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Dynamic effects of human-caused carbon dioxide emissions on atmospheric temperature via various variable-order fractional derivatives

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Received 4 May 2024; *Accepted 10 June 2024*; *Published online 20 June 2024*; *Published 1 December 2024*

Abstract
The intricate relationship between human-induced carbon dioxide (CO_2) emissions and the atmospheric temperature presents a multifaceted challenge with profound implications for the environment, ecosystems, and human societies. This research presents a novel fractional variable-order model that the impact of CO_2 emissions from human activities using Liouville-Caputo (LC), Caputo-Fabrizio (CF), and Atangana-Baleanu (AB) fractional derivatives. Existence and uniqueness of fractions solutions are established and numerical simulations have been carried out based on different set of parameters to study the effects of CO_2 emission caused by human activities leading to global warming.

Keywords carbon dioxide; fractional variable-order model; Liouville-Caputo; Caputo-Fabrizio; Atangana-Baleanu.

Computational Ecology and Software
ISSN 2220-721X
URL: http://www.iaees.org/publications/journals/ces/online-version.asp
RSS: http://www.iaees.org/publications/journals/ces/rss.xml
E-mail: ces@iaees.org
Editor-in-Chief: WenJun Zhang
Publisher: International Academy of Ecology and Environmental Sciences

1 Introduction

Carbon dioxide, among other greenhouse gases like methane, nitrous oxide, and water vapor, plays a vital role in regulating the Earth's temperature.The essential function of these gases is to regulate the Earth's temperature by trapping heat and preventing its escape into space, similar to how a greenhouse's glass enclosure retains heat within.Human activities have significantly altered the natural balance of greenhouse gases in the atmosphere, particularly since the beginning of the Industrial Revolution in the 18th century. The burning of fossil fuels such as coal, oil, and natural gas for energy production, transportation, and industrial processes releases large quantities of CO_2 into the air. The increase in atmospheric CO_2 concentrations due to these human activities has led to a phenomenon known as the enhanced greenhouse effect. Rising CO_2 levels result in increased retention of heat within the atmosphere, causing gradual warming of Earth's surface and lower atmosphere.This process is commonly referred to as global warming.The increase in temperature has caused changes in weather patterns, resulting in more frequent and intense extreme weather events like heat waves, droughts, heavy rainfall, and modifications in ecosystems and biodiversity.

The adverse effects of automotive emissions on air quality and climate are emphasized, highlighting the necessity of stringent regulations and cleaner technologies (Engr and Thomas, 2015). The impact of industrialization and pollution on natural resource biomass is addressed, and advocate the sustainable development strategies that balance economic growth with environmental conservation (Dubey et al., 2003; Dubey and Narayanan, 2010). Carbon emission-related issues are explored through modeling and simulation (Tsai, 2019). The link between population dynamics and the increment of global carbon dioxide level are discussed (Onozaki, 2009). The rise of sea level due to global warming are elaborated (Shukla et al., 2017) and the interconnected processes that influence atmospheric carbon levels and oceanic carbon storage are at (Nikol'skii, 2010). Several investigations are made to reduce the carbon dioxide level in the atmosphere by forest management programs and using technological options for controlling anthropogenic carbon emissions (Verma and Gautam, 2022; Verma and Misra, 2018). Factors influencing atmospheric carbon dioxide levels are discussed, contributing to our understanding of climate change dynamics and the impact of human activities on the environment (Misra and Verma, 2013)

Fractional calculus is a branch of mathematical analysis that extends traditional calculus to include non-integer or fractional orders of derivatives and integrals. It has become a valuable tool in modeling complex systems across fields like physics, engineering, biology, and economics etc., and also used to simulate real world problems (Guo et al., 2017; Kumar et al., 2016). Variable order fractional differential equations represent a sophisticated extension of traditional fractional calculus, allowing for dynamic modeling of complex systems with varying degrees of memory and long-range dependence. These fractional variable-order operators have many applications in signal transmission, communication theory, hydrogeology, chemical kinetics, reaction theory, control systems and cryptography (Atangana, 2015; Alkahtani et al., 2016; Atangana and Alqahtani, 2016; Coronel-Escamilla et al., 2017; Atangana, 2015).

2 Preliminaries

This section provides some basic definitions of variable order fractional derivatives which are used in subsequent sections.

Definition 2.1: The Liouville–Caputo (LC) fractional derivative with variable-order $\psi(\mathrm{t})$ is defined as

$${}_{0}^{LC}D_t^{\psi(\mathrm{t})} f(t)= \frac{1}{1-\psi(\mathrm{t})}\int_0^t (t-u)^{-\psi(\mathrm{t})} f(u)du \ , 0 <\psi(\mathrm{t}) \le 1$$

Definition 2.2: The Caputo–Fabrizio (CF) derivative with variable-order $\psi(\mathrm{t})$ in Liouville–Caputo sense is defined as follows

$${}_{0}^{CF}D_t^{\psi(\mathrm{t})} f(t) = \frac{(2-\psi(\mathrm{t}))M(\psi(\mathrm{t}))}{2(1-\psi(\mathrm{t}))}\int_0^t \exp\left[\frac{-\psi(\mathrm{t})}{(1-\psi(\mathrm{t}))}(t-u)\right]f'(u)du \ , 0 <\psi(\mathrm{t})<1$$

where $M(\psi(\mathrm{t})) = \frac{2}{2-\psi(\mathrm{t})}$ is a normalization function.

Definition 2.3: The Atangana–Baleanu (AB) fractional derivative with variable-order $\psi(\mathrm{t})$ in Liouville–Caputo sense is defined as follows

$${}_{0}^{AB}D_t^{\psi(\mathrm{t})} f(t) = \frac{B(\psi(\mathrm{t}))}{(1-\psi(\mathrm{t}))}\int_0^t E_{\psi(\mathrm{t})}\left[\frac{-\psi(\mathrm{t})}{(1-\psi(\mathrm{t}))}(t-u)^{\psi(\mathrm{t})}\right]f'(u)du \ , 0 <\psi(\mathrm{t}) \le 1$$

where $B(\psi(\mathrm{t})) = 1-\psi(\mathrm{t}) + \frac{\psi(\mathrm{t})}{\psi(\mathrm{t})}$ is a normalization function.

Remark: When $\psi(\mathrm{t})$ is a constant, then we retrieve the constant-order fractional derivative in

Liouville-Caputo, Caputo-Fabrizio and Atangana-Baleanu sense.

