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Fractional order generalized Richards growth model with delay arguments on coupled networks

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Abstract
This study aims to examine the dynamical behavior of Caputo fractional order generalized Richards growth model with delay. In order to transition from continuous time model to a discrete version, piecewise constant arguments are added to the model and thus system of difference equation are obtained from the solutions of the model in the sub-intervals. We obtain an algebraic condition where the positive equilibrium point of the discrete model with respect to changing parameter r is locally asymptotically stable. Moreover, it has been theoretically proven that the Neimark-Sacker bifurcation occurs at the positive equilibrium point of the discrete system and the direction of this bifurcation has been determined. In addition, discretized generalized Richards growth model is also considered on the globally coupled network with N=10 nodes. Numerical simulations have revealed that thecoupling strength parameter c plays a key role in the dynamical behavior of the node with the highest degree in such a complex network. Numerical simulations are performed to demonstrate the stability, bifurcations and dynamic transition of the coupled network.

Keywords fractional order model; complex network; discrete system; stability; Neimark-Sacker bifurcation.

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1 Introduction

The Gompertz model is one of the most frequently used models fitted to growth data and other data in the literature along with the logistic model. The Gompertz model includes more parameters than other growth models which makes it more flexible than other models and gives more consistent results with observed data. In the literature, there are many successful applications of the Richards growth model in fields such as plant growth, bird growth, fish growth, tumor growth, bacterial growth, epidemiology and other fields (Cornejoet al., 2023; Tjrve and Tjrve, 2017; Maunder et al., 2018; Golmankhaneh et al., 2024; Telekena et al., 2018; Hsieh, 2009; Hsieh and Chen, 2009; Lopez et al., 2004; Hsieh, 2010, He et al., 2023).

The classical Richards growth model can be defined via following differential equation:

$$\frac{dN(t)}{dt} = rN(t)\left(1 - \left(\frac{N(t)}{K}\right)^{\beta}\right) \tag{1}$$

The more general form of the model (1), named the generalized Richards growth model, is

$$\frac{dN(t)}{dt} = rN^{p}(t)\left(1 - \left(\frac{N(t)}{K}\right)^{\beta}\right) \tag{2}$$

where r is the maximum intrinsic rate of increase of N, K is the upper asymptote of N and β is an additional parameter in the Richards equation introduced as a power law so that it can define asymmetric curves and p is known as the deceleration of growth parameter which captures different early stages of the model.

In recent years, the increasing popularity of fractional order derivatives has brought a new perspective to the modeling of physical and biological processes. Dynamical systems with fractional order derivatives that can reflect the long memory and hereditary properties give more consistent results with real data than mathematical models with ordinary differential equations. Many biological and physical processes are successfully modeled in fields such as biology (Amilo et al., 2023), physics (Zhou et. al., 2020), chemistry (Veeresha, 2022) and complex network (Singh, 2022). In the study of Amilo et al. (2023), the authors considered dynamics of lung cancer and showed that the model with fractional order derivative contribute to a more accurate representation of the disease's progression and potential therapeutic interventions. On the other hand, the fact that fractional order differential equations are defined by an integral causes difficulty in finding analytical solutions of these equations. In addition, we need difference equations for describing the dynamical behaviors such as chaos in one and two dimensions. One of the procedure to transition from the continuous-time fractional order dynamical system to the discrete-time version is the use of piecewise constant arguments (El Sayed and Salman, 2013; Matouk et al., 2015; Zhang et al., 2016; El Sayed et al., 2016; El Sayed et al., 2017; El Sadany and Matouk, 2015; Matouk and Elsadany; 2016; Elsayed et al., 2007; Selvam and Janagaraj, 2016; Agarwal et al., 2013; El Sayed et al., 2014; El Rahem and Salman, 2014). El Sayed and Salman (2013) analyzed the dynamical behavior fractional order Riccati differential equation with piecewise constant arguments. Fractional order Hastings-Powell food chain model is discretized by using piecewise constant arguments and existence of some dynamical behaviors such as bifurcation and chaos in discrete time model are proved (El Sayed and Salman, 2013). On the other hand, differential equation with piecewise constant arguments are also closely related to delay differential equations. El Sayed et al. (2016) discretize fractional order delay Mackey-Glass equation using piecewise constant arguments and obtain system of difference equations. They provide that the model exhibits more complex dynamics after the discretization.

