied for $0 < \delta_1 < 0.064670$ and for higher values of δ_1 it will not be satisfied indicating that δ_1 has destabilizing effect on the model system.

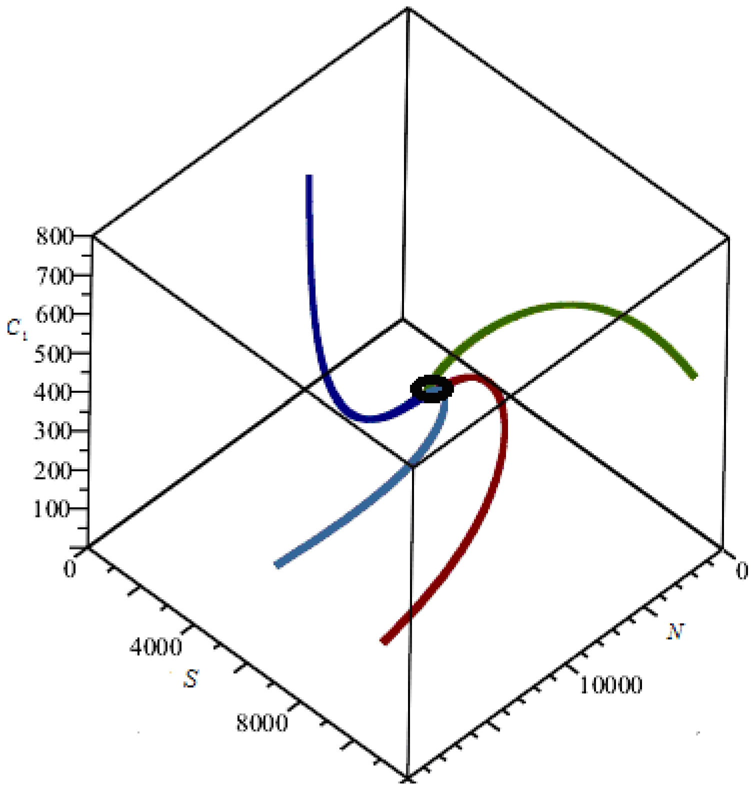

Fig. 1 Nonlinear stability in $N-S-C_1$ space.

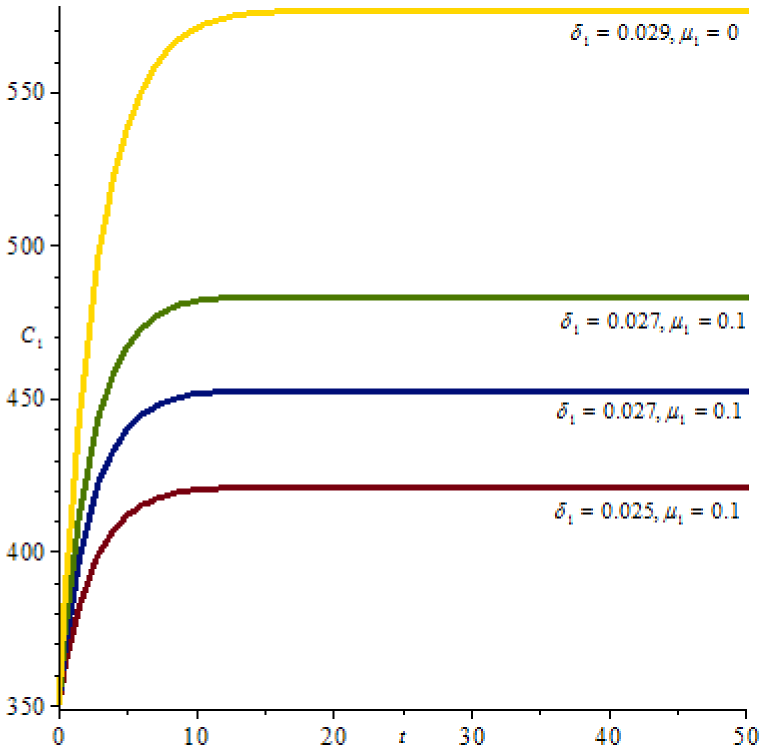

Fig. 2 Variation of C_1 with time t for different values of δ_1.

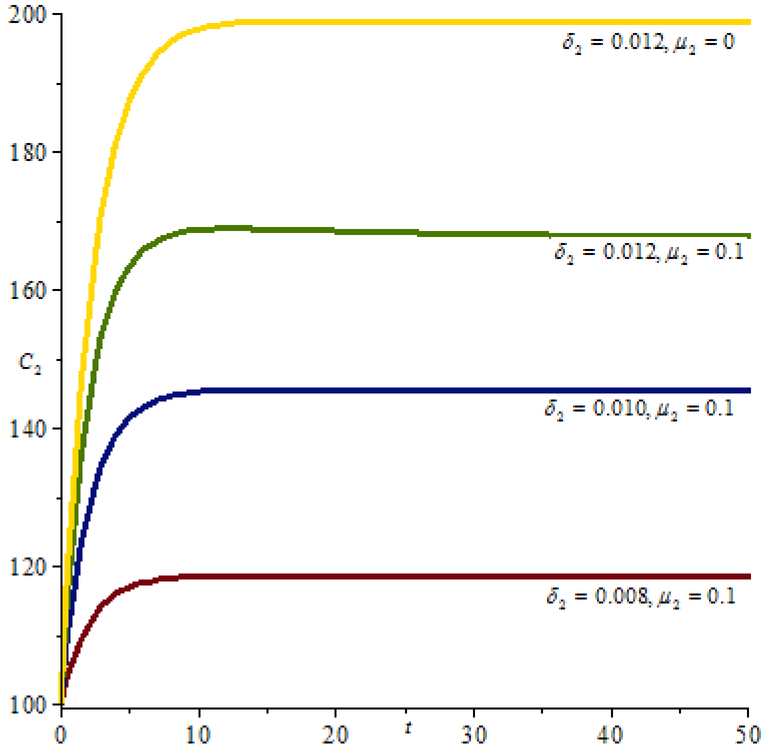

Fig. 3 Variation of C_2 with time t for different values of δ_2.