3 Model Formulation in Classical and Fractional Sense

3.1 Classical model of Artificial Rain Making

A mathematical model (ShyamSundar et al., 2022) has been developed to examine the effect of carbon dioxide to the environment caused by human activites. This model considers four state variables: $N(t)$ represents the density of human population at time t, $A(t)$ signifies the growth rate of cumulative density of human activities related factors at time t, $C(t)$ indicates the concentration of carbon dioxide in the atmosphere at time t and T(t) denotes the average atmospheric temperature at time t.

$$\frac{dN}{dt} = r\left(1-\frac{N}{K}\right)N - s(T-T_0)N$$

$$\frac{dA}{dt} = \lambda N - \gamma A$$

$$\frac{dC}{dt} = Q + \theta A - \delta C \qquad (1)$$

$$\frac{dT}{dt} = \varphi(C-C_0) - \omega(T-T_0)$$

where $N(0) \geq 0, A(0) \geq 0, C(0) \geq 0, T(0) = T_0 \geq 0$. The growth rate of human population is represented by r and carrying capacity is represented by K. The constant s is the death rate coefficient of population density due to average atmospheric temperature. The constant λ is the growth rate coefficient of cumulative density $A(t)$ of human activities related factors and γ is its depletion rate coefficient due to natural factors such as the inefficiency of these factors. Let Q be the constant input of carbon dioxide from natural sources and θ is its growth rate coefficient due to human activities related factors. δ represents natural depletion rate coefficient of carbon dioxide.The growth rate coefficient of temperature T with its natural depletion rate coefficient ω is represented by φ. C_0 is the threshold concentration of carbondioxide and T_0 is the initial concentration of atmospheric temperature.

3.2 Fractional version of the classical model

By substituting the classical derivative with the operator $\frac{d^{\psi(t)}f(t)}{dt}$, the fractional model of the system (1) is obtained.

$$\frac{d^{\psi(t)}N}{dt} = r^{\psi(t)}\left(1-\frac{N}{K}\right)N - s^{\psi(t)}(T-T_0)N$$

$$\frac{d^{\psi(t)}N}{dt} = \lambda^{\psi(t)}N - \gamma^{\psi(t)}A$$

$$\frac{d^{\psi(t)}C}{dt} = Q + \theta^{\psi(t)}A - \delta^{\psi(t)}C \qquad (2)$$

$$\frac{d^{\psi(t)}T}{dt} = \varphi^{\psi(t)}(C-C_0) - \omega^{\psi(t)}(T-T_0)$$

with initial conditions $N(0) \geq 0, A(0) \geq 0, C(0) \geq 0, T(0) = T_0 \geq 0$. The equilibria of the above fractional-order model can be obtained from

$$\frac{d^{\psi(t)}N}{dt} = 0\,,\ \frac{d^{\psi(t)}A}{dt} = 0\,, \frac{d^{\psi(t)}C}{dt} = 0 \ and\ \frac{d^{\psi(t)}T}{dt} = 0$$

It was observed that the system (1) has two equilibria, one of them is $E_0 = (0,0,C_0,T_0)$ and the other one is $E^* = (N^*, A^*, C^*, T^*)$.

The Jacobian matrix of the system (2) is as follows

$$J=\begin{bmatrix} r^{\psi(t)}\left(1-\frac{2N}{K}\right)-s^{\psi(t)}(T-T_0) & 0 & 0 & -s^{\psi(t)}N \\ \lambda^{\psi(t)} & -\gamma^{\psi(t)} & 0 & 0 \\ 0 & \theta^{\psi(t)} & -\delta^{\psi(t)} & 0 \\ 0 & 0 & \varphi^{\psi(t)} & -\omega^{\psi(t)} \end{bmatrix}$$

The eigen values are the solutions of the characteristic equation $\det(A_i-\lambda I)=0,$ where the matrix A_i and the unit matrix I with the eigen values calculated at E_0 and E^*. For further details of the results can be found in (Sundar et al., 2022)

Since the parameters are dimensionless, the fractional models within LC, CF and AB sense will be the same and it will not be necessary to investigate again.

4 Existence and Uniqueness of Fractional Solutions

4.1 Existence and uniqueness of fractional solutions by the Liouville-Caputo model

In this section, we establish the existence and uniqueness of solutions of the Liouville-Caputo model. Let us construct the system (2) as

$$\begin{aligned} {}^{LC}_{0}D_t^{\psi(t)}[N(t)] &= F_1(t,N)=r^{\psi(t)}\left(1-\frac{N}{K}\right)N-s^{\psi(t)}(T-T_0)N \\ {}^{LC}_{0}D_t^{\psi(t)}[A(t)] &= F_2(t,A)=\lambda^{\psi(t)}N-\gamma^{\psi(t)}A \\ {}^{LC}_{0}D_t^{\psi(t)}[C(t)] &= F_3(t,C)= Q+\theta^{\psi(t)}A-\delta^{\psi(t)}C \\ {}^{LC}_{0}D_t^{\psi(t)}[T(t)] &= F_4(t,T)=\varphi^{\psi(t)}(C-C_0)-\omega^{\psi(t)}(T-T_0) \end{aligned} \tag{3}$$

By using Liouville-Caputo fractional integral operator to the above system, we get

$$\begin{aligned} N(t)-N(0) &= \frac{1}{\psi(t)}\int_0^t (t-k)^{\psi(t)-1}F_1(K,N(t))dk \\ A(t)-A(0) &= \frac{1}{\psi(t)}\int_0^t (t-k)^{\psi(t)-1}F_2(K,A(t))dk \\ C(t)-C(0) &= \frac{1}{\psi(t)}\int_0^t (t-k)^{\psi(t)-1}F_3(K,C(t))dk \\ T(t)-T(0) &= \frac{1}{\psi(t)}\int_0^t (t-k)^{\psi(t)-1}F_4(K,T(t))dk \end{aligned} \tag{4}$$

We will show that the kernel F_i for $i=1,2,3,4$ follows the Lipschitz condition and contraction.

Theorem 4.1.1:

The kernel $F_i(K,N)$ for $i=1,2,3,4$ satisfies Lipschitz condition and contraction if the following inequality $0\le r_i<1$ holds.

Proof:

Consider two functions $N \; and \; \bar{N}$

$$\|F_1(t,N)-F_1(t,\bar{N})\|$$

$$=\left\|r^{\psi(t)}\left(1-\frac{N}{K}\right)N-s^{\psi(t)}(T-T_0)N-r^{\psi(t)}\left(1-\frac{\bar{N}}{K}\right)\bar{N}+s^{\psi(t)}(T-T_0)\bar{N}\right\|$$

$$=\left\|r^{\psi(t)}(N-\bar{N})-\frac{r^{\psi(t)}}{K}(N+\bar{N})(N-\bar{N})-s^{\psi(t)}(T-T_0)(N-\bar{N})\right\|$$

$$\le r^{\psi(t)}\|N-\bar{N}\|-\frac{r^{\psi(t)}}{K}\|N+\bar{N}\|\|N-\bar{N}\|-s^{\psi(t)}(T-T_0)\|N-\bar{N}\|$$

$$\leq \left[r^{\psi(t)} - \frac{r^{\psi(t)}}{K}(N+\bar{N}) - s^{\psi(t)}(T-T_0)\right]\|N-\bar{N}\|$$

$$\leq r_1\|N-\bar{N}\| \tag{5}$$

where $r_1 = \left[r^{\psi(t)} - \frac{r^{\psi(t)}}{K}(N+\bar{N}) - s^{\psi(t)}(T-T_0)\right]$ is a positive constant.As a result, the Lipschitz condition is met for $r_1$and if $0 \leq r_1 < 1$,then $r_1$follows contraction. Similarly, it can be exihibited and demonstrated in the other equations as follows

$$\|F_2(t,A) - F_2(t,\bar{A})\| \leq r_2\|A-\bar{A}\|$$
$$\|F_3(t,C) - F_3(t,\bar{C})\| \leq r_3\|C-\bar{C}\|$$
$$\|F_4(t,T) - F_4(t,\bar{T})\| \leq r_4\|T-\bar{T}\|$$