The complex network, that are a set of many connected nodes interacted in different ways, provides a useful method for analyzing complexity of a large number of real complex systems. There are many different types of networks such as globally coupled network, star network, Erdos-Renyi network in the literature where each node represents a nonlinear dynamical system (Wang et al., 2017; Huang et al., 2019; Li et al., 2004; Nepomuceno and Perc, 2015; Ahmed and Matouk, 2006; Zhang et al., 2006; Zhang, 2018). In these studies, complex dynamical behavior such as synchronization, control of dynamical networks, the epidemic of disease, transition chaos is studied.

The fractional-order delay Richards growth model can be defined as follows:

$$D^{\alpha}N(t) = rN^{p}(t-\tau)\left(1 - \left(\frac{N(t-\tau)}{K}\right)^{\beta}\right) \tag{3}$$

If the piecewise constant arguments are used in the place of the term $t-\tau$ as $\left[\!\left[\frac{t-h}{h}\right]\!\right]$ we obtain

$$D^{\alpha}N(t) = rN^{p}\left(\left[\!\left[\frac{t-h}{h}\right]\!\right]h\right)\left(1-\left(\frac{N\left(\left[\!\left[\frac{t-h}{h}\right]\!\right]h\right)}{K}\right)^{\beta}\right) \tag{4}$$

with the initial condition $N(0) = N_0$ where $N(t)$ is a value of a measure of size or density of an organism or population, β is additional shape parameter, p is deceleration of growth parameter,K is the carrying capacity, $[\![t]\!]$denotes the integer part of $t \in [0,\infty)$ and h is discretization parameter..

The purpose of this study is to examinestability and bifurcation analysis of the fractional order generalized Richards growth model (4) after the discretization procedure. In addition, the discretized model also considers the globally coupled network.

2 Local Stability Analysis

Let $t \in [0,h)$, then $\frac{t-h}{h} \in (-1,0)$. So we get

$$D^{\alpha}N(t) = rN_{-1}{}^{p}\left(1-\left(\frac{N_{-1}}{K}\right)^{\beta}\right), t \in [0,h),$$

and the solution (4) is given by

$$N_1(t) = N_0 + I^{\alpha}\left(rN_{-1}{}^{p}\left(1-\left(\frac{N_{-1}}{K}\right)^{\beta}\right)\right) = N_0 + \frac{t^{\alpha}}{\Gamma(1+\alpha)}\left(rN_{-1}{}^{p}\left(1-\left(\frac{N_{-1}}{K}\right)^{\beta}\right)\right).$$

Let $t \in [h,2h)$, then $\frac{t-h}{h} \in (0,1)$. So we get

$$D^{\alpha}N(t) = rN_0{}^{p}\left(1-\left(\frac{N_0}{K}\right)^{\beta}\right), t \in [h,2h),$$

and the solution (4) is given by

$$N_2(t) = N_1(h) + I^{\alpha}\left(rN_0{}^{p}\left(1-\left(\frac{N_{01}}{K}\right)^{\beta}\right)\right) = N_1(h) + \frac{(t-h)^{\alpha}}{\Gamma(1+\alpha)}\left(rN_0{}^{p}\left(1-\left(\frac{N_0}{K}\right)^{\beta}\right)\right).$$

Repeating the process we can easily deduce that the solution of (4) is given by

$$N_{n+1}(t) = N_n(nh) + \frac{(t-nh)^{\alpha}}{\Gamma(1+\alpha)}\left(rN^{p}{}_{n-1}(nh)\left(1-\left(\frac{N_{n-1(nh)}}{K}\right)^{\beta}\right)\right), \qquad t \in [nh,(n+1)h).$$

Let $t \to (n+1)h$, then we have

$$N_{n+1}((n+1)h) = N_n(nh) + \frac{h^{\alpha}}{\Gamma(1+\alpha)}\left(rN^{p}{}_{n-1}(nh)\left(1-\left(\frac{N_{n-1(nh)}}{K}\right)^{\beta}\right)\right).$$

If we introduce $x_n = N_n$ and $y_n = N_{n-1}$, then we have

$$\begin{cases} x_{n+1} = x_n + \dfrac{h^{\alpha}}{\Gamma(1+\alpha)}\left(ry_n{}^{p}\left(1-\left(\dfrac{y_n}{K}\right)^{\beta}\right)\right) \\ y_{n+1} = x_n \end{cases} \tag{5}$$

Now, the equilibrium point of the system (5) is easily calculated as $(x^*,y^*) = (K,K)$.The following theorem gives necessary and sufficient algebraic conditions where the equilibrium point is local asymptotically stable.