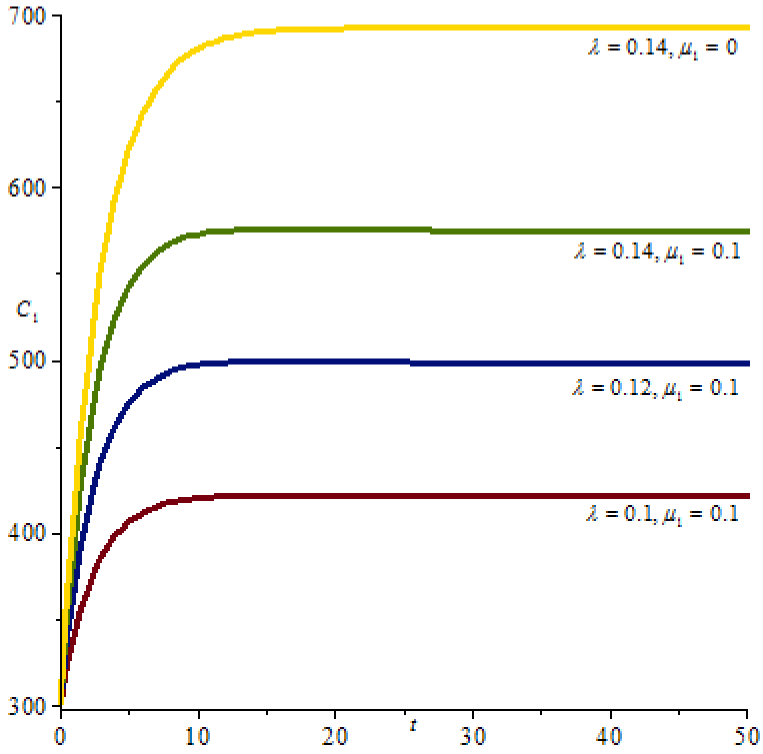

Fig. 4 Variation of C_1 with time t for different values of λ .

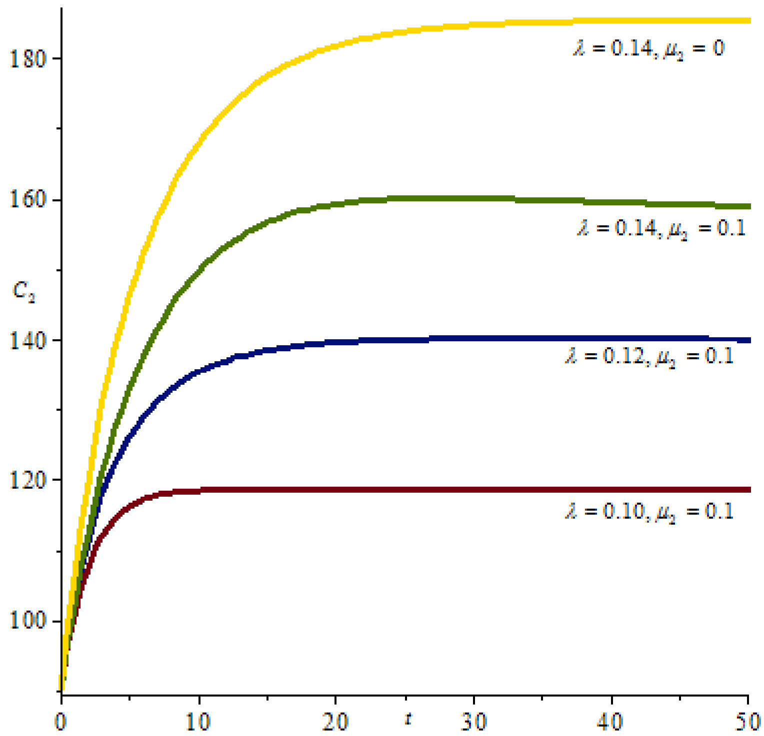

Fig. 5 Variation of C_2 with time t for different values of λ .

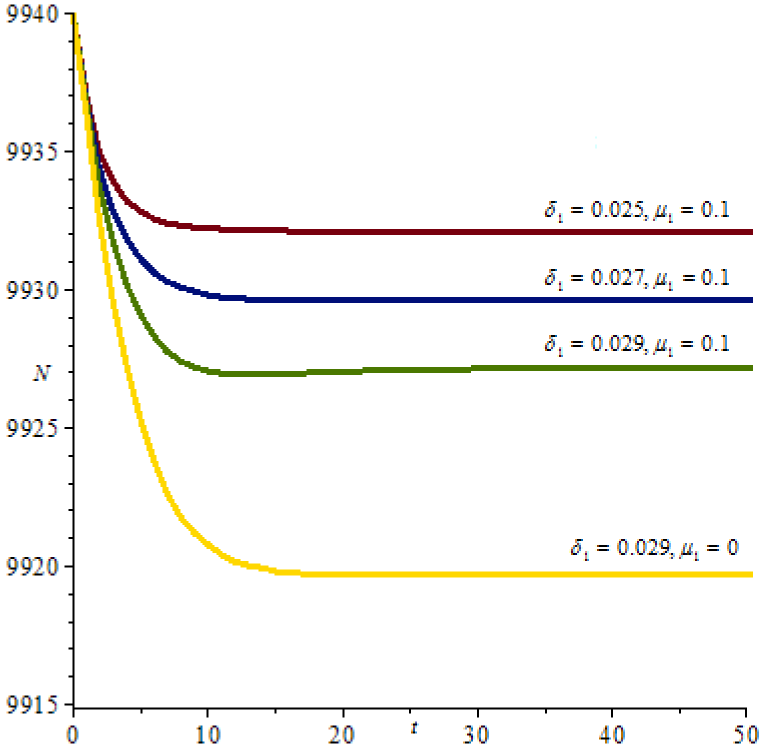

Fig. 6 Variation of N with time t for different values of δ_1.