Therefore F_i satisfies Lipschitz condition. Also, if $0 \leq r_i < 1$, then the kernels follows contractions. From system (3), the recurrent form can be written as follows

$$\Phi_{1n} = N_n(t) - N_{n-1}(t)$$
$$= \frac{1}{\psi(t)}\int_0^t (t-k)^{\psi(t)-1}\,[F_1(K,N_{n-1}) - F_1(K,N_{n-2})]dk$$

$$\Phi_{2n} = A_n(t) - A_{n-1}(t)$$
$$= \frac{1}{\psi(t)}\int_0^t (t-k)^{\psi(t)-1}\,[F_2(K,A_{n-1}) - F_2(K,A_{n-2})]dk$$

$$\Phi_{3n} = C_n(t) - C_{n-1}(t)$$
$$= \frac{1}{\psi(t)}\int_0^t (t-k)^{\psi(t)-1}\,[F_3(K,C_{n-1}) - F_3(K,C_{n-2})]dk$$

$$\Phi_{4n} = T_n(t) - T_{n-1}(t)$$
$$= \frac{1}{\psi(t)}\int_0^t (t-k)^{\psi(t)-1}\,[F_4(K,T_{n-1}) - F_4(K,T_{n-2})]dk$$

Now taking norm for $\|\Phi_{1n}(t)\|$, we get

$$\|\Phi_{1n}(t)\| = \|N_n(t) - N_{n-1}(t)\|$$
$$= \left\|\frac{1}{\psi(t)}\int_0^t (t-k)^{\psi(t)-1}\,[F_1(K,N_{n-1}) - F_1(K,N_{n-2})]dk\right\|$$
$$\leq \frac{1}{\psi(t)}\int_0^t \left\|(t-k)^{\psi(t)-1}[F_1(K,N_{n-1}) - F_1(K,N_{n-2})]\right\| dk$$

Now using Lipschitz condition in the above equation, we obtain

$$\|\Phi_{1n}(t)\| \leq \frac{r_1}{\psi(t)}\int_0^t \left\|\Phi_{1(n-1)}(k)\right\|dk$$

similarly

$$\|\Phi_{2n}(t)\| \leq \frac{r_2}{\psi(t)}\int_0^t \left\|\Phi_{2(n-1)}(k)\right\|dk$$

$$\|\Phi_{3n}(t)\| \leq \frac{r_3}{\psi(t)}\int_0^t \|\Phi_{3(n-1)}(k)\| dk$$

$$\|\Phi_{4n}(t)\| \leq \frac{r_4}{\psi(t)}\int_0^t \|\Phi_{4(n-1)}(k)\| dk \qquad (6)$$

which implies that it can be written as

$N_n(t) = \sum_{i=1}^{n} \Phi_{1i}(t)$; $A_n(t) = \sum_{i=1}^{n} \Phi_{2i}(t)$,

$C_n(t) = \sum_{i=1}^{n} \Phi_{3i}(t)$,$T_n(t) = \sum_{i=1}^{n} \Phi_{4i}(t)$

Theorem 4.1.2:

The Liouville-Caputo model (3) has system of solutions if there exists *t* >1 such that $\frac{r_i t}{\psi(t)} \leq 1$ for *i*=1,2,3,4

Proof:

Consider,

$$\|\Phi_{1n}(t)\| \leq \frac{r_1}{\psi(t)}\int_0^t \|\Phi_{1(n-1)}(k)\| dk$$

Replacing *n* by *n*-1 in the above inequality

$$\|\Phi_{1(n-1)}(t)\| \leq \frac{r_1}{\psi(t)}\int_0^t \|\Phi_{1(n-2)}(k)\| dk$$

$$\leq \left[\frac{r_1}{\psi(t)}\right]^2 \int_0^t \|\Phi_{1(n-2)}(k)\| dk$$

Again, replacing *n* by *n*-2 in the given inequality

$$\|\Phi_{1(n-2)}(t)\| \leq \left[\frac{r_1}{\psi(t)}\right]^3 \int_0^t \|\Phi_{1(n-3)}(k)\| dk$$

On substituting in this way and using the initial condition we obtain

$$\|\Phi_{1n}(t)\| \leq \|N_n(0)\| \left[\frac{r_1 t}{\psi(t)}\right]^n$$

Similarly, we get ν(t)

$$\|\Phi_{2n}(t)\| \leq \|A_n(0)\| \left[\frac{r_2 t}{\psi(t)}\right]^n$$

$$\|\Phi_{3n}(t)\| \leq \|C_n(0)\| \left[\frac{r_3 t}{\psi(t)}\right]^n$$

$$\|\Phi_{4n}(t)\| \leq \|T_n(0)\| \left[\frac{r_4 t}{\psi(t)}\right]^n$$

This result proved the existence and continuity of solutions.

To show that $N(t)$, $A(t), C(t)$,and$T(t)$ are the solutions of (3) ,we consider the following equations

$$N(t) - N(0) = N_n(t) - R_{1n}(t)$$
$$A(t) - A(0) = A_n(t) - R_{2n}(t)$$
$$C(t) - C(0) = C_n(t) - R_{3n}(t) \qquad (7)$$
$$T(t) - T(0) = T_n(t) - R_{4n}(t)$$

$$\|R_{1n}(t)\| = \left\|\frac{1}{\psi(t)}\int_0^t [F_1(K, N_n) - F_1(K, N_{n-1})]\, dk\right\|$$

$\leq \frac{1}{\psi(t)} \int_0^t \|[F_1(K,N_n) - F_1(K,N_{n-1})]\| \, dk$

$\leq \frac{1}{\psi(t)} r_1 \|N_n - N_{n-1}\| t$

Applying the above process recursively,

$\|R_{1n}(t)\| = \left[\frac{r_1 t}{\psi(t)}\right]^{n+1} . M$

where M is the Lipschitz constant.

when $n \to \infty$, $\|R_{1n}(t)\| \to 0$

similarly we prove for

$\|R_{2n}(t)\| \to 0, \|R_{3n}(t)\| \to 0 \; and \|R_{4n}(t)\| \to 0$ as $n \to \infty$

Theorem 4.1.3:

If the condition $\left[1 - \frac{r_i t}{\psi(t)}\right] \geq 0$, for $i = 1,2,3,4$ holds then Liouville-Caputo model have unique solution.