Theorem 1: The equilibrium point $(x^*, y^*) = (K, K)$of system (5) is local asymptotically stable if and only if

$$0 < r < \frac{h^{-\alpha}\Gamma(\alpha+1)K^{1-p}}{\beta}. \quad (6)$$

Proof 1: The Jacobian matrix calculated at positive equilibrium point $(x^*, y^*) = (K, K)$ of system (5) is

$$J = \begin{pmatrix} 1 & -\frac{h^{\alpha}r\beta K^{-1+p}}{\Gamma(\alpha+1)} \\ 1 & 0 \end{pmatrix}$$

and the corresponding characteristic equation is

$$\lambda^2 + p_1\lambda + p_2 = 0 \quad (7)$$

where

$$p_1 = -1 \quad (8)$$

and

$$p_2 = \frac{h^{\alpha}r\beta K^{-1+p}}{\Gamma(\alpha+1)}. \quad (9)$$

Now, we can use the following conditions, which are called Schur-Cohn criterions, to studied asymptotic stability analysis of the positive equilibrium point of the system (5).

i)$1 + p_1 + p_2 > 0$,

ii)$1 - p_1 + p_2 > 0$,

iii)$1 - p_2 > 0$.

From i) and ii) we have

$$1 + p_1 + p_2 = \frac{h^{\alpha}r\beta K^{-1+p}}{\Gamma(\alpha+1)} > 0,$$

and

$$1 - p_1 + p_2 = 2 + \frac{h^{\alpha}r\beta K^{-1+p}}{\Gamma(\alpha+1)} > 0$$

respectively. Under the condition $r < \frac{h^{-\alpha}\Gamma(\alpha+1)K^{1-p}}{\beta}$, we hold

$$1 - p_2 = 1 - \frac{h^{\alpha}r\beta K^{-1+p}}{\Gamma(\alpha+1)} > 0.$$

This completes our proof.

3 Neimark Sacker Bifurcation Analysis

In this section, we deal with the existence and direction of Neimark-Sacker bifurcation about the positive fixed point for the discrete system (5). The theoretical background of this bifurcation and its applications can be found in studies (Guckenheimer and Holmes, 1983; Din 2017; Din 2017).

Theorem 2: System (5) undergoes Neimark-Sacker bifurcation at the fixed point $(x^*, y^*) = (K, K)$. Moreover, if k < 0 then an attracting invariant cycle will appear for $0 < r < r^*$, if k > 0 then a repelling invariant cycle will appear for $r > r^*$.

Proof 2: From the solution of equation $1 - p_2 = 0$ with respect to parameter r, the critical Neimark-sacker bifurcation point is obtained as

$$r = r^* = \frac{h^{-\alpha}\Gamma(\alpha+1)K^{1-p}}{\beta}. \quad (10)$$

On the other hand, characteristic equation of the linearized system at the positive fixed point gives the eigenvalues as follows.

$$\lambda_{1,2} = \frac{1}{2} \mp i\frac{\sqrt{-(\Gamma(\alpha+1))^2K^2 + 4h^\alpha r\beta\Gamma(\alpha+1)K^{1+p}}}{2\Gamma(\alpha+1)K}. \quad (11)$$

For $r = r^*$, these eigenvalues leads to

$$|\lambda_{1,2}| = \left|\frac{1}{2} \pm \frac{i\sqrt{3}}{2}\right| = |a \pm ib| = 1.$$

The transversality condition

$$\frac{d|\lambda_{1,2}(r)|}{dr}\Big|_{r=r^*} = \frac{h^\alpha \beta K^{-1+p}}{2\Gamma(\alpha+1)} \neq 0 \quad (12)$$

is always satisfied for all parameter values. In addition, nonresonance conditions is always satisfied for $p_1 \neq 0,1$.

Let $u = x - x^*$and $v = y - y^*$, then the system (5) is transformed into

$$\begin{pmatrix} u \\ v \end{pmatrix} \to \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} f_1(u,v) \\ f_2(u,v) \end{pmatrix} \quad (13)$$

where

$$f_1(u,v) = m_{13}u^2 + m_{14}uv + m_{15}v^2 + m_{16}u^3 + m_{17}u^2v + m_{18}uv^2 + m_{19}v^3 + O((|u|+|v|)^4)$$
$$f_2(u,v) = m_{23}u^2 + m_{24}uv + m_{25}v^2 + m_{26}u^3 + m_{27}u^2v + m_{28}uv^2 + m_{29}v^3 + O((|u|+|v|)^4)$$

and

$$m_{13} = m_{14} = m_{16} = m_{17} = m_{18} = m_{23} = m_{24} = m_{25} = m_{26} = m_{27} = m_{28} = m_{29} = 0$$
$$m_{15} = \frac{\alpha(-1+2p+\beta)}{2\Gamma(\alpha+1)K}$$
$$m_{19} = -\frac{(2+3(-2+p)p - 3\beta + 3p\beta + \beta^2)\Gamma(\alpha+1)}{6\alpha K^2}.$$