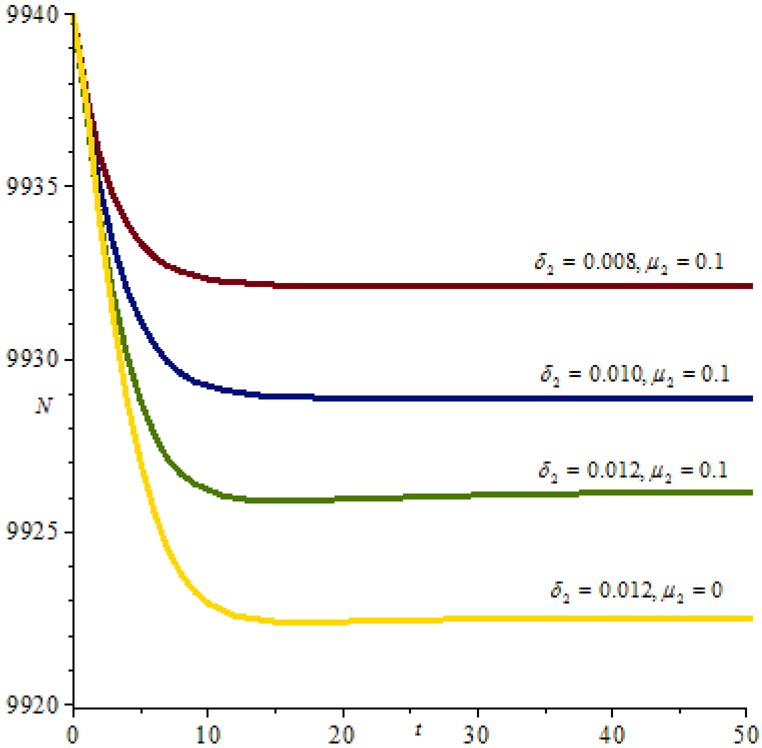

Fig. 7 Variation of N with time t for different values of δ_2.

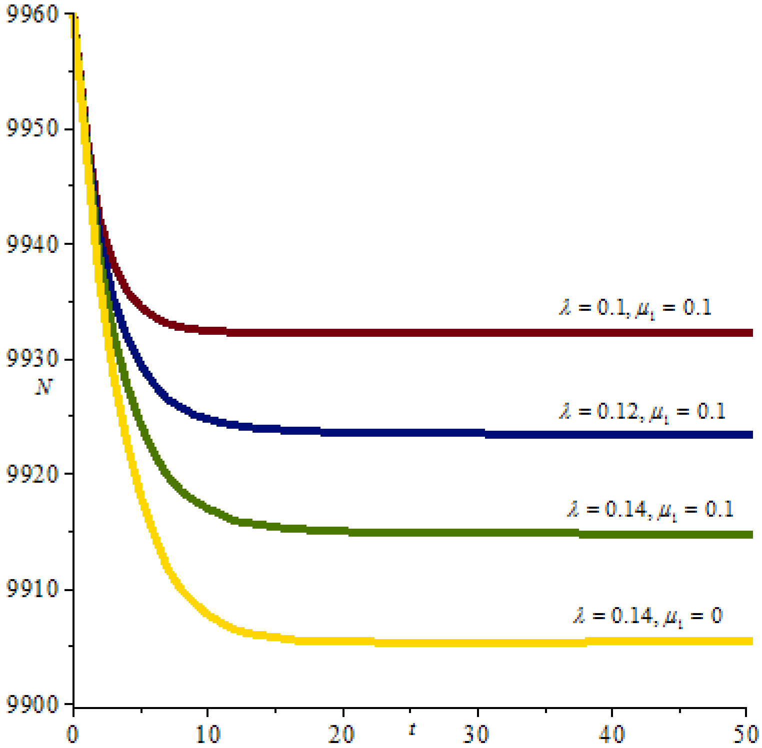

Fig. 8 Variation of N with time t for different values of λ.

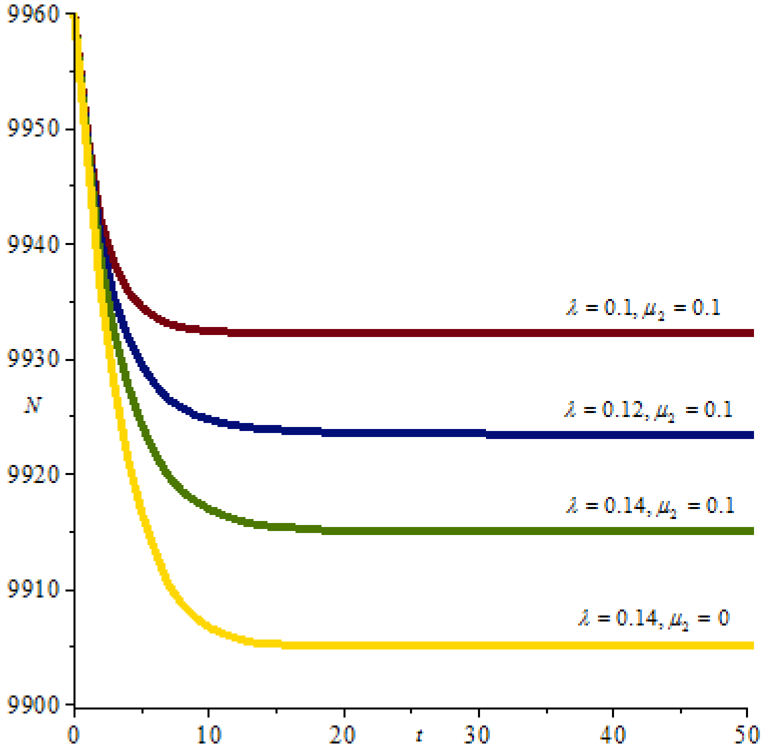

Fig. 9 Variation of N with time t for different values of λ.

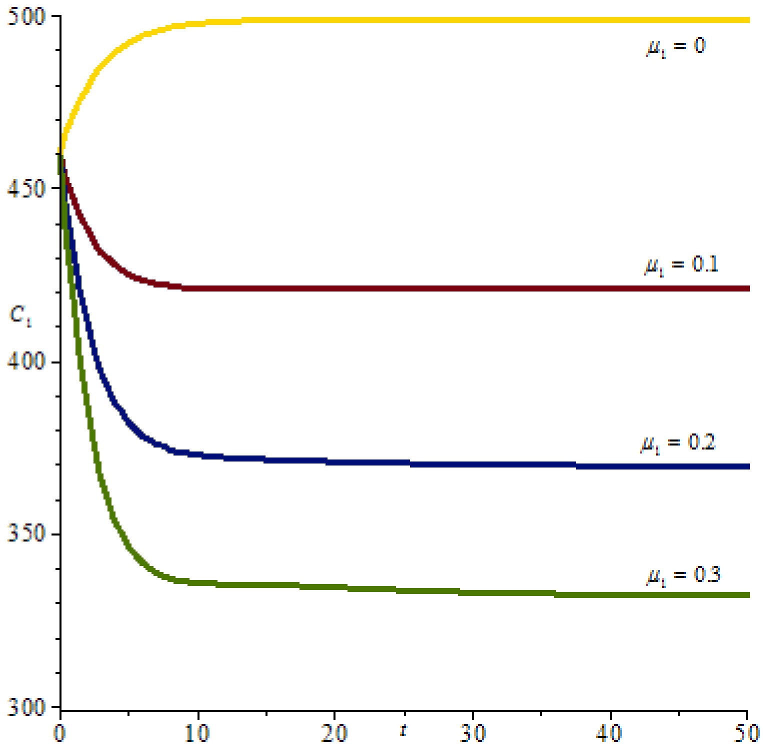

Fig. 10 Variation of C_1 with time t for different values of μ_1.