Proof:

To establish the uniqueness for a solution of the system (3), consider the different set of solutions for the system (3), say $\bar{N}$, $\bar{A}$, $\bar{C}$ and $\bar{T}$. Then as an outcome of the first equation of (3), we write

$$N(t) - \bar{N}(t) = \frac{1}{\psi(t)} \int_0^t [F_1(K,N) - F_1(K,\bar{N})] \, dk$$

Using the norm of above equation

$\|N(t) - \bar{N}(t)\| = \left\| \frac{1}{\psi(t)} \int_0^t [F_1(K,N) - F_1(K,\bar{N})]] \, dk \right\|$

Now by applying Lipschitz condition

$$\|N(t) - \bar{N}(t)\| = \frac{1}{\psi(t)} r_1 t \|N(t) - \bar{N}(t)\|$$

Consequently

$\|N(t) - \bar{N}(t)\| - \frac{1}{\psi(t)} r_1 t \|N(t) - \bar{N}(t)\| \leq 0$

$\|N(t) - \bar{N}(t)\| [1 - \frac{1}{\psi(t)} r_1 t] \leq 0$ (8)

Since $[1 - \frac{1}{\psi(t)} r_i t] > 0$, equation (7) becomes the form

$\|N(t) - \bar{N}(t)\| = 0$

i.e., $N(t) = \bar{N}(t)$

similarly we prove

$A(t) = \bar{A}(t)$, $C(t) = \bar{C}(t)$and $T(t) = \bar{T}(t)$

4.2 Existence and Uniqueness of Fractional solutions by the Caputo-Fabrizio model

Let us construct the system (2) in the sense of Caputo-Fabrizio, we have

${}^{CF}_{0}D_t^{\psi(t)}[N(t)] = F_1(t,N) = r^{\psi(t)} \left(1 - \frac{N}{K}\right) N - s^{\psi(t)} (T - T_0) N$

${}^{CF}_{0}D_t^{\psi(t)}[A(t)] = F_2(t,A) = \lambda^{\psi(t)} N - \gamma^{\psi(t)} A$

$${}_{0}^{\mathrm{CF}}\mathrm{D}_{\mathrm{t}}^{\psi(t)}[C(\mathrm{t})] = \mathrm{F}_3(\mathrm{t}, C) = \ Q + \theta^{\psi(t)} A - \delta^{\psi(t)} C \quad (9)$$
$${}_{0}^{\mathrm{CF}}\mathrm{D}_{\mathrm{t}}^{\psi(t)}[T(\mathrm{t})] = \mathrm{F}_4(\mathrm{t}, T) = \varphi^{\psi(t)}(C - C_0) - \omega^{\psi(t)}(T - T_0)$$

The Caputo-Fabrizio integral form of the above system is

$$N(t) - N(0) = \frac{1-\psi(t)}{M(\psi(t))} F_1(t, N) + \frac{\psi(t)}{M(\psi(t))} \int_0^t F_1(\tau, N) d\tau$$

$$A(t) - A(0) = \frac{1-\psi(t)}{M(\psi(t))} F_2(t, A) + \frac{\psi(t)}{M(\psi(t))} \int_0^t F_2(\tau, A) d\tau$$

$$C(t) - C(0) = \frac{1-\psi(t)}{M(\psi(t))} F_3(t, C) + \frac{\psi(t)}{M(\psi(t))} \int_0^t F_3(\tau, C) d\tau$$

$$T(t) - T(0) = \frac{1-\psi(t)}{M(\psi(t))} F_4(t, T) + \frac{\psi(t)}{M(\psi(t))} \int_0^t F_4(\tau, T) d\tau \quad (10)$$

Here we have to prove the kernel F_i for $i = 1,2,3,4$ follows the Lipschitz condition and a contraction.

Theorem 4.2.1:

The kernel $F_i(\tau, N)$, for $i = 1,2,3,4$ satisfies the Lipschitz condition and a contraction if the following inequality $0 \le \rho_i < 1$ holds.

Proof:

This theorem is proved as similar as theorem 4.1.1

The recurrent form of (9) for the first equation is

$$\xi_{1n} = N_n(t) - N_{n-1}(t)$$
$$= \frac{1-\psi(t)}{M(\psi(t))} [F_1(t, N_{n-1}) - F_1(t, N_{n-2})]$$
$$+ \frac{\psi(t)}{M(\psi(t))} \int_0^t [F_1(\tau, N_{n-1}) - F_1(\tau, N_{n-2})] d\tau$$

Similarly ξ_{2n} and ξ_{3n} are also be derived

Using the initial condition and taking norm, we get

$$\|\xi_{1n}\| \le \frac{1-\psi(t)}{M(\psi(t))} \|[F_1(t, N_{n-1}) - F_1(t, N_{n-2})]\|$$
$$+ \frac{\psi(t)}{M(\psi(t))} \int_0^t \|[F_1(\tau, N_{n-1}) - F_1(\tau, N_{n-2})]\| \, d\tau$$

Since $\rho_1$satisfies Lipschitz condition

$$\|\xi_{1n}(t)\| \le \frac{1-\psi(t)}{M(\psi(t))} \rho_1 \left\|\left[\xi_{1(n-1)}(t)\right]\right\| + \frac{\psi(t)}{M(\psi(t))} \rho_1 \int_0^t \left\|\xi_{1(n-1)}(\tau)\right\| d\tau \quad (11)$$

Similarly $\| \xi_{2n}(t)\|$, $\| \xi_{3n}(t)\|$ and$\| \xi_{4n}(t)\|$ can also be obtained.

Therefore,

$N_n(t) = \sum_{i=1}^{n} \xi_{1i}(t)$, $A_n(t) = \sum_{i=1}^{n} \xi_{2i}(t)$, $C_n(t) = \sum_{i=1}^{n} \xi_{3i}(t)$and

$$T_n(t) = \sum_{i=1}^{n} \xi_{4i}(t)$$

Theorem 4.2.2:

The Caputo Fabrizio fractional derivative model (9) has system of solutions if there exists $v > 1$ such that $\left[\frac{1-\psi(t)}{M(\psi(t))}\rho_i + \frac{\psi(t)}{M(\psi(t))}\rho_i\, v\right] \leq 1$, for i=1,2,3,4

Proof:

Operating (11) recursively and using the initial conditions we have

$$\|\xi_{1n}(t)\| \leq \|N_n(0)\|\left[\frac{1-\psi(t)}{M(\psi(t))}\rho_1 + \frac{\psi(t)}{M(\psi(t))}\rho_1 v\right]^n$$

Similarly we have for $\|\xi_{2n}(t)\|$, $\|\xi_{3n}(t)\|$, $\|\xi_{4n}(t)\|$ and $\|\xi_{5n}(t)\|$this result proved the existence and continuity of solution.