By using transformation $\begin{pmatrix} u \\ v \end{pmatrix} = T\begin{pmatrix} X \\ Y \end{pmatrix}$, then the map (13) rewritten as the following form

$$\begin{pmatrix} X \\ Y \end{pmatrix} \to \begin{pmatrix} a & -b \\ b & a \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix} + \begin{pmatrix} F_1(X,Y) \\ F_2(X,Y) \end{pmatrix} \quad (14)$$

where

$$T = \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ 0 & 1 \end{pmatrix}$$

$$F_1(X,Y) = -\frac{Y^3(2 - 6p + 3p^2 - 3\beta + 3p\beta + \beta^2)\Gamma(\alpha+1)}{3\sqrt{3}\alpha K^2} + \frac{Y^2\alpha(-1+2p+\beta)}{\sqrt{3}\Gamma(\alpha+1)K} + O((|u|+|v|)^4)$$

$$F_2(X,Y) = 0.$$

The constant k, which will determine the direction of the Neimark-Sacker bifurcation, can be calculated using the following equation.

$$k = -\mathrm{Re}\left[\frac{(1-2\lambda)\bar{\lambda}^2}{1-\lambda}\xi_{11}\xi_{20}\right] - \frac{1}{2}|\xi_{11}|^2 - |\xi_{02}|^2 + \mathrm{Re}[\bar{\lambda}\xi_{21}]$$
$$= \frac{-2\alpha^3(-1+2p+\beta)^2 + (2+3(-2+p)p - 3\beta + 3p\beta + \beta^2)(\Gamma(\alpha+1))^3}{16\alpha(\Gamma(\alpha+1))^2K^2} \quad (15)$$

where

$$\xi_{20} = \frac{1}{8}\big((F_{1XX} - F_{1YY} + 2F_{2XY}) + i(F_{2XX} - F_{2YY} - 2F_{1XY})\big) = -\frac{\alpha(-1+2p+\beta)}{4\sqrt{3}\Gamma(\alpha+1)K},$$
$$\xi_{11} = \frac{1}{4}\big((F_{1XX} + F_{1YY}) + i(F_{2XX} + F_{2YY})\big) = \frac{\alpha(-1+2p+\beta)}{2\sqrt{3}\Gamma(\alpha+1)K},$$

$$\xi_{02} = \frac{1}{8}\left((F_{1XX} - F_{1YY} - 2F_{2XY}) + i(F_{2XX} - F_{2YY} + 2F_{1XY})\right) = -\frac{\alpha(-1+2p+\beta)}{4\sqrt{3}\Gamma(\alpha+1)K},$$

$$\xi_{21} = \frac{1}{16}\left((F_{1XXX} + F_{1XYY} + F_{2XXY} + F_{2YYY}) + i(F_{2XXX} + F_{2XYY} - F_{1XXY} - F_{1YYY})\right)$$
$$= \frac{i(2+3p^2+3p(-2+\beta)-3\beta+\beta^2)\Gamma(\alpha+1)}{8\sqrt{3}\alpha K^2}.$$

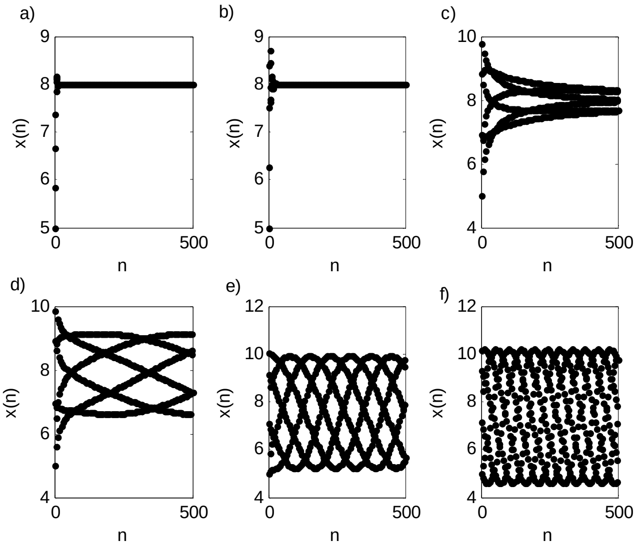

Fig. 1 Phase portraits of the discrete model (5) with respect to parameter r for $r = 1.5$ (a), $r = 2.1$ (b), $r = 2.29504$ (c), $r = 2.35$ (d) $r = 2.5$ (e), $r = 2.6$ (f) where α=0.99, β=0.7, K=8, h=0.5, p=1.1, x(1)=y(1)=5.

