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Dynamic analysis of a Leslie-Gower model with additive Allee effect on the prey population and predator harvesting including stochastic effect on each population

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Abstract
In this study, we proposed a Leslie-Gower prey-predator model, whose dynamics includes a constant effort harvesting rate in predators and an additive Allee impact on prey. A system of stochastic differential equations is also used to study its behaviour, with the assumption that each population's exposure to environmental unpredictability is represented by noise terms. This kind of randomness is considerably more reasonable and realistic in the proposed paradigm. Due to the paucity of studies on the dynamics of this kind of model, this investigation is being believed to be a way to advance the subject of literature. First, we establish the system's positivity and boundlessness. Next, we look into the dynamics of every one of the stable states, the form of the positive equilibrium point, and the continued existence of every species in the system.It is established that the equilibrium levels of prey and predator are impacted by the Allee effect parameter as well as the impact of harvesting. The positive steady state point's global stability criterion is derived. By selecting the Allee effect and harvesting effort as the bifurcation parameters, it has been established that a Hopf bifurcation exists close to the interior steady state.This study is novel since it incorporates various ecological factors into a single model, potentially opening up new perspectives on predator-prey relationships. To support the mathematical conclusions, rigorous numerical visualisations of the key parameters are provided below using specific hypothetical data. In conclusion, we may state that our model is a project that aims to preserve the ecological equilibrium of the natural world.

Keywords Leslie-Gower scheme; Allee effect; harvesting; stochastic effect; stability; Hopf bifurcation.

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1 Introduction

The dynamic interactions among distinct populations and related intricate characteristics have received a lot of interest from biologists and ecologists. Predator-prey dynamics study has received constant attention ever since Lotka and Volterra's pioneering study.These models can, as is well known, accurately reflect fluctuations in population quantities. The pertinent theories in this area have made significant strides in recent years (Comins and Blatt, 1974; Xiao and Chen, 2001; Lu et al., 2003; Agiza et al., 2009; Chakraborty et al., 2012) by becoming more and more complete. Leslie-Gower (Leslie, 1948) proposed an additional component to the predator-prey scheme, referring to the logistic rule-based development of the connected population and the relationship between the number of prey and the predator's carrying capacity. When prey is scarce, predators can relocate to find different sources of food, but this will limit their ability to evolve because their preferred prey is not accessible. The authors of (Aziz-Alaoui and Okiye, 2003) proposed an enhanced Leslie-Gower model to address this issue by including a positive number that measures the predator's environmental defence. Since then, the Leslie-Gower model has been studied by a number of authors with various functional responses (Altendorf et al., 2001; Korobeinikov, 2001; Yang et al., 2008; Yu, 2012, 2014; Singh and Gakkhar, 2014; Feng and Kang, 2015; Xu et al., 2020; Liu and Huang, 2020).

In the fields of fisheries, forestry, and regulating wildlife, the utilization of ecological resources and the harvesting of species are standard practices. Thus, the behavior of the prey-predator scheme are giving harvesting models (Wei et al., 2017; Pal and Mahapatra., 2014, 2018a-b; Asfaw et al., 2018; Liu et al., 2020). It is significant to remember that harvesting goes on well beyond the population's extinction. Three approaches to harvesting are possible: (a) a constant-yield $H\left(n_2\right)=$ constant, (b) a constant-effort $H\left(n_2\right)=\beta E n_2$ or (c) a Michaelis–Menten type $H\left(n_2\right)=\dfrac{\beta E n_2}{cE+\ln_2}$.

The Allee phenomenon is another modern propensity to employ these kinds of models. The Allee effect is a term used to describe how population growth at low densities depends on positive density (Sen et al., 2012; Wei and Chen, 2014; Bodine and Yust, 2017; Hastings et al., 2017; Elaydi et al., 2018; Manna and Banerjee, 2018; Sen et al., 2018; Du et al., 2019). It could be caused by a wide variety of biological processes, including lowered predator alertness, genetic trends, difficult mating, and inadequate nutrition at low population densities (Allee, 1931; Odum and Allee, 1954). According to Pal and Mandal's (Pal and Mandal, 2014) investigation using the improved Leslie-Gower predator-prey system, the increasing function of the prey population was regulated by a multiplicative strong Allee effect, usingthe Beddington-DeAngelis response functional (Pal et al., 2018; Majhi and Mandal, 2019). The modified Leslie-Gower predator-prey scheme that includes the additive Allee effect was explored by Cai et al. (2015) by applying the Holling type II functional response. They made some fascinating observations, one of which is the possibility of ecological extinction being raised by the Allee influence.

Inspired by the existing literature, it makes sense to model and analyse how a changing environment affects a predator-prey interaction, allowing both sides to have an additive Allee effect on the prey advancement function with a constant rate of effort harvesting in predators. This would undoubtedly be an interesting area of study. But no one has yet explored into this region.

The article is organised as follows.Boundedness and positivity of the scheme's solutions are established in Section 2 of the paper, where the mathematical model is presented. Section 3 describes the presence of equilibrium and stability. Section 4 develops an investigation for global stability and Hopf bifurcation. Section 5 formulates the model in a stochastic fashion and derives a stochastic stability condition. Computer simulations of several system solutions are shown in Section 6 to support our analytical conclusions. The full scope of the paper is discussed in Section 7, as well as the biological importance of our analytical results.