To show that $(t), A(t), C(t)$and $T(t)$ are the solutions of (9)

consider

$$N(t) - N(0) = N_n(t) - D_{1n}(t)$$
$$A(t) - A(0) = A_n(t) - D_{2n}(t)$$
$$C(t) - C(0) = C_n(t) - D_{3n}(t) \qquad (12)$$
$$T(t) - T(0) = T_n(t) - D_{4n}(t)$$

Now

$$\|D_{1n}(t)\| \leq \frac{1-\psi(t)}{M(\psi(t))}\|[F_1(t, N_n) - F_1(t, N_{n-1})]\|$$

$$+\frac{\psi(t)}{M(\psi(t))}\int_0^t \|[F_1(\tau, N_n) - F_1(\tau, N_{n-1})]\|\, d\tau$$

$$\leq \frac{1-\psi(t)}{M(\psi(t))}\rho_1\|N_n - N_{n-1}\| + \frac{\psi(t)}{M(\psi(t))}\rho_1\|N_n - N_{n-1}\|v$$

$$\leq \left[\frac{1-\psi(t)}{M(\psi(t))}\rho_1 + \frac{\psi(t)}{M(\psi(t))}\rho_1 v\right]\|N_n - N_{n-1}\|$$

Applying the above process recursively

$$\|D_{1n}(t)\| \leq \left[\frac{1-\psi(t)}{M(\psi(t))}\rho_1 + \frac{\psi(t)}{M(\psi(t))}\rho_1 v\right]^{n+1} .S$$

where S is the Lipschitz constant

when $n \to \infty$, $\|D_{1n}\| \to 0$

Similarly we prove for $\|D_{2n}\| \to 0$,$\|D_{3n}\| \to 0$ and$\|D_{4n}\| \to 0$ as $n \to \infty$

Theorem 4.2.3:

If the condition $\left[1 - \left[\frac{1-\psi(t)}{M(\psi(t))}\rho_i + \frac{\psi(t)}{M(\psi(t))}\rho_i v\right]\right] \geq 0$, for i=1,2,3,4 holds then the Caputo-Fabrizio fractional derivative model have unique solutions.

Proof:

Suppose the system (8) has another solution $\bar{N},\ \bar{A},\ \bar{C}\ and\ \overline{T}$

$$\|N(t) - \bar{N}(t)\| = \left\|\frac{1}{\psi(t)}\int_0^t [F_1(K, N) - F_1(K, \bar{N})]]\, dk\right\|$$

$$N(t) - \bar{N}(t) = \frac{1-\psi(t)}{M(\psi(t))}[F_1(t, N) - F_1(t, \bar{N})] + \frac{\psi(t)}{M(\psi(t))}\int_0^t [F_1(\tau, N) - F_1(\tau, \bar{N})]\, d\tau$$

Using norm and applying Lipschitz condition

$\|N(t)-\bar{N}(t)\| \leq \left[\frac{1-\psi(t)}{M(\psi(t))}\rho_1 + \frac{\psi(t)}{M(\psi(t))}\rho_1 v\right]\|N(t)-\bar{N}(t)\|$

Consequently we have

$\|N(t)-\bar{N}(t)\|$

$\left[1-\left[\frac{1-\psi(t)}{M(\psi(t))}\rho_1 + \frac{\psi(t)}{M(\psi(t))}\rho_1 v\right]\right] \leq 0$

Since $\left[1-\left[\frac{1-\psi(t)}{M(\psi(t))}\rho_1 + \frac{\psi(t)}{M(\psi(t))}\rho_1 v\right]\right] > 0$, we have

$\|N(t)-\bar{N}(t)\| = 0$

i.e.,$N(t) = \bar{N}(t)$

Similarly we prove

$A(t) = \bar{A}(t),\ C(t) = \bar{C}(t)$and $T(t) = \bar{T}(t)$

4.3 Existence and uniqueness of solutions for the Atangana-Baleanu fractional model

Let us construct (2) in AtanganaBaleanu fractional derivative in Caputo sense

$${}^{AB}_{0}D_t^{\psi(t)}[N(t)] = F_1(t,N) = r^{\psi(t)}\left(1-\frac{N}{K}\right)N - s^{\psi(t)}(T-T_0)N$$

$$\begin{aligned}
{}^{AB}_{0}D_t^{\psi(t)}[A(t)] &= F_2(t,A) = \lambda^{\psi(t)}N - \gamma^{\psi(t)}A \\
{}^{AB}_{0}D_t^{\psi(t)}[C(t)] &= F_3(t,C) = Q + \theta^{\psi(t)}A - \delta^{\psi(t)}C \\
{}^{AB}_{0}D_t^{\psi(t)}[T(t)] &= F_4(t,T) = \varphi^{\psi(t)}(C-C_0) - \omega^{\psi(t)}(T-T_0)
\end{aligned} \tag{13}$$

The Atangana-Baleanu integral form of the above system is

$$N(t)-N(0) = \frac{1-\psi(t)}{AB(\psi(t))}F_1(t,N) + \frac{\psi(t)}{AB(\psi(t))\psi(t)}\int_0^t (t-\gamma)^{\psi(t)-1}F_1(\gamma,N)d\gamma$$

$$A(t)-A(0) = \frac{1-\psi(t)}{AB(\psi(t))}F_2(t,A) + \frac{\psi(t)}{AB(\psi(t))\psi(t)}\int_0^t (t-\gamma)^{\psi(t)-1}F_2(\gamma,A)d\gamma$$

$$C(t)-C(0) = \frac{1-\psi(t)}{AB(\psi(t))}F_3(t,C) + \frac{\psi(t)}{AB(\psi(t))\psi(t)}\int_0^t (t-\gamma)^{\psi(t)-1}F_3(\gamma,C)d\gamma$$

$$T(t)-T(0) = \frac{1-\psi(t)}{AB(\psi(t))}F_4(t,T) + \frac{\psi(t)}{AB(\psi(t))\psi(t)}\int_0^t (t-\gamma)^{\psi(t)-1}F_4(\gamma,T)d\gamma$$

Now to prove the kernel F_i for $i = 1,2,3,4$ follows the Lipschitz condition and contraction.

Theorem 4.3.1:

The kernel $F_i(\gamma,N)$, for $i = 1,2,3,4$ satisfies the Lipschitz condition and contraction if $0 \leq \delta_i < 1$ holds.