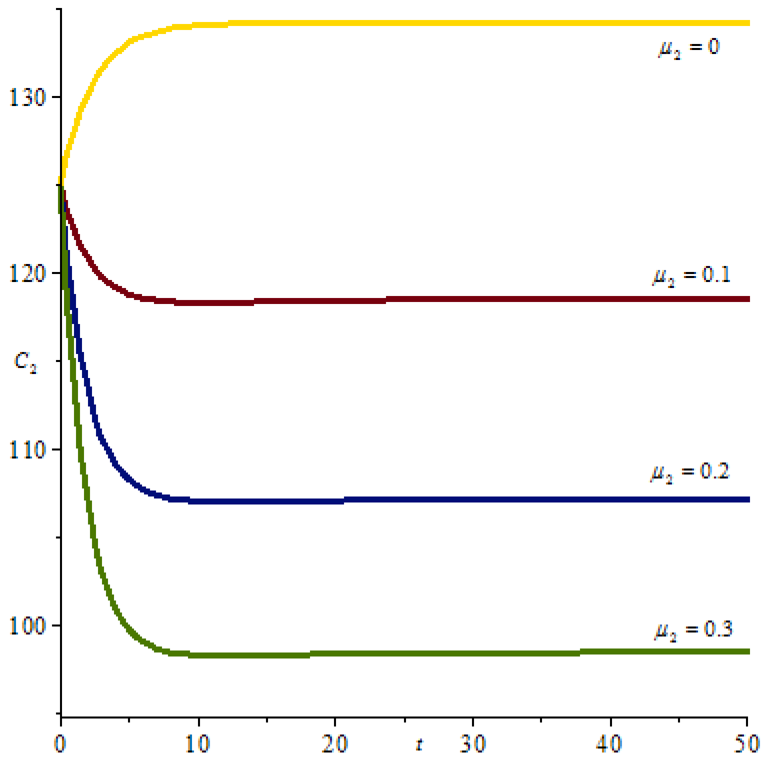

Fig. 11 Variation of C_2 with time t for different values of μ_2.

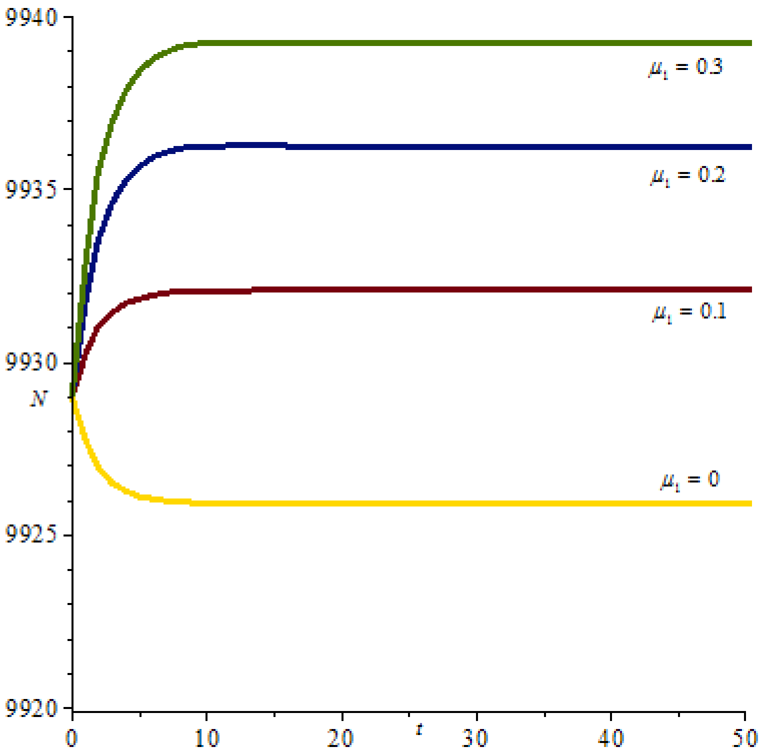

Fig. 12 Variation of N with time t for different values of μ_1.

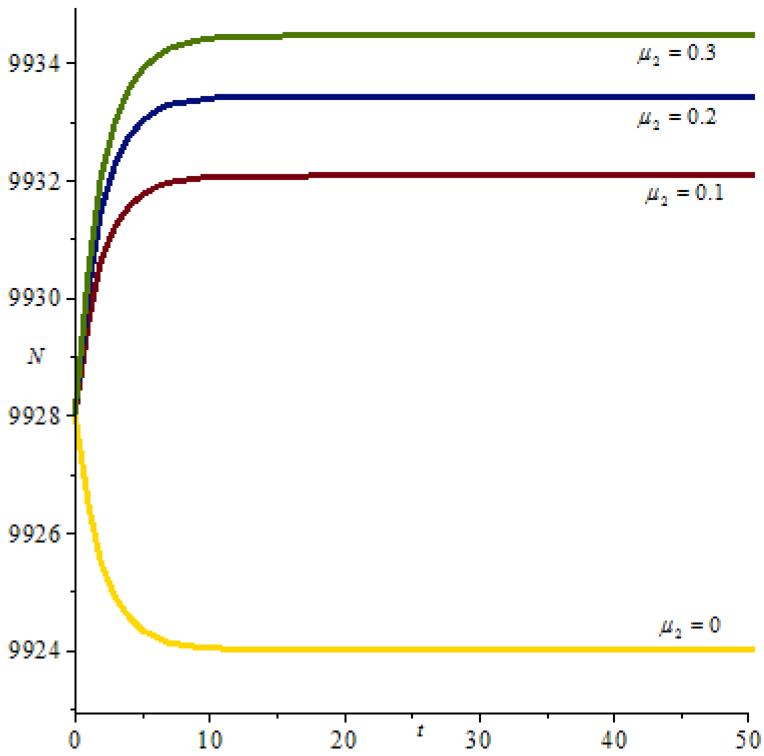

Fig. 13 Variation of N with time t for different values of μ_2.