2 Mathematical Formulation of Proposed System

In order to develop the Leslie-Gower schemes, coupled, nonlinear ordinary differential equations are used. These equations show how prey and predators interact in the following way:

$$\frac{dn_1}{dt} = \rho n_1 \left(1 - \frac{n_1}{\delta}\right) - \alpha\, n_1\, n_2,$$

$$\frac{dn_2}{dt} = \gamma\, n_2 \left(1 - \frac{n_2}{b\, n_1}\right) \tag{1}$$

with $n_1(0) \geq 0, n_2(0) \geq 0.$

It is expected that prey will logistically advance and have a linear growth in the consumption ratio with food saturation, and $\alpha > 0$ signifiesan invasionquantity. In addition, $\frac{dn_2}{dt} = \gamma\, n_2 \left(1 - \frac{n_2}{b\, n_1}\right)$ indicates the logistic growth of the predator, but the usual δ_1, which evaluatesthe carrying capacity based on the ecosystem's resources and is $\delta_1 = b\, n_1$, proportionate to the quantity of prey (b is the ratio of prey to predators). In the given equation, the Lesie-Gower factor isindicatedas $n_2 / b\, n_1$. It evaluates the decrease in thenumber of predatorscaused by the low population density (per capita n_2 / n_1) of the prey. In case there is a significant unavailability, n_2 is able to look out alternate species, however this will restrict its progresssince its favourite food (n_1) cannot beabundantly available.This issue can be resolved by introducing a positive quantity into the denominator (Table 1). The model is thus known as the improved Leslie-Gower scheme. For the sake of this study, we only consider the improved Leslie-Gower scheme incorporating the additive Allee impact $\frac{\theta}{n_1 + A}$ on the prey development function and a constant effort harvesting rate $\beta\, E\, n_2$ in predators, which is of the form

$$\frac{dn_1}{dt} = \rho n_1 \left(1 - \frac{n_1}{\delta} - \frac{\theta}{n_1 + A}\right) - \alpha\, n_1\, n_2,$$

$$\frac{dn_2}{dt} = \gamma\, n_2 \left(1 - \frac{n_2}{b\, n_1 + \omega}\right) - \beta\, E\, n_2 \tag{2}$$

$n_1(0) \geq 0, n_2(0) \geq 0.$

Table 1 Biological meaning of the parameters.

Parameter	Description
$n_1(t)$	Amount of prey population
$n_2(t)$	Amount of predator population
ρ	Intrinsic growth rate of prey
γ	Intrinsic growth rate of predator
δ	Carrying capacity
α	Predator's capture/attack level
b	The progress of predators in relation to the effectiveness of prey use
ω	Alternative food that might be present in the environment for newborns
β	The harvest coefficient
E	The harvest effort

Here, θ and A explore the Allee effect'sefficiency.If criteria $0 < A < \theta$ and $0 < \theta < A$ aremet, the Allee effect is strong; otherwise, it is weak. In terms of biology, a high Allee effect means that advantages are outweighed by losses due to low levels of overcrowding, whereas a weak Allee effect does not meet this condition (Gonza´lez-Olivares et al., 2006). The additive term $\frac{\theta}{n_1 + A}$ was chosen since it is most straightforward and providesa combined strong and weak Allee effect.

Let $n_1(t) \geq 0$ and $n_2(t) \geq 0$ indicates the quantity of the prey and predator populations, correspondingly. These populations behaviours are given using system (2) and are specified in the area of biological sense $\Lambda = \left\{(n_1, n_2) \in \mathbb{R}_+^2 : 0 \leq n_1 \leq \delta, 0 \leq n_2 \leq b\delta + \omega\right\}$.

It is essential to confirm that the scheme (2) is mathematically valid biologically realistic, and that its trajectories ϕ_n are uniformly bounded prior to doing a mathematical study to identify potential behaviours in the changing nature of prey and predators.

Theorem 1. The trajectory ϕ_n of the scheme (2) is unique, along with the initial condition $(n_1(0), n_2(0)) \in \Lambda.$ Furthermore, Λ is unchangeable.

Proof. The continuity of the vector field in scheme (2) ensures the fulfilment of both the existence and uniqueness conditions for the trajectory ϕ_n. However, it needs to be ensured that the model (2)'s ϕ_n trajectories do not deviate from Λ. This is accomplished by analysing how the model's (2) boundary trajectory changes.In fact, if $n_1 = 0$, then $\frac{dn_1}{dt} = 0$ for any $n_2 \geq 0$. Similarly, if $n_1 = \delta$ and $\frac{dn_1}{dt} = -\rho\,\delta \frac{\theta}{\delta + A} - \alpha\,\delta\, n_2 \leq 0$ for all $n_2 \geq 0$. On the other hand, $n_2 = 0$, then $\frac{dn_2}{dt} = 0$ for all $n_1 \geq 0$. If $n_2 = b\delta + \omega$ and $n_1 = \delta$, we have $\frac{dn_2}{dt} = -\beta\, E\, n_2$ each $n_2 \geq 0$. As a result, the trajectories ϕ_n stay within Λ.

Theorem 2. The trajectories ϕ_n of the scheme (2) are uniformly bounded.

Proof. Since $\frac{dn_1}{dt} = \rho n_1\left(1 - \frac{n_1}{\delta} - \frac{\theta}{n_1 + A}\right) - \alpha\, n_1\, n_2 \leq \rho n_1\left(1 - \frac{n_1}{\delta}\right)$,

that is, $n_1 \leq \frac{\delta n_1(0)e^{\rho t}}{\delta + n_1(0)\left(e^{\rho t} - 1\right)}$,

with $0 \leq n_1(0) \leq \delta$ the initial condition of the solution $n_1(t)$, then $n_1(t) \leq \delta$ for all $t \geq 0$.

Similarly,

$$\frac{dn_2}{dt} = \gamma\, n_2\left(1 - \frac{n_2}{b\, n_1 + \omega}\right) - \beta\, E\, n_2 \leq \gamma\, n_2\left(1 - \frac{n_2}{b\, n_1 + \omega}\right),$$

and $n_2(t) \leq L,$ where $L \leq b\,\delta + \omega.$

Thus, when taking into account the function $\psi(t): \mathbb{R}_+ \to \mathbb{R}_+$,

$\psi(t) = n_1(t) + n_2(t)$ then $0 \leq \psi(t) \leq \delta + L$ and the trajectories ϕ_n of the scheme (2) become uniformly bounded.

3 Local and Global Stability at Equilibrium Points

The following criteria must be satisfied for system (2) prior to computing and establishing the local and global stability of the potential equilibrium points.

(a) The trivial equilibrium $E_0 = (0,\ 0)$ is always feasible.