Proof:

The proof is similar to the proof of 4.1.1

For the first equation, the recurrent form of (13) is

$$\theta_{1n} = N_n(t) - N_{n-1}(t) = \frac{1-\psi(t)}{AB(\psi(t))}[F_1(t,N_{n-1}) - F_1(t,N_{n-2})]$$

$$+\frac{\psi(t)}{AB(\psi(t))\psi(t)}\int_0^t (t-\gamma)^{\psi(t)-1}[F_1(\gamma,N_{n-1}) - F_1(\gamma,N_{n-2})]d\gamma$$

Similarly θ_{2n},θ_{3n} and θ_{4n} are also be derived.
Using the initial condition and taking norm, we get

$$\|\theta_{1n}\| \leq \frac{1-\psi(t)}{AB(\psi(t))}\|[F_1(t,N_{n-1})-F_1(t,N_{n-2})]\|$$

$$+\frac{\psi(t)}{AB(\psi(t))\psi(t)}\int_0^t (t-\gamma)^{\psi(t)-1}\|[F_1(\gamma,N_{n-1})-F_1(\gamma,N_{n-2})]\|\,d\gamma$$

Since δ_1 satisfies Lipschitz condition

$$\|\theta_{1n}\| \leq \frac{1-\psi(t)}{AB(\psi(t))}\delta_1\left\|\theta_{1(n-1)}\right\| + \frac{\psi(t)}{AB(\psi(t))\psi(t)}\delta_1\int_0^t (t-\gamma)^{\psi(t)-1}\left\|\theta_{1(n-1)}(\gamma)\right\|d\gamma$$

Similarly for $\|\theta_{2n}\|$, $\|\theta_{3n}\|$ and $\|\theta_{4n}\|$
which implies that it can be written as

$$N_n(t)=\sum_{i=1}^n \theta_{1i}(t)\,, A_n(t)=\sum_{i=1}^n \theta_{2i}(t)\,, C(t)=\sum_{i=1}^n \theta_{3i}(t)\ \ ,T_n(t)=\sum_{i=1}^n \theta_{4i}(t) \qquad (14)$$

Theorem 4.3.2:
The Atangana-Baleanu derivative model (13) have system of solutions, if there exists μ>1 such that

$$\left[\frac{1-\psi(t)}{AB(\psi(t))}\delta_i + \frac{\psi(t)}{AB(\psi(t))\psi(t)}\delta_i\mu\right] \leq 1\ for\ \ i=1,2,3,4$$

Proof:
Consider,

$$\|\theta_{1n}\| \leq \|N_n(0)\|\left[\frac{1-\psi(t)}{AB(\psi(t))}\delta_1 + \frac{\psi(t)}{AB(\psi(t))\psi(t)}\delta_1\mu\right]^n$$

Similarly, $\|\theta_{2n}\|$, $\|\theta_{3n}\|$ and$\|\theta_{4n}\|$can also be obtained
These results proved the existence and continuity of solution.
Now to show that (t), $C(t)$, and $T(t)$are solutions of (13)
Consider

$$N(t)-N(0)=N_n(t)-E_{1n}(t)$$
$$A(t)-A(0)=A_n(t)-E_{2n}(t)$$
$$C(t)-C(0)=C_n(t)-E_{3n}(t) \qquad (15)$$
$$T(t)-T(0)=T_n(t)-E_{4n}(t)$$

Now

$$\|E_{1n}(t)\| \leq \frac{1-\psi(t)}{AB(\psi(t))}\|[F_1(t,N_n)-F_1(t,N_{n-1})]\|$$

$$+\frac{\psi(t)}{AB(\psi(t))\psi(t)}\int_0^t (t-\gamma)^{\psi(t)-1}\|[F_1(\gamma,N_n)-F_1(\gamma,N_{n-1})]\|\,d\gamma$$

$$\leq \frac{1-\psi(t)}{AB(\psi(t))}\delta_1\|N_n-N_{n-1}\| + \frac{\psi(t)}{AB(\psi(t))\psi(t)}\delta_1\|N_n-N_{n-1}\|\mu$$

$$\leq \left[\frac{1-\psi(t)}{AB(\psi(t))}\delta_1 + \frac{\psi(t)}{AB(\psi(t))\psi(t)}\delta_1\mu\right]\|N_n-N_{n-1}\|$$

Applying the above process recursively

$$\|E_{1n}(t)\| \leq \left[\frac{1-\psi(t)}{AB(\psi(t))}\delta_1 + \frac{\psi(t)}{AB(\psi(t))\psi(t)}\delta_1\mu\right]^{n+1}.W$$

where W is the Lipschitz constant
when $n\to\infty$, $\|E_{1n}\| \to 0$

Similarly we prove for $\|E_{2n}\| \to 0$, $\|E_{3n}\| \to 0$ and $\|E_{4n}\| \to 0$ as $n \to \infty$

Theorem 4.3.3:

If the condition $\left[1 - \left[\frac{1-\psi(t)}{AB(\psi(t))}\delta_i + \frac{\psi(t)}{AB(\psi(t))\psi(t)}\delta_i\mu\right]\right] \geq 0$ for *i*=1,2,3,4 holds then the Atangana-Baleanu fractional derivative model have unique solutions.

Proof:

Suppose the system (13) has another solution $\bar{N}$, $\bar{A}$, $\bar{C}$ $and\ \overline{T}$

then

$$N(t) - \bar{N}(t) = \frac{1-\psi(t)}{AB(\psi(t))}[F_1(t,N) - F_1(t,\bar{N})]$$

$$+\frac{\psi(t)}{AB(\psi(t))\psi(t)}\int_0^t (t-\gamma)^{\psi(t)-1}[F_1(\gamma,N) - F_1(\gamma,\bar{N})]d\gamma$$

Using norm and apply Lipschitz condition

$$\|N(t) - \bar{N}(t)\| \leq (\left[\frac{1-\psi(t)}{AB(\psi(t))}\delta_1 + \frac{\psi(t)}{AB(\psi(t))\psi(t)}\delta_1\mu\right]\|N(t) - \bar{N}(t)\|)$$

Consequently we have

$$\left[1 - \left[\frac{1-\psi(t)}{AB(\psi(t))}\delta_1 + \frac{\psi(t)}{AB(\psi(t))\psi(t)}\delta_1\mu\right]\right]\|N(t) - \bar{N}(t)\| \leq 0$$

Since $\left[1 - \left[\frac{1-\psi(t)}{AB(\psi(t))}\delta_1 + \frac{\psi(t)}{AB(\psi(t))\psi(t)}\delta_1\mu\right]\right] > 0$, we have

$\|N(t) - \bar{N}(t)\| = 0$

i.e $N(t) = \bar{N}(t)$

Similarly we prove

$A(t) = \bar{A}(t)$, $C(t) = \bar{C}(t)$and $T(t) = \bar{T}(t)$

5 Numerical Scheme

In this section, the Numerical scheme (Solís-Pérez et al., 2018) is considered in the sense of Liouville-Caputo, Caputo-Fabrizio and Atangana –Baleanu fractional derivatives.

Let us consider our fractional model as

${}_0^*D_t^\alpha u(t) = f(t, u(t))$

where * denotes LC,CF and AB terms and $u(t) = (N(t), A(t), C(t), T(t))$.