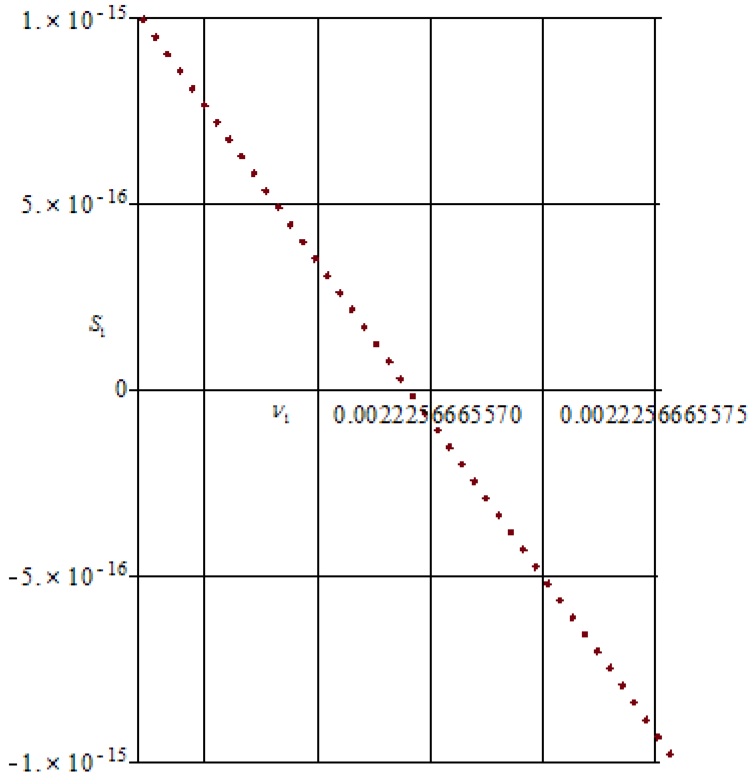

Fig. 14 Variation of stability condition S_1 with time ν_1.

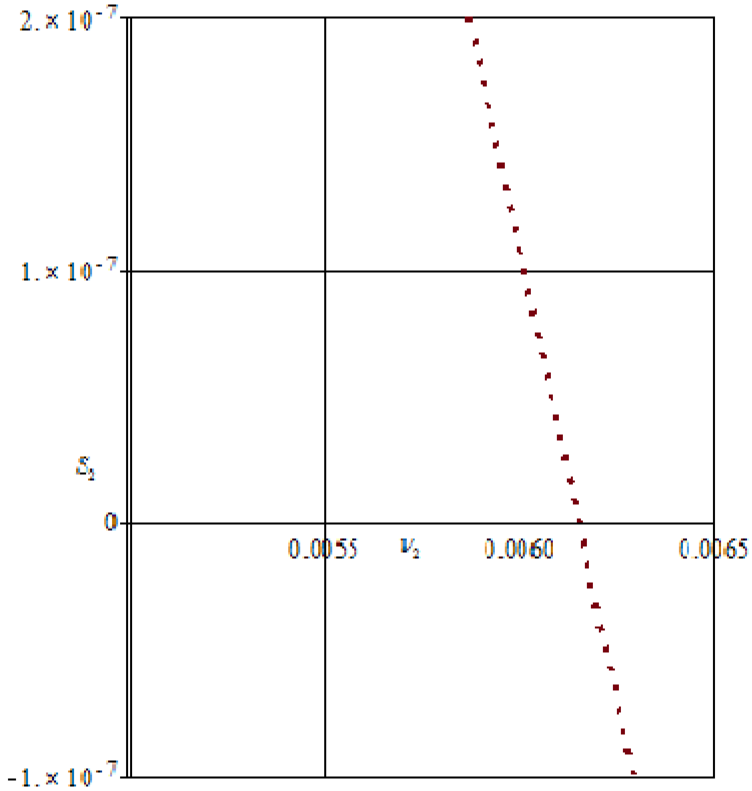

Fig. 15 Variation of stability condition S_2 with time ν_2.

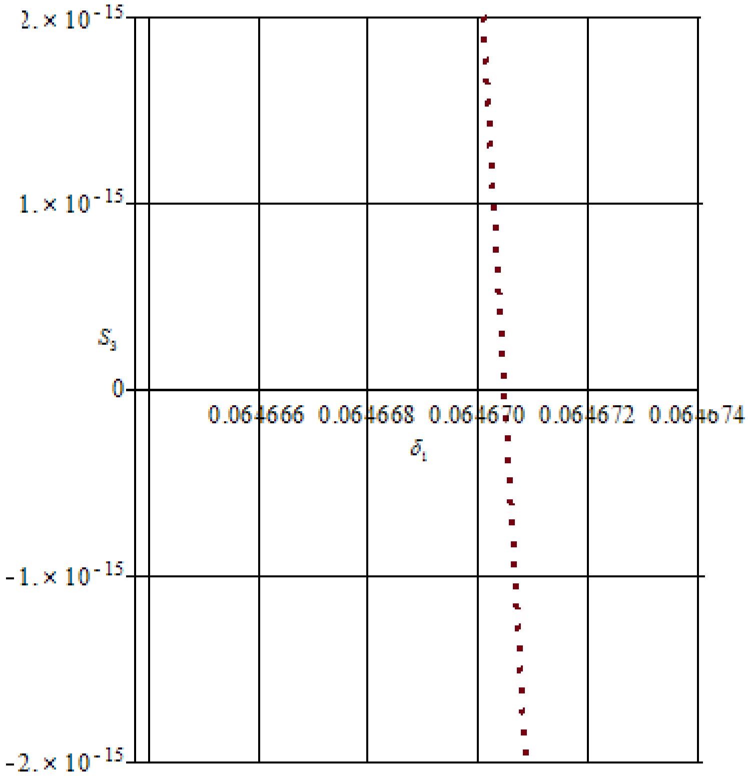

Fig. 16 Variation of stability condition S_3 with time.