(b) The axial equilibria are $E^{\pm}=\left(n_1^{\pm},\,0\right)$, where $n_1^{\pm}=\dfrac{(\delta-A)\pm\sqrt{(\delta-A)^2-4\delta(\theta-A)}}{2}$. $E^{\pm}$ are feasible if $\delta>A$ and $(\delta+A)^2>4\delta A$.

(c) The coexistence equilibrium point $E^{*}=\left(n_1^{*},\,n_2^{*}\right)$ is feasible if $A_1^2-4A_2A_0>0$, $-\dfrac{A_1}{A_2}>0$ and $\dfrac{A_0}{A_2}>0$.

2nd equation of (2) implies $\gamma\left(1-\dfrac{n_2}{bn_1+\omega}\right)-\beta E=0$

$$\left(1-\frac{n_2}{bn_1+\omega}\right)=\frac{\beta E}{\gamma}\Rightarrow 1-\frac{\beta E}{\gamma}=\frac{n_2}{bn_1+\omega}$$

$$\Rightarrow\frac{(\gamma-\beta E)}{\gamma}=\frac{n_2}{bn_1+\omega}\Rightarrow n_2^*=\left(bn_1^*+\omega\right)\frac{(\gamma-\beta E)}{\gamma}\qquad(3)$$

1st equation of (2) implies $\rho\left(1-\dfrac{n_1^*}{\delta}-\dfrac{\theta}{n_1^*+A}\right)-\alpha\,n_2^*=0\Rightarrow n_2^*=\dfrac{\rho}{\alpha}\left(1-\dfrac{n_1^*}{\delta}-\dfrac{\theta}{n_1^*+A}\right)$ (4)

From (3) and (4), we have

$$\left(bn_1^*+\omega\right)\frac{(\gamma-\beta E)}{\gamma}=\frac{\rho}{\alpha}\left(1-\frac{n_1^*}{\delta}-\frac{\theta}{n_1^*+A}\right)$$

n_1^* is the positive root of the quadratic equation

$$A_2n_1^{*2}+A_1n_1^*+A_0=0\qquad(5)$$

where

$$A_2=\alpha\delta b(\gamma-\beta E)+\gamma\rho$$

$$A_1=\alpha\delta(bA+\omega)(\gamma-\beta E)-(\delta-A)\gamma\rho$$

$$A_0=\alpha\delta\omega A(\gamma-\beta E)-(A-\theta)\gamma\rho\delta$$

For (5) has positive equilibrium points if and only if $A_1^2-4A_2A_0>0,-\dfrac{A_1}{A_2}>0$ and $\dfrac{A_0}{A_2}>0$.

The variational matrix of the scheme (2) at $E_0=(0,\,0)$ as follows

$$J=\begin{pmatrix} \rho-\dfrac{2\rho n_1}{\delta}-\dfrac{A\rho\theta}{(n_1+A)^2}-\alpha n_2 & -\alpha n_1 \\ \dfrac{b\gamma n_2^2}{(bn_1+\omega)^2} & \gamma-\dfrac{2\gamma n_2}{bn_1+\omega}-\beta E \end{pmatrix} \tag{6}$$

$J=\begin{pmatrix} -\rho\left(\dfrac{\theta-A}{A}\right) & 0 \\ 0 & \gamma-\beta E \end{pmatrix}$, the latent values are $-\rho\left(\dfrac{\theta-A}{A}\right)$ and $\gamma-\beta E$. Thus, the trivial equilibrium point $E_0=(0,\,0)$ is locally asymptotically stable if $\theta>A$ (strong Allee effect) , $\gamma<\beta E$ and unstable if $\theta<A$ (weak Allee effect), $\gamma>\beta E$.

(i) The axial equilibria are $E^{\pm}=(n_1^{\pm},\,0)$, the corresponding Jacobian matrix is

$$J=\begin{pmatrix} \rho-\dfrac{2\rho n_1^{\pm}}{\delta}-\dfrac{A\rho\theta}{(n_1^{\pm}+A)^2} & -\alpha n_1^{\pm} \\ 0 & \gamma-\beta E \end{pmatrix}$$

The axial equilibrium point is locally asymptotically stable if $\dfrac{2n_1^{\pm}}{\delta}+\dfrac{A\theta}{(n_1^{\pm}+A)^2}>1$ and $\gamma<\beta E.$

(ii) The Jacobian matrix at an interior equilibrium point $E^*=(n_1^*,\,n_2^*)$

$$J(E^*)=\begin{pmatrix} \rho n_1^*\left(\dfrac{\theta}{n_1^*+A}-\dfrac{1}{\delta}\right) & -\alpha n_1^* \\ \dfrac{b\gamma n_2^{*2}}{(bn_1^*+\omega)^2} & -\dfrac{\gamma n_2^*}{bn_1^*+\omega} \end{pmatrix}, \tag{7}$$

we have that

$$\operatorname{tr} J(E^*)=\frac{\rho n_1^*(bn_1^*+\omega)\left[(\theta-\delta)-(n_1^*+A)\right]-\gamma\delta n_2^*(n_1^*+A)}{\delta(n_1^*+A)(bn_1^*+\omega)}<0,$$

$$\det J(E^*)=\frac{\rho(bn_1^*+\omega)\left[(n_1^*+A)-\theta\delta\right]+b\alpha\delta n_2^*(n_1^*+A)}{\delta(n_1^*+A)(bn_1^*+\omega)}>0,$$

that is, $E^*=(n_1^*,\,n_2^*)$ is locally stable. Let $\Delta=\operatorname{tr} J(E^*)^2-4\det J(E^*)$. If $\Delta<0$, then E^* is locally a stable focus. If $\Delta>0$, E^* is locally a stable node.

4 Global Stability

The Lyapunov procedure is applied to explore the model's (2) global stability of the coexistence equilibrium point.