Now we use the numerical scheme (Solís-Pérez et al., 2018) represented for Liouville-Caputo (16), Caputo-Fabrizio (17) and Atangana-Baleanu (18) fractional derivatives in (2)

$$u_{n+1}(t) = u(0) + \frac{1}{\psi(t)}\sum_{m=0}^{n}\left(\begin{array}{c}\frac{h^{\psi(t)}f(t_m,u_m)}{\psi(t)(\psi(t)+1)} \\ ((n-m+2+2\alpha) - \frac{h^{\psi(t)}f(t_{m-1},u_{m-1})}{\psi(t)(\psi(t)+1)} \\ \left(\begin{array}{c}(n+1-m)^{\psi(t)+1} - \\ (n-m)^{\psi(t))}(n-m+1+\psi(t)\end{array}\right)\end{array}\right) \tag{16}$$

$$(u_{n+1}) = (u_n) + \left[\frac{(2-\psi(t))(1-\psi(t))}{2} + \frac{3h}{4}\psi(t)(2-\psi(t))\right]f(t_n,u_n) - \left[\begin{array}{c}\frac{(2-\psi(t))(1-\psi(t))}{2} + \\ \frac{h}{4}\psi(t)\ (2-\psi(t))\end{array}\right]f(t_{n-1},u_{n-1}) \tag{17}$$

$$u_{n+1}(t) = u(0) + \frac{\psi(t)(1-\psi(t))}{\psi(t)(1-\psi(t))+\psi(t)} f(t_n, u_n)$$
$$+ \frac{1}{(\psi(t)+1)((1-\psi(t))\psi(t))+\psi(t)} \sum_{m=0}^{n} \left(\begin{array}{c} h^{\psi(t)} f(t_m, u_m) \\ \left(\begin{array}{c} (n+1-m)^{\psi(t)}(n-m+2+\psi(t)) \\ -(n-m)^{\psi(t)}(n-m+2+2\alpha(t)) \end{array} \right) \\ -h^{\psi(t)} f(t_{m-1}, u_{m-1}) \\ \left(\begin{array}{c} (n+1-\ m)^{\psi(t)+1} \\ -(n-m)^{\psi(t)}(n-m+1+\psi(t)) \end{array} \right) \end{array} \right) \quad (18)$$

6 Results and Discussion

The primary objective of our proposed fractional model is to observe the impact of CO_2emission caused by human activities on atmospheric temperature through various variables as well as fractional orders.

Case 1(Variable-order case): The growth of human population *N*(*t*) is affected adversely by the atmospheric temperature contributing global warming is represented by the variable order function $\psi(t)$ =P+Q $e^{(-N.t)}$, where P represents the initial value while Q indicates the difference between initial and steady state value and Nsymbolizes the decay constant with respect to time *t*. *A*(*t*) takes the variable order form as $\psi(t)$=$K/(1+e^{-rt})$, where *K* represents the carrying capacity and *r* is the growth rate parameter. This type of function is more appropriate for situations where pollution increases with increasing automobile usage until it reaches a limit imposed by environmental factors or regulations. *C*(*t*) takes the variable order as $\psi(t)$=S_0e^{-rt}, where S_0signifies the initial concentration of CO_2 and *r* exemplifies the growth rate of CO_2 concentration. *T*(*t*) takes the variable order as $\psi(t)$=T_0e^{rt}, where T_0 depicts the initial temperature of the atmosphere and r conveys the growth rate constant.

Figs 1, 2, 3 and 4 represents the above discussed variable-order fractional forms of *N*(*t*), *A*(*t*), *C*(*t*) and *T*(*t*) in LC,CF and AB senses by taking possible parameter values from the literature. We utilized MATLAB R2023a programming language to perform numerical simulation of our fractional model.

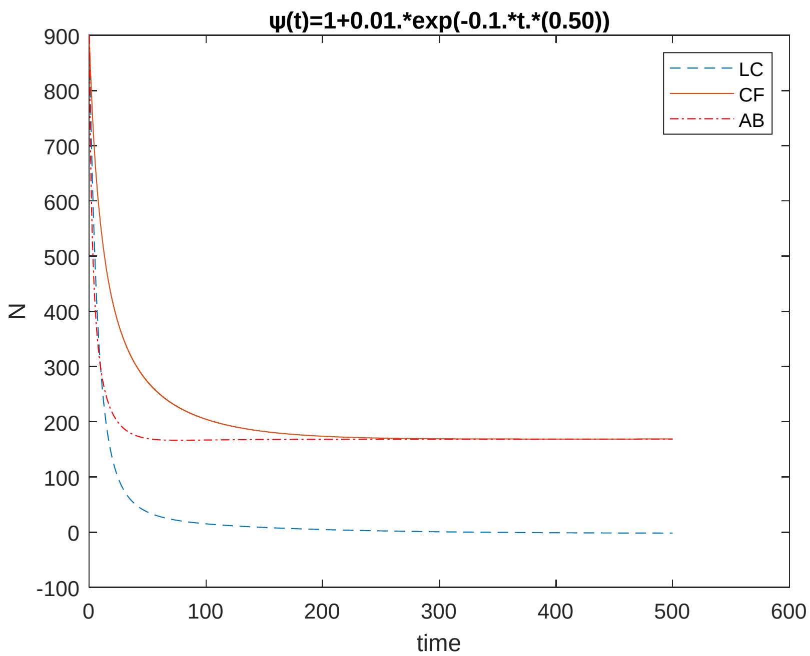


Fig. 1 Comparison graph of *N*(*t*) via LC, CF and AB.

Fig. 1 presents a graphic representation of the distribution of human population according to a variable order ψ(t)=1+0.01exp(-0.1 x t x 0.50). It is observed that in the cases of CF and AB, the density of the population rapidly decreases and stabilizes over time. However, in the case of LC, the population density decreases until it reaches a saturation point, which is considered unusual due to the singularity of the LC derivative.