5 Conclusion

In this paper, a nonlinear mathematical model has been proposed to study the effects of gaseous pollutants and particulate matters on human population in a city caused by vehicular traffic and other pollutants emitting sources with and without a control mechanism. Two external species such as aerosols and water have been introduced into the atmosphere to remove gaseous pollutants and particulate matters as control mechanisms. In the modeling process, six variables have been considered, namely, the cumulative density of human population, the various pollution emitting sources etc., the cumulative concentration of gaseous pollutants, the cumulative concentration of particulate matters, the cumulative concentration of suitable aerosols to remove gaseous pollutants from the atmosphere and the number density of water drops sprayed in the atmosphere to remove particulate matters from the atmosphere. The concentrations of gaseous pollutants and particulate matters are assumed to be dependent on the cumulative density of various pollutants emitting sources in the city. The cumulative density of pollutants emitting sources has been assumed to be proportional to human population density which follows logistic model. The growth rate of human population density decreases due to concentrations of gaseous pollutants and particulate matters. The rates of introduction of aerosols and water for removal of gaseous pollutants and particulate matters have been assumed to be directly proportional to the cumulative concentrations of gaseous pollutants and particulate matters respectively. The model has been proposed in the form of nonlinear ordinary differential equations which has been analyzed by using the theory of stability under two situations *viz* without control mechanism and with control mechanism. The existence of interior equilibrium has been established and its local as well as nonlinear stability have been studied in both the cases. It has been shown that as the cumulative concentrations of gaseous pollutants and particulate matters increase in the atmosphere, the human population density decreases in both the cases but the equilibrium level of population density is higher in the case when control mechanism is used. The study implies that it is absolutely necessary to apply control mechanisms to control emission from traffic and other pollutant emitting sources in a city to protect human health.

Appendix A

Proof of Theorem 3.1.1

To establish the local stability of $\widetilde{E}$, let us consider the following positive definite function,

$$V = \frac{1}{2}m_1 N_1^2 + \frac{1}{2}m_2 S_1^2 + \frac{1}{2}m_3 C_{11}^2 + \frac{1}{2}m_4 C_{22}^2 \tag{A1}$$

where N_1, S_1, C_{11}, C_{22} are the small perturbations about $\widetilde{E}(\widetilde{N},\widetilde{S},\widetilde{C}_1,\widetilde{C}_2)$ described as $N = \widetilde{N} + N_1$, $S = \widetilde{S} + S_1$, $C_1 = \widetilde{C}_1 + C_{11}$, $C_2 = \widetilde{C}_2 + C_{22}$ and $m_i\ (i = 1,2,3,4)$ are positive constants to be chosen appropriately.

Differentiating (A1) with respect to t we get

$$\frac{dV}{dt} = m_1 N_1 \frac{dN_1}{dt} + m_2 S_1 \frac{dS}{dt} + m_3 C_{11} \frac{dC_{11}}{dt} + m_4 C_{22} \frac{dC_{22}}{dt} \tag{A2}$$

The linearized system corresponding to $\widetilde{E}(\widetilde{N},\widetilde{S},\widetilde{C}_1,\widetilde{C}_2)$ is given as,

$$\begin{bmatrix} \dot{N}_1 \\ \dot{S}_1 \\ \dot{C}_{11} \\ \dot{C}_{22} \end{bmatrix} = \begin{bmatrix} -\frac{r\widetilde{N}}{K} & 0 & -d_1\widetilde{N} & -d_2\widetilde{N} \\ \lambda & -\lambda_0 & 0 & 0 \\ -\nu_1\widetilde{C}_1 & \delta_1 & -(\delta_{10} + \nu_1\widetilde{N}) & 0 \\ -\nu_2\widetilde{C}_2 + \delta_{21} & \delta_2 & 0 & -(\delta_{20} + \nu_2\widetilde{N}) \end{bmatrix} \begin{bmatrix} N_1 \\ S_1 \\ C_{11} \\ C_{22} \end{bmatrix}$$

Using above linearized system in equation (A2) and on simplification we have,

$$\frac{dV}{dt} = -m_1\frac{r\widetilde{N}}{K}N_1^2 - m_2\lambda_0 S_1^2 - m_3(\delta_{10} + \nu_1\widetilde{N})C_{11}^2 - m_4(\delta_{20} + \nu_2\widetilde{N})C_{22}^2 + m_2\lambda N_1 S_1$$
$$- (m_1 d_1\widetilde{N} + m_3\nu_1\widetilde{C}_1)N_1 C_{11} - (m_1 d_2\widetilde{N} - m_4(\delta_{21} - \nu_2\widetilde{C}_2))N_1 C_{22} + m_3\delta_1 S_1 C_{11} + m_4\delta_2 S_1 C_{22} \tag{A3}$$

where $m_4 = \dfrac{d_2\widetilde{N}}{\delta_{21} - \nu_2\widetilde{C}_2}$ is chosen

In above equation, $\dfrac{dV}{dt}$ will be negative definite if the following conditions are satisfied

$$(m_2\lambda)^2 < \frac{4}{9}m_2\lambda_0 m_1\frac{r\widetilde{N}}{K} \tag{A4}$$

$$(m_1 d_1\widetilde{N} + m_3\nu_1\widetilde{C}_1)^2 < \frac{2}{3}m_3(\delta_{10} + \nu_1\widetilde{N})m_1\frac{r\widetilde{N}}{K} \tag{A5}$$

$$(m_3\delta_1)^2 < \frac{2}{3}m_3(\delta_{10} + \nu_1\widetilde{N})m_2\lambda_0 \tag{A6}$$

$$(m_4\delta_2)^2 < \frac{2}{3}m_4(\delta_{20} + \nu_2\widetilde{N})m_2\lambda_0 \tag{A7}$$

After performing some algebraic calculations and choosing,

$$m_1 = 1,\ m_2 < \frac{4}{9}\frac{r\tilde{N}}{K}\frac{\lambda_0}{\lambda^2}\ m_3 = \frac{d_1\tilde{N}}{\nu_1\tilde{C}_1}$$

$\frac{dV}{dt}$ is a negative definite under the conditions (25) - (27).

Appendix B

Proof of Theorem 3.1.2

To establish the nonlinear stability of $\tilde{E}$, let us consider the following positive definite function,

$$U = k_1\left(N - \tilde{N} - \tilde{N}\log\frac{N}{\tilde{N}}\right) + \frac{1}{2}k_2(S-\tilde{S})^2 + \frac{1}{2}k_3(C_1-\tilde{C}_1)^2 + \frac{1}{2}k_4(C_2-\tilde{C}_2)^2 \tag{B1}$$

where $k_i\ (i = 1,2,3)$ are positive constants to be chosen appropriately.