Thus, the Lyapunov function is defined as;

$$L\left(n_1,n_2\right)=n_1-n_1^*-n_1^*\ln\frac{n_1}{n_1^*}+n_2-n_2^*-n_2^*\ln\frac{n_2}{n_2^*} \tag{8}$$

The time derivative of (8) is given by:

$$\frac{dL}{dt}=\frac{n_1-n_1^*}{n_1}\frac{dn_1}{dt}+\frac{n_2-n_2^*}{n_2}\frac{dn_2}{dt}. \tag{9}$$

Substituting (2) in (9) and simplifying, results to

$$\frac{dL}{dt}=\left(n_1-n_1^*\right)\left[\rho-\frac{\rho n_1}{\delta}-\frac{\rho\theta}{n_1+A}-\alpha n_2\right]+\left(n_2-n_2^*\right)\left[\gamma-\frac{\gamma n_2}{bn_1+\omega}-\beta E\right]$$

$$=\left(n_1-n_1^*\right)\left[\frac{\rho n_1^*}{\delta}+\frac{\rho\theta}{n_1^*+A}+\alpha n_2^*-\frac{\rho n_1}{\delta}-\frac{\rho\theta}{n_1+A}-\alpha n_2\right]+\left(n_2-n_2^*\right)\left[\frac{\gamma n_2^*}{bn_1^*+\omega}-\frac{\gamma n_2}{bn_1+\omega}\right]$$

$$=-\left[\frac{\rho}{\delta}-\frac{\rho\theta}{(n_1+A)\left(n_1^*+A\right)}\right]\left(n_1-n_1^*\right)^2-\frac{\gamma\left(\omega+bn_1^*\right)}{(bn_1+\omega)\left(bn_1^*+\omega\right)}\left(n_2-n_2^*\right)^2-\left[\alpha-\frac{b\gamma n_2^*}{(bn_1+\omega)\left(bn_1^*+\omega\right)}\right]\left(n_1-n_1^*\right)\left(n_2-n_2^*\right)$$

$$=-N^TBN$$

where $N=\left\{\left|n_1-n_1^*\right|,\left|n_2-n_2^*\right|\right\}$ and $B=\left[b_{ij}\right]_{2\times 2}$. The components of the matrix B are

$$b_{11}=\left[\frac{\rho}{\delta}-\frac{\rho\theta}{(n_1+A)\left(n_1^*+A\right)}\right],\ b_{12}=b_{21}=\frac{1}{2}\left[\alpha-\frac{b\gamma n_2^*}{(bn_1+\omega)\left(bn_1^*+\omega\right)}\right],$$

$$b_{22}=\frac{\gamma\left(\omega+bn_1^*\right)}{(bn_1+\omega)\left(bn_1^*+\omega\right)}.$$

Hence B is positive definite if

$$\frac{4\gamma\left(\omega+bn_1^*\right)}{(bn_1+\omega)\left(bn_1^*+\omega\right)}\left[\frac{\rho}{\delta}-\frac{\rho\theta}{(n_1+A)\left(n_1^*+A\right)}\right]>\left[\alpha-\frac{b\gamma n_2^*}{(bn_1+\omega)\left(bn_1^*+\omega\right)}\right]^2$$, which in turns

$\frac{dL}{dt}\le 0$ and hence E^* globally asymptotically stable.

Hopf-bifurcation is a localphenomena whereby changing a parameter causes a periodic solution to appear around the system's equilibrium point.We discuss the Hopf-bifurcation by using the factor A as the bifurcation parameter in system (2) because it appears to be essential. As a result, we have some $A = A_{HP}$. The Hopf bifurcation threshold is a positive root of $\left[TrJ\left(E^*\right)\right]_{A=A_{HP}} = 0,$ and can satisfy $\left[DetJ\left(E^*\right)\right]_{A=A_{HP}} > 0.$ Therefore we conclude the result from the subsequent theorem.

Theorem 3. Suppose the positive equilibrium E^* exists and there is a Hopf bifurcation around E^* when $A = A_{HP}$ where $A_{HP} = \dfrac{\rho n_1^*\left(bn_1^* + \omega\right)\left[\delta - \theta - 1\right] - \gamma\,\delta n_1^* n_2^*}{\rho n_1^*\left(bn_1^* + \omega\right) + \gamma\,\delta n_2^*}$.

Proof. The characteristic equation of matrix $J\left(E^*\right)$ is

$\lambda^2 - T(J_{E^*})\lambda + D\left(J_{E^*}\right) = 0,$ and the circumstancesunder which the Hopf bifurcation occurs are stated below.

i. $\left[TrJ\left(E^*\right)\right]_{A=A_{HP}} = 0,$

ii. $\left[DetJ\left(E^*\right)\right]_{A=A_{HP}} > 0,$

iii. $\dfrac{d}{dA}\left[TrJ\left(E^*\right)\right]_{A=A_{HP}} \neq 0.$

The criteria in (i) and (ii) are met for $\mathrm{A}=\mathrm{A}_{HP} = \dfrac{\rho n_1^*\left(bn_1^* + \omega\right)\left[\delta - \theta - 1\right] - \gamma\,\delta n_1^* n_2^*}{\rho n_1^*\left(bn_1^* + \omega\right) + \gamma\,\delta n_2^*}$ and

$n_1^* + A > \theta\delta$. Following that, we mustconfirm the transversality criterion (iii). Obviously,

$$\frac{d}{dA}\left[TrJ\left(E^*\right)\right]_{A=A_{HP}} = \frac{\theta n_1^*}{\left(n_1^* + A\right)^2} \neq 0.$$

Consequently, condition (iii) is satisfied, which guarantees the occurrence of Hopf bifurcation about E^* at $A = A_{HP}$. Similarly, we can find the Hopf-bifurcation condition for the parameter harvesting $E = E_{HP}$.