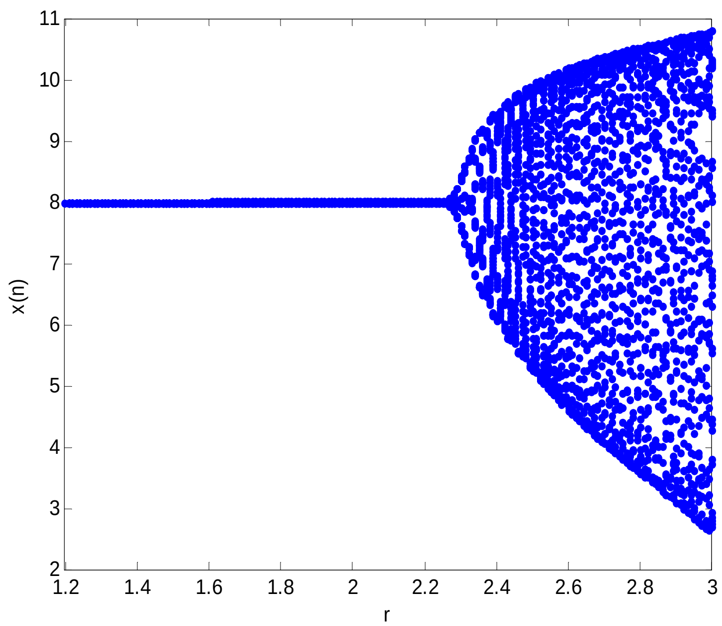

Fig. 2 Bifurcation diagram of the discrete system (5) in accordance with parameter r, where $\alpha = 0.99, \beta = 0.7, K = 8,\ h = 0.5, p = 1.1, x(1) = y(1) = 5$.

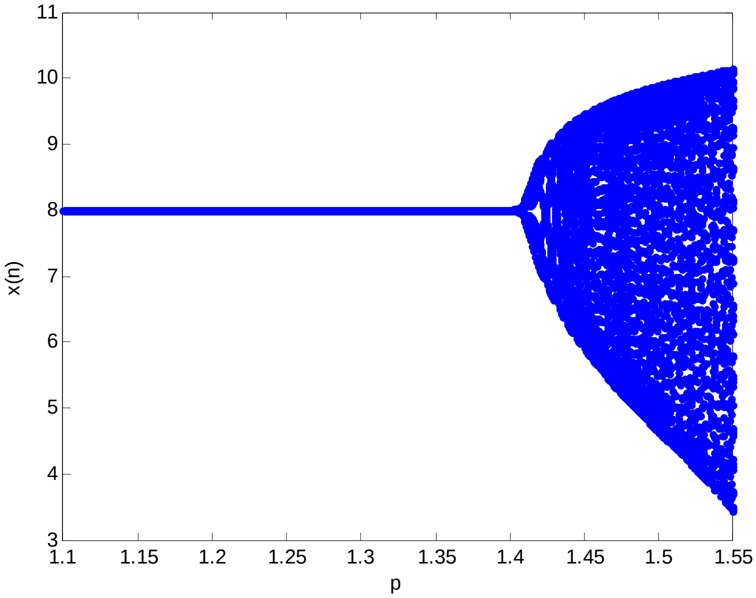

Fig. 3 Bifurcation diagram of the discrete system (5) in accordance with parameter p, where $\alpha = 0.99, \beta = 0.7, K = 8,\ h = 0.5, r = 1.2, x(1) = y(1) = 5$.

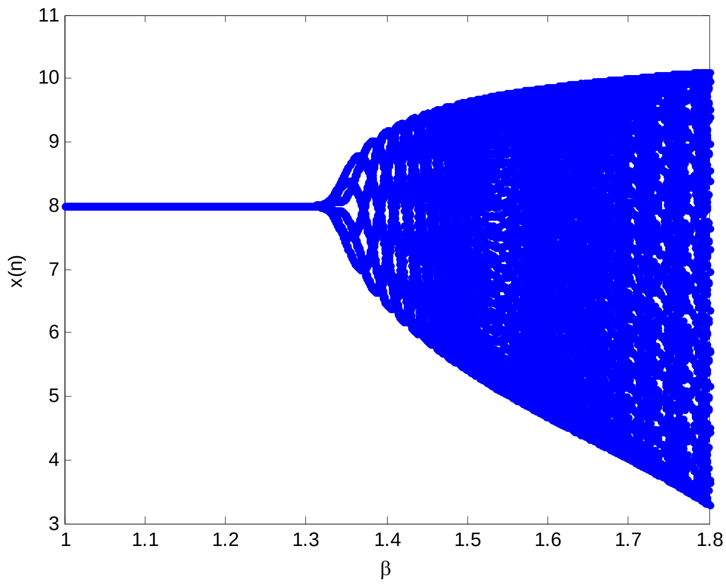

Fig. 4 Bifurcation diagram of the discrete system (5) in accordance with parameter β, where $\alpha = 0.99, p = 1.1, K = 8,\ h = 0.5, r = 1.2, x(1) = y(1) = 5$.