Differentiating above equation with respect to t we get

$$\frac{dU}{dt} = -k_1\frac{r}{K}(N-\tilde{N})^2 - k_2\lambda_0(S-\tilde{S})^2 - k_3(\delta_{10}+\nu_1 N)(C_1-\tilde{C}_1)^2 - k_4(\delta_{20}+\nu_2 N)(C_2-\tilde{C}_2)^2$$

$$+k_2\lambda(N-\tilde{N})(S-\tilde{S}) - (k_1 d_1 + k_3\nu_1\tilde{C}_1)(N-\tilde{N})(C_1-\tilde{C}_1) - (k_1 d_2 - k_4(\delta_{21}-\nu_2\tilde{C}_2))(N-\tilde{N})(C_2-\tilde{C}_2)$$

$$+k_3\delta_1(S-\tilde{S})(C_1-\tilde{C}_1) + k_4\delta_2(S-\tilde{S})(C_2-\tilde{C}_2) \tag{B2}$$

where $k_4 = \frac{d_2}{\delta_{21} - \nu_2\tilde{C}_2}$

In the above equation, $\frac{dU}{dt}$ will be negative if the following conditions are satisfied,

$$(k_2\lambda)^2 < \frac{4}{9}k_1 k_2 \lambda_0 \frac{r}{K} \tag{B3}$$

$$(k_1 d_1 + k_3\nu_1\tilde{C}_1)^2 < \frac{2}{3}k_1 k_3 \frac{r}{K}(\delta_{10}+\nu_1 N) \tag{B4}$$

$$(k_3\delta_1)^2 < \frac{2}{3}k_3 k_2(\delta_{10}+\nu_1 N)\lambda_0 \tag{B6}$$

$$(k_4\delta_2)^2 < \frac{2}{3}k_4 k_2(\delta_{20}+\nu_2 N)\lambda_0 \tag{B7}$$

Maximizing the left hand side and minimizing the right hand side and choosing,

$k_1 = 1, k_2 < \frac{4}{9}\frac{r}{K}\frac{\lambda_0}{\lambda^2}, k_3 = \frac{d_1}{\nu_1\tilde{C}_1}$ the above inequalities are satisfied. Thus $\frac{dU}{dt}$ is negative definite provided the conditions (28) - (30) are satisfied inside the region of attraction Ω.

Appendix C

Proof of Theorem 3.2.1

To establish the local stability of $\overline{E}$, let us consider the following positive definite function,

$$W_1 = \frac{1}{2}l_1N_1^2 + \frac{1}{2}l_2S_1^2 + \frac{1}{2}l_3C_{11}^2 + \frac{1}{2}l_4C_{22}^2 + \frac{1}{2}l_5C_{d11}^2 + \frac{1}{2}l_6C_{d22}^2 \quad \text{(C1)}$$

where $N_1, S_1, C_{11}, C_{22}, C_{d11}, C_{d22}$ are the small perturbations about $E^* = (N^*, S^*, C_1^*, C_2^*, C_{d1}^*, C_{d2}^*)$ described as

$N = N^* + N_1$, $S = S^* + S_1$, $C_1 = C_1^* + C_{11}$, $C_2 = C_2^* + C_{22}$, $C_{d1} = C_{d2}^* + C_{d11}$, $C_{d2} = C_{d2}^* + C_{d22}$ and

$l_i\ (i = 1...6)$ are positive constants to be chosen appropriately.

Differentiating (C1) with respect to t we get

$$\frac{dW_1}{dt} = l_1N_1\frac{dN_1}{dt} + l_2S_1\frac{dS_1}{dt} + l_3C_{11}\frac{dC_{11}}{dt} + l_4C_{22}\frac{dC_{22}}{dt} + l_5C_{d11}\frac{dC_{d11}}{dt} + l_6C_{d22}\frac{dC_{d22}}{dt} \quad \text{(C2)}$$

The linearized system corresponding to $E^* = (N^*, S^*, C_1^*, C_2^*, C_{d1}^*, C_{d2}^*)$ is given as,

$$\begin{bmatrix} \dot{N}_1 \\ \dot{S}_1 \\ \dot{C}_{11} \\ \dot{C}_{22} \\ \dot{C}_{d11} \\ \dot{C}_{d22} \end{bmatrix} = \begin{bmatrix} -\frac{rN^*}{K} & 0 & -d_1N^* & -d_2N^* & 0 & 0 \\ \lambda & -\lambda_0 & 0 & 0 & 0 & 0 \\ -\nu_1C_1^* & \delta_1 & -\begin{pmatrix}\delta_{10} + \mu_{11}C_{d1}^* \\ +\nu_1N^*\end{pmatrix} & 0 & -\mu_{11}C_1^* & 0 \\ -\nu_2C_2^* + \delta_{21} & \delta_2 & 0 & -\begin{pmatrix}\delta_{20} + \mu_{22}C_{d2}^* \\ +\nu_2N^*\end{pmatrix} & 0 & -\mu_{22}C_2^* \\ 0 & 0 & \begin{pmatrix}\mu_1 \\ -\mu_{11}C_{d1}^*\end{pmatrix} & 0 & -\begin{pmatrix}\mu_{10} \\ +\mu_{11}C_1^*\end{pmatrix} & 0 \\ 0 & 0 & 0 & \begin{pmatrix}\mu_2 \\ -\mu_{22}C_{d2}^*\end{pmatrix} & 0 & -\begin{pmatrix}\mu_{20} \\ +\mu_{22}C_2^*\end{pmatrix} \end{bmatrix} \begin{bmatrix} N_1 \\ S_1 \\ C_{11} \\ C_{22} \\ C_{d11} \\ C_{d22} \end{bmatrix}$$

Using above linearized system in equation (C2) and on simplification we have,

$$\begin{aligned} \frac{dW_1}{dt} = & -l_1\frac{rN^*}{K}N_1^2 - l_2\lambda_0S_1^2 - l_3(\delta_{10} + \mu_{11}C_{d1}^* + \nu_1N^*)C_{11}^2 - l_4(\delta_{20} + \mu_{22}C_{d2}^*\nu_2N^*)C_{22}^2 \\ & - l_5(\mu_{10} + \mu_{11}C_1^*)C_{d1}^2 - l_6(\mu_{20} + \mu_{22}C_2^*)C_{d2}^2 + l_2\lambda N_1S_1 - (l_3\nu_1C_1^* + l_1d_1N^*)N_1C_{11} \\ & - (l_1d_2N^* - l_4(\nu_2C_2^* - \delta_{21}))N_1C_{22} + l_3\delta_1S_1C_{11} + l_4\delta_2S_1C_{22} \\ & + ((l_5(\mu_1 - \mu_{11}C_{d1}^*) - l_3\mu_{11}C_1^*)C_{11}C_{d11} + ((l_6(\mu_2 - \mu_{22}C_{d2}^*) - l_4\mu_{22}C_2^*)C_{22}C_{d22} \end{aligned} \quad \text{(C3)}$$