5 Stochastic Analysis

In this part, we accept stochastic perturbations of the parameters n_1 and n_2 about their values at the positive equilibrium E^* for the scenarioif it is locally asymptotically stable. We take into account the white noise stochastic disturbances that are proportional to the distances of n_1, n_2 from n_1^*, n_2^*. Hence, the stochastically perturbed scheme (2) is given by

$$dn_1 = \rho n_1\left(1-\frac{n_1}{\delta}-\frac{\theta}{n_1+A}\right)-\alpha\, n_1\, n_2+\sigma_1\left(n_1-n_1^{*}\right)d\xi_t^1$$

$$dn_2 = \gamma\, n_2\left(1-\frac{n_2}{b\, n_1+\omega}\right)-\beta\, E\, n_2+\sigma_2\left(n_2-n_2^{*}\right)d\xi_t^2 \tag{10}$$

where $\sigma_j,\ j=1,\,2$ are real quantities, $\xi_t^j=\xi_j\left(t\right),\ j=1,\,2$ are independent regular Wiener mechanisms. To analyze the stochastic stability of E^*, we take the linear system of (10) about E^* as shown below:

$$d\mu(t)= f(\mu(t))\,dt+g(\mu(t))\,d\xi(t) \tag{11}$$

where $\mu(t)=\mathrm{col}(\mu_1(t),\mu_2(t),\mu_3(t))$, $f(\mu(t))=J\,\mu(t)$, $g(\mu)=\begin{bmatrix}\sigma_1\mu_1 & 0\\ 0 & \sigma_2\mu_2\end{bmatrix}$,

$d\xi\left(t\right)=\mathrm{col}(\xi_1(t),\xi_2(t),\xi_3(t))$, $\mu_1=x-x^*$, $\mu_2=y-y^*$.

Let $\Omega=\left\{\left(t\ge t_0\right)\times\square^{\,n},t_0\in\square^{\,+}\right\}$. Therefore $V\in C_2^0(\Omega)$ is a continuous function w.r.t t and a twice continuously differentiable function w.r.t μ. The differential operator L with a function $V\left(t\right)$ is stated as

$$LV(t,\mu)=\frac{\partial V(t,\mu)}{\partial t}+f^T(\mu)\frac{\partial V(t,\mu)}{\partial\mu}+\frac{1}{2}Tr\left(g^T(\mu)\frac{\partial^2V(t,\mu)}{\partial\mu^2}g(\mu)\right), \tag{12}$$

where $\dfrac{\partial V}{\partial\mu}=\mathrm{col}\left(\dfrac{\partial V}{\partial\mu_1},\dfrac{\partial V}{\partial\mu_2}\right)$; $\dfrac{\partial^2V(t,\mu)}{\partial\mu^2}=\dfrac{\partial^2V}{\partial\mu_j\partial\mu_i}$; $i,\,j=1,2$ and T means transposition.

with reference to Afanasev et al. [38], the subsequent theorem is true.

Theorem 4. Suppose a function $V(\mu,t)\in C_2^0(\Omega)$ occurs that meets the criteriaoutlined below

$$K_1\left|\mu\right|^p\le V(t,\mu)\le K_2\left|\mu\right|^p;\,LV(t,\mu)\le -K_3\left|\mu\right|^p,\ K_i>0,\ p>0 \tag{13}$$

Then the trivial solution of (14) is exponentially p-stable for $t\ge 0$.

Also, if $p=2$ in (13), then the trivial solution of (11) is globally asymptotically stable.

Theorem 5. Assuming that $\sigma_1^2\le 2\rho n_1^*\left(\dfrac{1}{\delta}-\dfrac{\theta}{n_1^*+A}\right),\sigma_2^2\le\dfrac{\gamma\, n_2^*}{bn_1^*+\omega}$ then the zero solution of (11) is asymptotically mean square stable.

Proof: We assume the Lyapunov function $V(\mu)=\dfrac{1}{2}\left(w_1\mu_1^2+w_2\mu_2^2\right),w_i>0\in\square$ (14)

The inequality of (13) holds valid for $p=2$ and there is

$$LV(\mu) = w_1\left[\left(\rho n_1^*\left(\frac{\theta}{n_1^*+A}-\frac{1}{\delta}\right)\right)u_1 - \alpha n_1^* \mu_2\right]\mu_1$$

$$+ w_2\left[\left(\frac{b\gamma n_2^{*2}}{\left(bn_1^*+\omega\right)^2}\right)\mu_1 - \frac{\gamma n_2^*}{bn_1^*+\omega}\mu_2\right]\mu_2 + \frac{1}{2}Tr\left[g^T(\mu)\frac{\partial^2 V}{\partial \mu^2}g(\mu)\right]$$

(15)

We can easily observe that $\frac{\partial^2 V}{\partial \mu^2} = \begin{bmatrix} w_1 & 0 \\ 0 & w_2 \end{bmatrix}$ and thus $g^T(\mu)\frac{\partial^2 V}{\partial \mu^2}g(\mu) = \begin{bmatrix} w_1\sigma_1^2\mu_1 & 0 \\ 0 & w_2\sigma_2^2\mu_2 \end{bmatrix}$

with $\frac{1}{2}Tr\left[g^T(\mu)\frac{\partial^2 V}{\partial \mu^2}g(\mu)\right] = \frac{1}{2}\left[w_1\sigma_1^2\mu_1^2 + w_2\sigma_2^2\mu_2^2\right]$ (16)

If in (15) we choose $\alpha n_1^* w_1 = \frac{b\gamma n_2^{*2}}{\left(bn_1^*+\omega\right)^2}w_2$

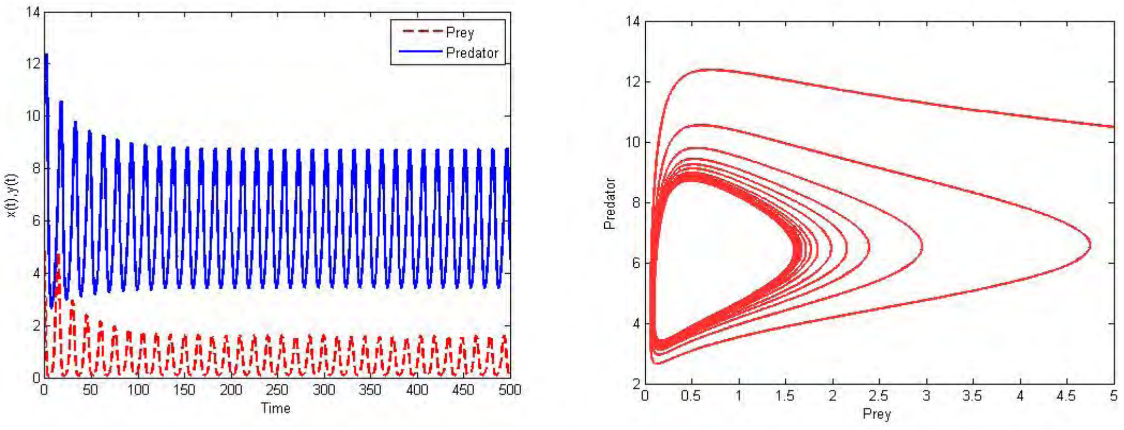
$$LV(\mu) = -w_1\left(\rho n_1^*\left(\frac{1}{\delta}-\frac{\theta}{n_1^*+A}\right)-\frac{1}{2}\sigma_1^2\right)\mu_1^2 - w_2\left(\frac{\gamma n_2^*}{bn_1^*+\omega}-\frac{1}{2}\sigma_2^2\right)\mu_2^2$$