4 Dynamical Analysis of The Model on Globally Coupled Network

Let's consider the equation (5) as the following form:

$$\begin{cases} x(k+1) = x(k) + r(y(k))^p \left(1 - \left(\frac{y(k)}{K}\right)^\beta\right) \frac{h^\alpha}{\Gamma(\alpha+1)} = f\big(x(k), y(k)\big) \\ y(k+1) = x(k) = g\big(x(k), y(k)\big) \end{cases} \tag{16}$$

Consider a globally coupled dynamical network of N linearly and diffusively coupled nodes, where each node is a two-dimensional discretized fractional-order delay Richards growth model (5). So, the state equations this network is defined by

$$\begin{cases} x_i(k+1) = f\big(x_i(k), y_i(k)\big) - c\sum_{j=1}^{N} a_{ij} f\left(x_j(k), y_j(k)\right) \\ y_i(k+1) = g\big(x_i(k), y_i(k)\big) - c\sum_{j=1}^{N} a_{ij} g\left(x_j(k), y_j(k)\right) \end{cases} \tag{17}$$

where i and j are the sequence number of the nodes and c describes the coupling strength of the network. The coupling matrix $A \in R^{NxN}$ can be expressed by

$$A = \begin{pmatrix} 1-N & 1 & 1 & \cdots & 1 \\ 1 & 1-N & 1 & \cdots & 1 \\ 1 & 1 & 1-N & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \ldots \\ 1 & 1 & 1 & \cdots & 1-N \end{pmatrix} \tag{18}$$

The system (17) can be written in matrix form as follows:

$$\begin{cases} X_{k+1} = (I - cA) f\big(X(k), Y(k)\big) \\ Y_{k+1} = (I - cA) g\big(X(k), Y(k)\big) \end{cases} \tag{19}$$

where $X_k = \big(x_1(k), x_2(k), \ldots . x_N(k)\big), Y_k = \big(y_1(k), y_2(k), \ldots . y_N(k)\big)$ and $I \in R^{NXN}$ identity matrix.

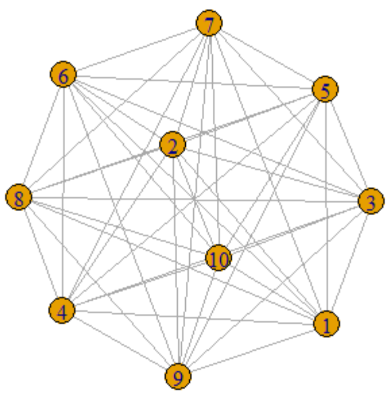

Fig. 5 Globally coupled network with $N = 10$ nodes.

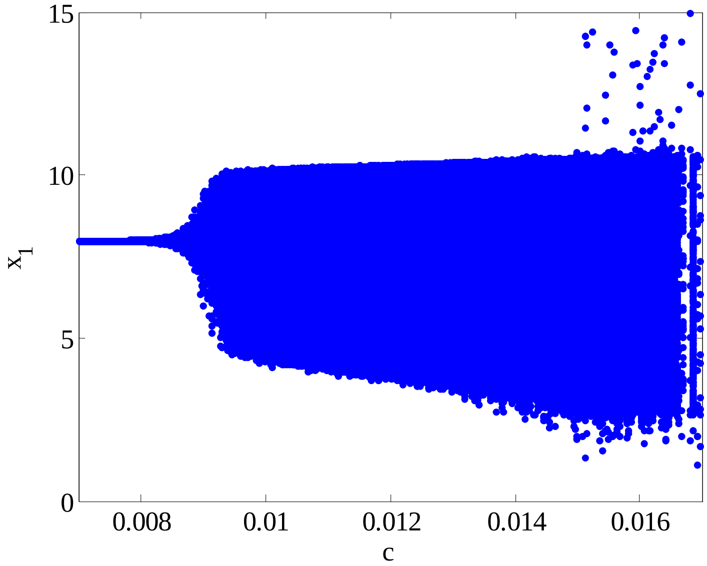

Fig. 6 Neimark-Sacker bifurcation in the global coupled network in accordance with parameter c, where α=0.99, β=0.7, K=8, h=0.5, p=1.1, $r = 2.2$ and $N = 10$.