Now after choosing $l_5 = \frac{\mu_{11}C_1^*}{\mu_1 - \mu_{11}C_{d1}^*}l_3$, $l_6 = \frac{\mu_{22}C_2^*}{\mu_2 - \mu_{22}C_{d2}^*}l_4$, $l_4 = \frac{d_2N^*}{\delta_{21} - \nu_2C_2^*}$ from equation (C3), $\frac{dW_1}{dt}$ is negative definite if the following conditions are satisfied,

$$(l_2\lambda)^2 < \frac{4}{9}l_1l_2\frac{rN^*}{K}\lambda_0 \quad \text{(C4)}$$

$$(l_3\nu_1C_1^* + l_1d_1N^*)^2 < \frac{2}{3}l_1l_3\frac{rN^*}{K}(\delta_{10} + \nu_1N^* + \mu_{11}C_{d1}^*) \quad \text{(C5)}$$

$$(l_3\delta_1)^2 < \frac{2}{3} l_2 l_3 \lambda_0 (\delta_{10} + \mu_{11} C_{d1}^* + \nu_1 N^*) \tag{C6}$$

$$(l_4\delta_2)^2 < \frac{2}{3} l_2 l_4 \lambda_0 (\delta_{20} + \nu_2 N^* + \mu_{22} C_{d2}^*) \tag{C7}$$

After performing some algebraic manipulations and choosing

$$l_1 = 1, l_2 < \frac{4}{9} \frac{rN^*}{K} \frac{\lambda_0}{\lambda^2}, l_3 = \frac{d_1 N^*}{\nu_1 C_1^*}$$

$\frac{dW_1}{dt}$ will be negative definite under the conditions (48) – (50)

Appendix D

Proof of Theorem 3.2.2

To establish the nonlinear stability of $\widetilde{E}$, let us consider the following positive definite function,

$$W_2 = p_1\left(N - N^* - N^* \log\frac{N}{N^*}\right) + \frac{1}{2} p_2 (S - S^*)^2 + \frac{1}{2} p_3 (C_1 - C_1^*)^2 + \frac{1}{2} p_4 (C_2 - C_2^*)^2$$
$$+ \frac{1}{2} p_5 (C_{d1} - C_{d1}^*)^2 + \frac{1}{2} p_6 (C_{d2} - C_{d2}^*)^2 \tag{D1}$$

where p_i $(i = 1...6)$ are positive constants to be chosen appropriately.

Differentiating above equation with respect to t we get,

$$\frac{dW_2}{dt} = -p_1 \frac{r}{K}(N - N^*)^2 - p_2 \lambda_0 (S - S^*)^2 - p_3 (\delta_{10} + \nu_1 N + \mu_{11} C_{d1})(C_1 - C_1^*)^2$$
$$- p_4 (\delta_{20} + \nu_2 N + \mu_{22} C_{d2})(C_2 - C_2^*)^2 - p_5 (\mu_{10} + \mu_{11} C_1)(C_{d1} - C_{d1}^*)^2$$
$$- p_6 (\mu_{20} + \mu_{22} C_2)(C_{d2} - C_{d2}^*)^2 + p_2 \lambda (N - N^*)(S - S^*)$$
$$- (p_1 d_1 + p_3 \nu_1 C_1^*)(N - N^*)(C_1 - C_1^*) - (p_1 d_2 + p_4 (\nu_2 C_2^* - \delta_{21}))(N - N^*)(C_2 - C_2^*)$$
$$+ p_3 \delta_1 (S - S^*)(C_1 - C_1^*) + p_4 \delta_2 (S - S^*)(C_2 - C_2^*)$$
$$+ (-p_3 \mu_{11} C_1^* + p_5 (\mu_1 - \mu_{11} C_{d1}^*))C_1 - C_1^*)(C_{d1} - C_{d1}^*)$$
$$+ (-p_4 \mu_{22} C_2^* + p_6 (\mu_2 - \mu_{22} C_{d2}^*))(C_2 - C_2^*)(C_{d2} - C_{d2}^*) \tag{D2}$$

Now after choosing $p_5 = \frac{\mu_{11} C_1^*}{(\mu_1 - \mu_{11} C_{d1}^*)} p_3$, $p_6 = \frac{\mu_{22} C_2^*}{(\mu_2 - \mu_{22} C_{d2}^*)} p_4$, $p_4 = \frac{d_2}{\delta_{21} - C_2^* \nu_2}$ in the above equation,

$\frac{dW_2}{dt}$ will be negative if the following conditions are satisfied,

$$(p_2 \lambda)^2 < \frac{4}{9} p_1 p_2 \frac{r}{K} \lambda_0 \tag{D3}$$

$$(p_1 d_1 + p_3 \nu_1 C_1^*)^2 < \frac{4}{6} p_1 p_3 \frac{r}{K}(\delta_{10} + \nu_1 N + \mu_{11} C_{d1}) \tag{D4}$$

$$(p_3 \delta_1)^2 < \frac{4}{6} p_2 p_3 \lambda_0 (\delta_{10} + \nu_1 N + \mu_{11} C_{d1}) \tag{D5}$$

$$(p_4 \delta_2)^2 < \frac{4}{6} p_2 p_4 \lambda_0 (\delta_{20} + \nu_2 N + \mu_{22} C_{d2}) \tag{D6}$$

Maximising left hand side and minimizing right hand side and choosing,

$p_1 = 1$, $p_2 < \frac{4}{9}\frac{r\lambda_0}{K\lambda^2}$ $p_3 = \frac{d_1}{C_1^* \nu_1}$ the above inequalities are satisfied. Thus $\frac{dW_2}{dt}$ is negative definite provided

the conditions (51) - (53) are satisfied inside the region of attraction Ω showing that W_2 is Lyapunov function. This proves the theorem.

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