We get the conclusion that the model (10)'s trivial solution is globally stable using Theorem 5.

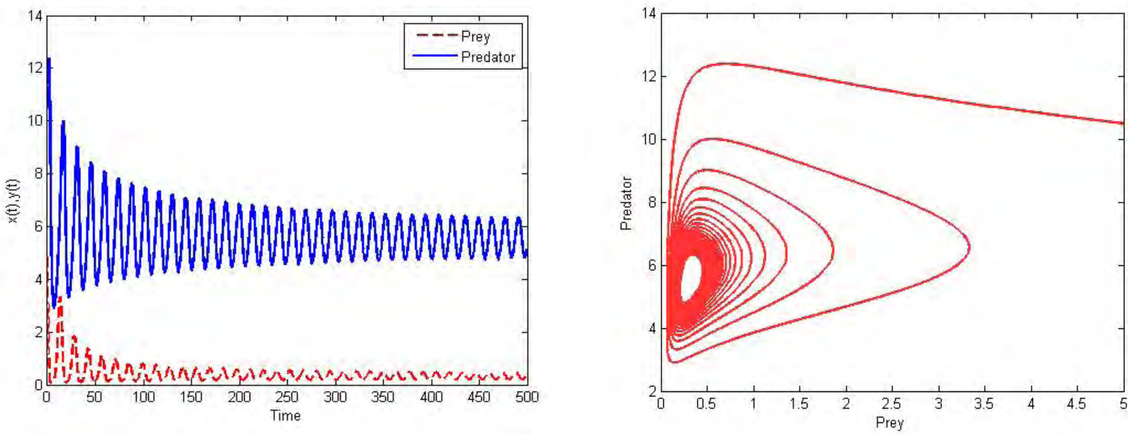
6 Numerical Simulations

Real-world data's significance cannot be disputed. However, gathering information from the field requires a lot of effort and time. This may also be significantly impacted by the current economic situation. Therefore, we are using certainhypothetical information here to validate the results of our analysis. The ecological growth of this kind of model is always significantly influenced by numerical validation. In order to simulate this study, we therefore adopted a qualitativeapproach instead of a quantitative perspective. These simulations were carried out using the MATLAB programme.

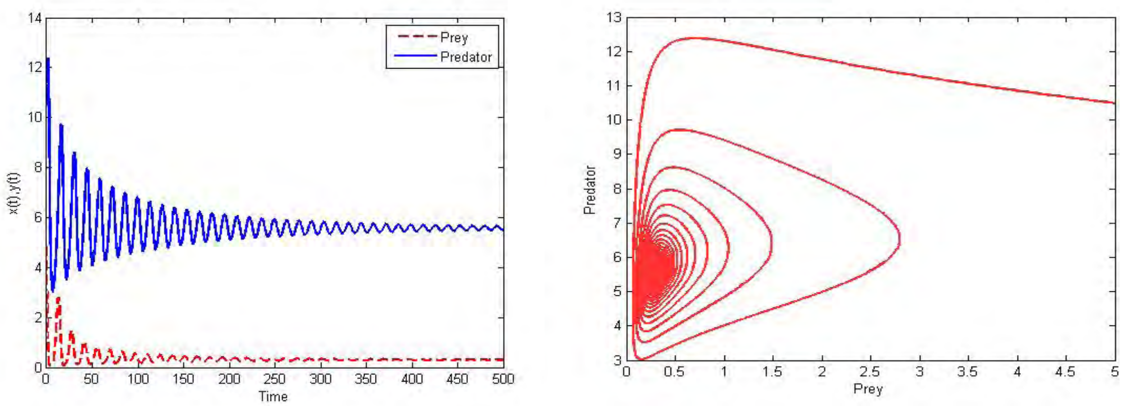
Example 1. For the scheme (2), we assume that the basic variables have amounts of $\rho = 1.76144,\ \delta = 309.6,\ \theta = 0.101,\ \alpha = 0.258, \gamma = 0.9048, b = 81.1,\ \omega = 2.4,\ \beta = 0.988$, $E = 0.725$, and various values of Allee effect A. Fig. 1 shows that the system lacks its stability at an interior equilibriumpoint if $A = 0.2, 0.23,$ and 0.25. Fig. 2 shows Hopf bifurcation for threshold value of Allee parameter $A = 0.2$.