5 Results and Discussion

In this study, fractional order delay Richards growth model (3), which is a more general version of the logistics model, is considered. By replacing delay parameter in model (3) with piecewise constant arguments leads to system of difference equation (5). The algebraic condition (6) in Theorem 1 gives the local asymptotic stable region of the positive equilibrium point $(x^*, y^*) = (K, K)$ of the discretized model (5) with respect to growth rate parameter r . For the numerical simulations, we take the parameter values as $\alpha = 0.99, \beta = 0.7, K = 8$, $h = 0.5, p = 1.1$ and $x(1) = y(1) = 5$. The stability region according to the change of the parameter r is obtained as $r < 2.29504$. It can be easily seen in the Figure 1a and 1b that, the fixed point (8,8) of the system (5) is local asymptotically stable.

To demonstrate the existence of the Neimark-Sacker bifurcation at the positive equilibrium point and determine its direction, we analysis the eigenvalue assignment, transversality and non-resonance conditions and compute value of k. All of these conditions show that system (5) undergoes Neimark-Sacker bifurcation about the positive fixed point. For the above numerical values of the parameters, the critical Neimark-Sacker bifurcation is obtained as $r^* = 2.29504$ from the solution of the equation $1 - p_0 = 0$. In addition from the equation (12) and the condition $p_1 \neq 0,1$, transversality and non-resonance conditions are always satisfied for all of the positive parameter values. Fig. 1c and Fig. 2, show the Neimak-Sacker bifurcation in discretized model (5) at the positive equilibrium point. From the equation (15), the value of k that determines the direction of the Neimark-Sacker bifurcation is calculated as $k = -0.00723392$ which show the existence of supercritical Neimark-Sacker bifurcation.

In order to see the effect of the deceleration of growth parameter parameterp and additional shape parameterβ on the dynamic structure of the system, we can again plot bifurcation diagram with respect to parameter p and β respectively. Fig. 3 and Fig. 4 show the similar dynamical behavior, which is Neimark-sacker bifurcation, with respect to changing the parameter p and β where bifurcation point is $p^* = 1.41183$ and $\beta^* = 1.33877$.

We also deal with globally coupled network with $N = 10$ nodes where each node in a complex network represents the discrete dynamical system (5). Such a network can be seen in Figure 5. Now, the state equations this complex network are the system (19) with $N = 10$ dimension where the coupling matrix A is

$$A = \begin{pmatrix} -9 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -9 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -9 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -9 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -9 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & -9 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & -9 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & -9 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -9 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -9 \end{pmatrix}$$

Now let's consider the nodes with the highest degree in the globally coupled network with $N = 10$ nodes. In the global coupled network with $N = 10$, all nodes have equal degrees and their degrees are 9. Fig. 6 shows that if the coupling parameter c reaches the some critical value where it is interval $c \in [0.007, 0.017]$, then Neimark-Sacker bifurcation occurs about the positive fixed point.

References

Agarwal RP, EL-Sayed AMA, Salman SM. 2013. Fractional order Chuas system: discretization, bifurcation and chaos. Advences in Differences Equations, 2013: 320

Ahmed E, Matouk AE. 2006. Complex dynamics of some models of antimicrobial resistance on complex networks. Mathematical Methods in the Applied Sciences, 44: 1896-1912

Amilo D, Kaymakamzade B, Hincal. 2023. A fractional order mathematical model for lung cancer incorporating integrated therapeutic approaches. Scientific Reports, 13: 12426

Cornejo O, Herrera SM, Baesler F, et al. 2023. The application of the random time transformation method to estimate richards model for tree growth prediction. Mathematics, 11(20): 4233

Din Q. 2017. Neimark-Sacker bifurcation and chaos control in Hassell-Varley model. Journal of Difference Equations and Applications, 23(4): 741-762

Din Q. 2017. Complexity and chaos control in a discrete-time prey-predator model. Communications in Nonlinear Science and Numerical Simulation, 23(4): 741-762

El-Rahem ZFE, Salman SM. 2014.On a discretization process of fractional order logistic differential equation. Journal of the Egyptian Mathematical Society, 22(3): 407-412

El-Sayed AMA, Salman SM. 2013. On a discretization process of fractonal order ricatti differential equation. Journal of Fractional Calculus and Applications, 4: 251-259

El-Sayed AMA, Salman SM, Elabd NA. 2016. On a Fractional-order delay Mackey-Glass equation. Advences in Difference Equations, 137: 2016

El-Sayed AMA, Salman SM, Elabd NA. 2017. Stability analysis and chaos control of the discretized fractional order Mackey-Glass equation. Journal of Fractional Calculus and Applications, 8(1): 16-28.

El-Sayed AMA, EL Mesiry AEM, El-Saka HAA. 2007. On the fractional order logistic equation. Applied Mathematics Letters, 20(7): 817-823

El-Sayed AMA, EL-Rahem ZFE, Salman SM. 2014. Discretization of forced Dung system with fractional order damping.Advences in Differences Equations, 66: 2014

Elsadany AA, Matouk AE. 2015. Dynamical behaviors of fractional order Lotka-Volterra predator-prey model and its discretization. Journal of Applied Mathematics and Computing, 49: 269-283.

Golmankhaneh AK, Tun S, Schlichtinger AM, et al. 2024. Modeling tumor growth using fractal calculus: Insights into tumor dynamics. Biosystems, 235: 105071

He J, Mao S, Adolf KYN. 2023. Neural computing for grey Richards differential equation to forecast traffic parameters with various time granularity. Neurocomputing, 549: 126394

Hsieh YH. 2009. Modeling and Dynamics of Infectious Diseases Series in Contemporary Applied Mathematics. Higher Education Press, Beijing, China

Hsieh YH, Chen CWS. 2009. Turning points, reproduction number, and impact of climatological events for multi-wave dengue outbreaks. Tropical Medicine and International Health, 14: 628-638

Hsieh YH, Fisman DN, Wu J. 2010. On epidemic modeling in real time: an application to the 2009 novel A (H1N1) influenza outbreak in Canada. BMC Research Notes, 5(3): 283

Huang T, Zhang H, Ma S. 2019. Bifurcations, complex behaviors, and dynamic transition in a coupled network of discrete predator-prey system. Discrete Dynamics in Nature and Society, 2019: 2583730

Matouk AE, ELsadany AA. 2016. Dynamical analysis, stabilization and discretization of a chaotic fractional order GLV model. Nonlinear Dynamics, 85: 1597-1612

Li X, Chen G, Ko KT. 2004. Transition to chaos in complex dynamical networks. Physica A: Statistical Mechanics and its Applications, 338: 367-378

Lopez S, Prieto M, Dijkstra J, et al. 2004. Statistical evaluation of mathematical models for microbial growth. International Journal of Food Microbiology, 96(3): 289-300

Matouk AE, Elsadany AA, Ahmed E, et al. 2015. Dynamical behavior of fractional-order Hasting-Powell food chain model and its discretization. Communications in Nonlinear Science and Numerical Simulation, 27(1-3): 153-167

Maunder M, Deriso RB, Schaefer KM, et al. 2018. The growth cessation model: a growth model for species showing a near cessation in growth with application to bigeye tuna. Marine Biology, 165: 76

Nepomuceno EG, Perc M. 2015. Computational chaos in complex networks. Journal of Complex Networks, 8: cnz015

Guckenheimer J, Holmes P. 1983. Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields. Springer

Selvam AGM, Janagaraj R. 2016. Dynamics in a discrete fractional order Lorenz system, Advances in Applied Science Research, 7(1): 189-195

Singh R, Rehman AU, Masud M, et al. 2022. Fractional order modeling and analysis of dynamics of stem cell differentiation in complex network. AIMS Mathematics, 7: 5175-5198

Telekena JT, Galvo AC, Robazza WS. 2018. Use of modifedRichards model to predict isothermal and non-isothermal microbial growth. Brazilian Journal of Microbiology, 49(3): 614-620

Tjrve KMC, Tjrve E. 2017. Modelling avian growth with the Unied-Richards: as exemplied by wader-chick growth. Journal of Avian Biology, 48(6): 770-784

Veeresha P. 2022. The efficient fractional order based approach to analyze chemical reaction associated with pattern formation. Chaos Solitons and Fractals, 165: 112862

Wang Z, Jiang G, Yu W, et al. 2017. Synchronization of coupled heterogeneous complex networks. Journal of the Franklin Institute, 354: 4102-4125

Zhang HF, Rui XW, Fu XC. 2006. The emergence of chaos in complex dynamical networks, Chaos Soliton and Fractals, 28: 472-479

Zhang WJ. 2018. Fundamentals of Network Biology. World Scientific Europe, London, UK

Zhang Y, Zhang X, Zhang Y. 2016. Dynamical Behavior Analysis and Control of a Fractional-order Discretized Tumor Model, International Conference on Information Engineering and Communications Technology (IECT 2016).

Zhou P, Ma J, Tang J. 2020. Clarify the physical process for fractional dynamical systems. Nonlinear Dynamics, 100: 2353-2364