(a)



(b)



(c)

Fig. 1 (a), (b), and (c) are time trajectories and phase portraits of scheme (2) with variousamounts of Allee effect

$A = 0.2, 0.23,$ and $0.25.$

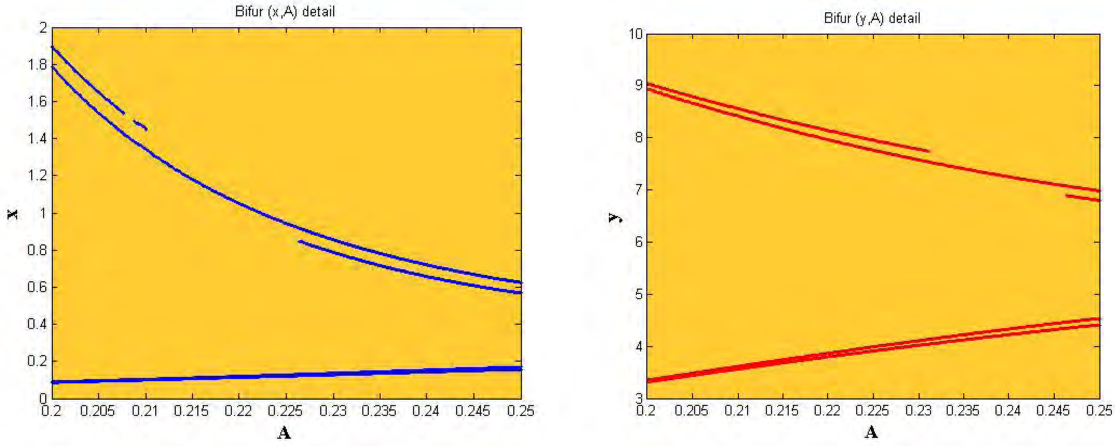
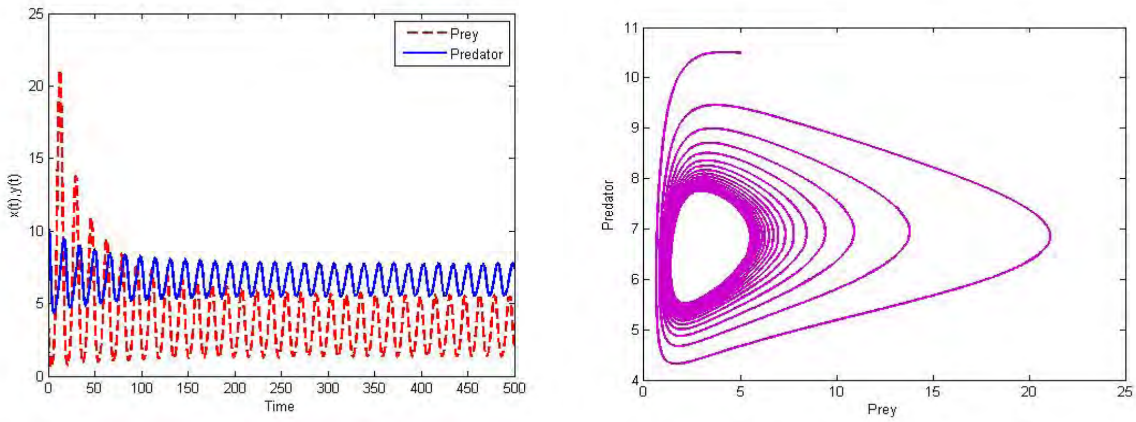
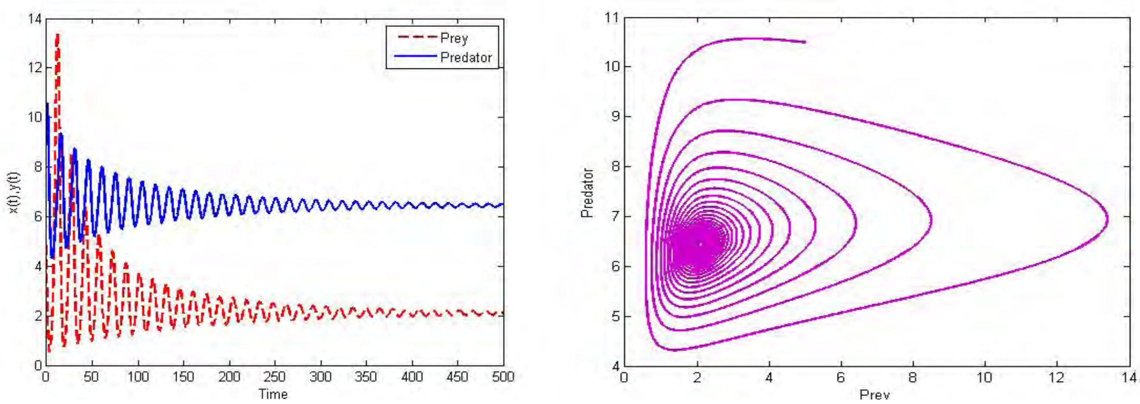


Fig. 2 The diagram for Hopf-bifurcation of system (2) for threshold value of $A = 0.2$.

Example 2. We take the values of the basic parametersfor the system (2) as $\rho = 1.96144,\ \delta = 309.6,\ \theta = 0.702, A = 2.72,\ \alpha = 0.258, \gamma = 1.9545, b = 45.5,\ \omega = 4.65,\ \beta = 0.978.$ and various values of Harvesting effort $E = 1.87$, and 1.89. Fig. 3 shows that the system lacks its stability at an interior equilibrium point if $E = 1.89$. Fig. 4 shows Hopf bifurcation for threshold value of harvesting effort $E = 1.89$.



(a)



(b)

Fig. 3 (a) and (b) are time trajectories phase portraits of scheme (2) for variousamounts of harvesting effect $E = 1.89,\ 1.87.$

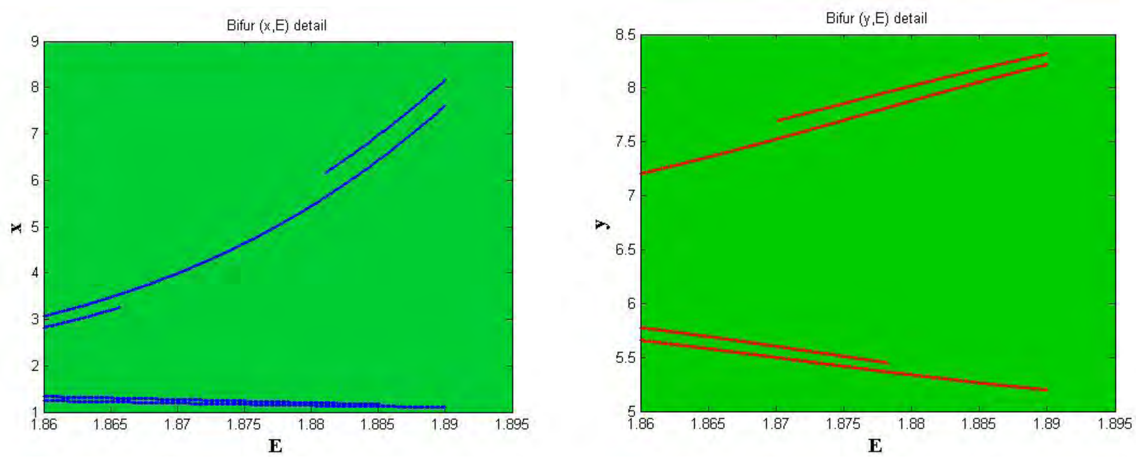
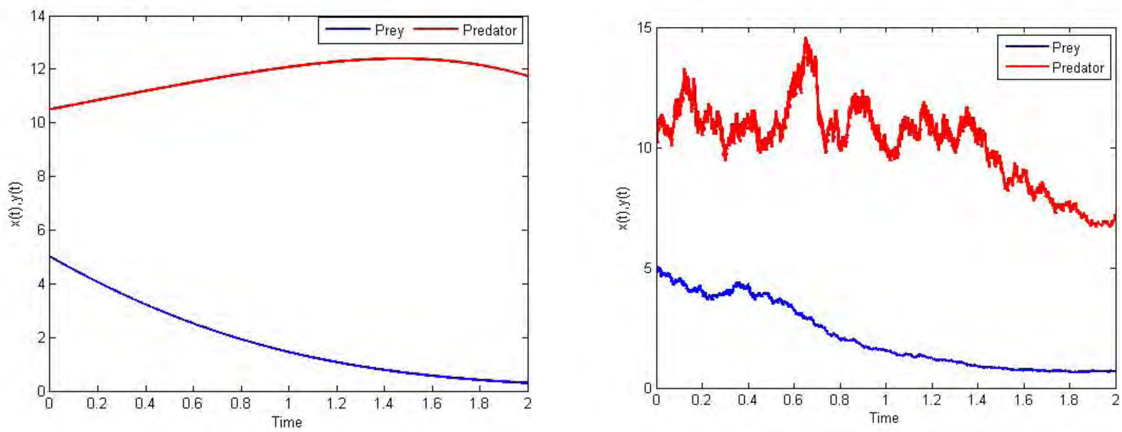
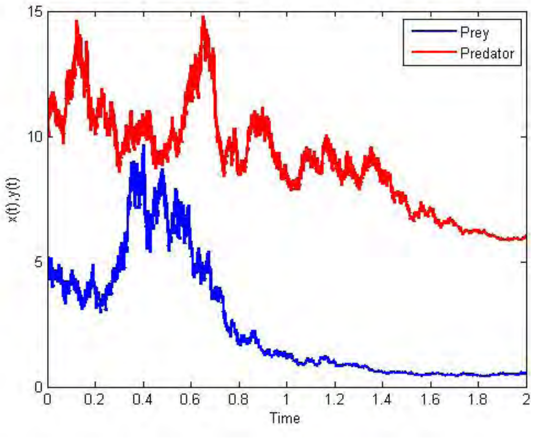


Fig. 4 The diagram for Hopf-bifurcation of system (2) for threshold value of $E = 1.89$.

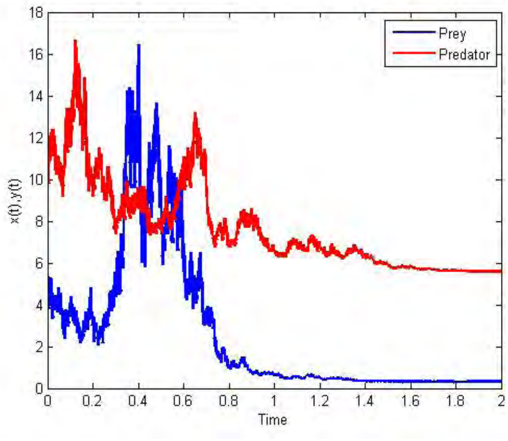
Example 3. Consider the parameter values as being comparable to Example (1), and as indicated in Fig. 5, we selected various white noise strengths. When the noise level is not too high, the deterministic model's characteristics are preserved by the stochastic model. The population may begin to fluctuate significantly when the noise levels are excessivelyhigh. We use the stochastic model (10) in this case to show how the population is dynamic.



(a) (b)



(c)



(d)

Fig. 5 Numerical simulation of scheme (10) with Allee effect for (a) $\sigma_1 = \sigma_2 = 0$, (b) $\sigma_1 = 0.1,\ \sigma_2 = 0.4,$ (c) $\sigma_1 = 0.4,\ \sigma_2 = 0.6,$ and (d) $\sigma_1 = 0.7,\ \sigma_2 = 0.9.$ The other parameters are same as in Fig. 1.

The stability changes whenthe parameter values for A and E are slightly altered, as seen in Figs 2 and 4. Consequently, when $E = 1.89$ a change in harvesting policy is also essential. Fig. 3 illustrates how the population reactions to varying amounts of harvesting vary.

7 Conclusions

This study examines the effects of an additive Allee effect on the prey growth function with a constant effort harvesting rate in predators using animproved Lesi-Gower predator-prey system. The system (2)'s positivity and boundedness are discussed first. The local stability of the requirements for scheme (2)'s potential equilibrium are developed and discussed. Also covered is the examination of global stability and harvesting standards. We produced different bifurcation diagrams with variable values of the system's characteristics to study the Allee effect and the harvesting effect on the system. Due to the important roles that Allee and harvesting effects play in a system's dynamic behaviour, it is crucial to study both species' long-term survival.We have developed the stochastic form of the scheme (2) in order to account for the impact of the changing environment. Asymptotic mean square stability requirements were then defined for the resulting model (10) and derived. According to the conclusions mentioned above (theorem 5), for ecological balance to be maintained in nature, both the deterministic and stochastic stability conditions must be met. It might be intriguing and useful for future work to examine how Allee effects affect the behaviours of predator-prey relationships in both populations and in a model of optimal control for harvesting.

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