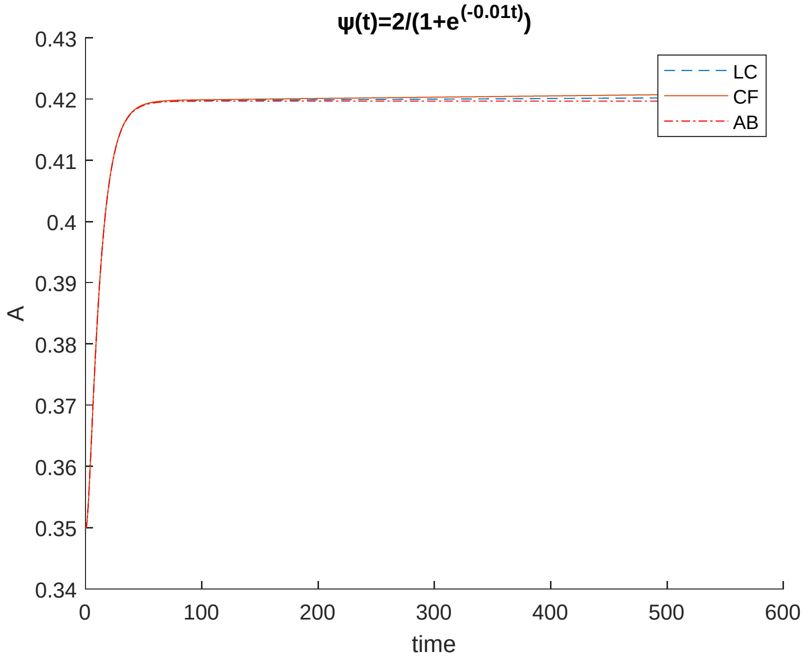


Fig. 2 Comparison graph of $A(t)$ via LC, CF and AB

Fig. 2 depicts the variation in cumulative density of human activities by $\psi(t)=2/(1+e^{-0.01t})$.Here 2(tons of CO_2 per unit area) represents the carrying capacity and 0.01 signifies the growth rate parameter. It is evident that in all instances of LC, CF and AB, the total density of human impacts on the environment increases rapidly before eventually stabilizing in the atmosphere.

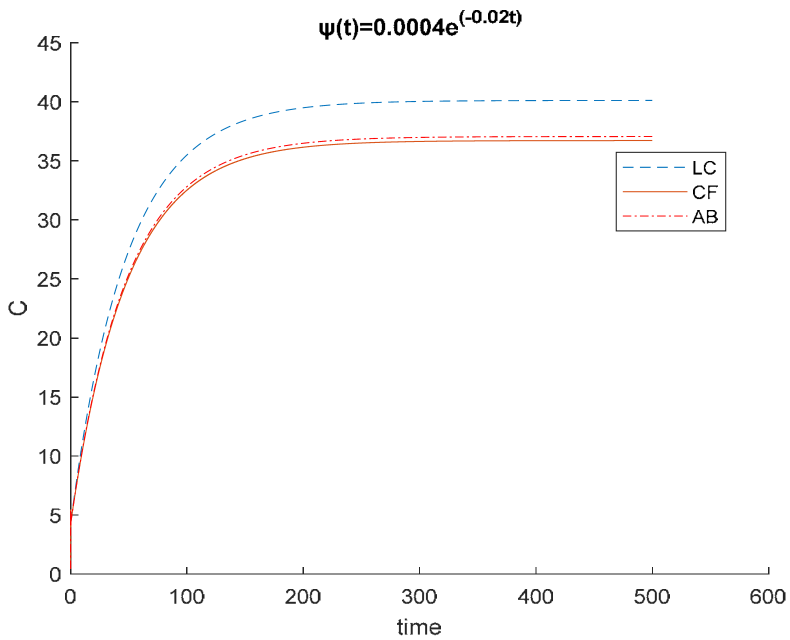


Fig. 3 Comparison graph of $C(t)$ via LC, CF and AB.

Fig. 3 displays the progression of concentration of CO_2 by the variable order function $\psi(t)$=0.0004e$^{-0.02t}$ where 0.0004 (parts per million) represents the initial concentration of CO_2 and 0.02 indicates the rate of CO_2concentration growth in the atmosphere. It is observed that the CO_2 concentration gradually increases and stabilizes more slowly in the cases of CF and AB compared to LC.

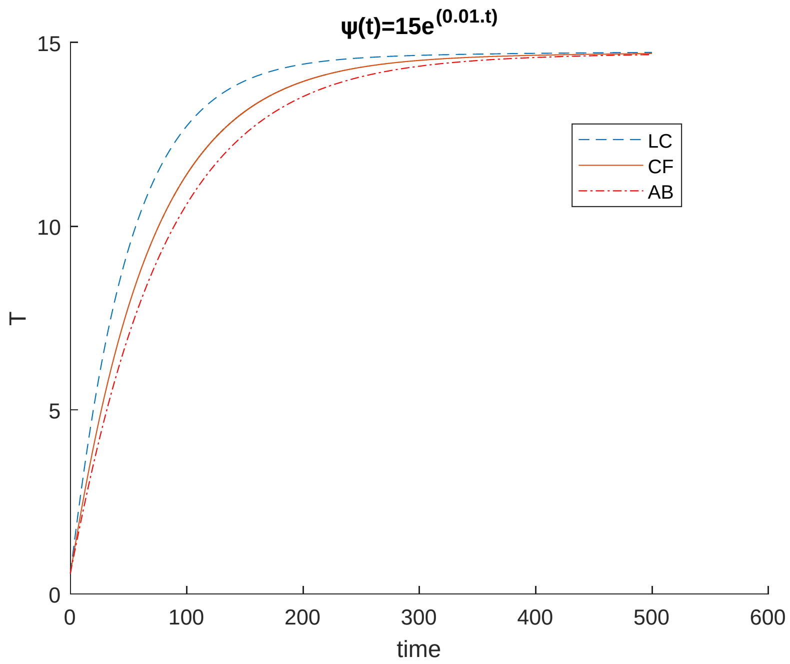
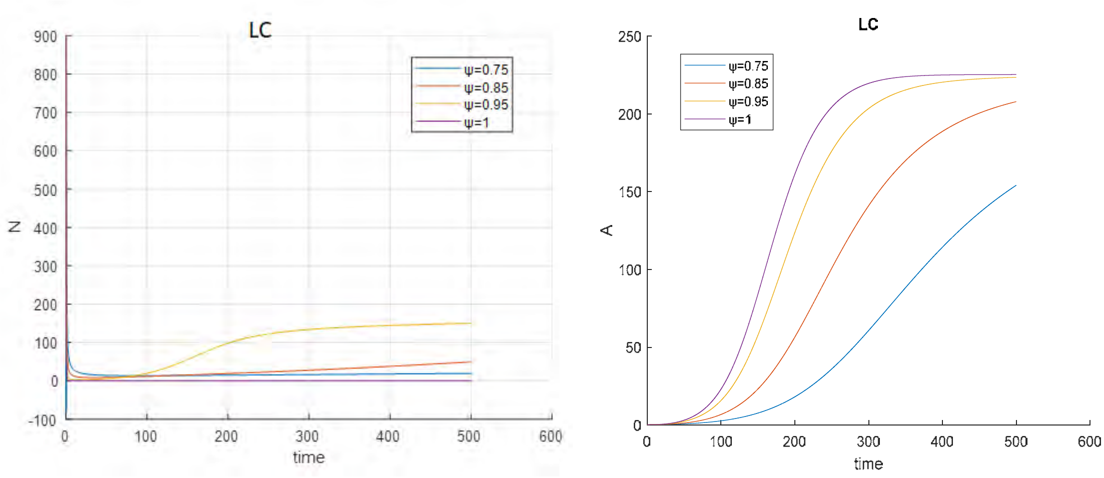
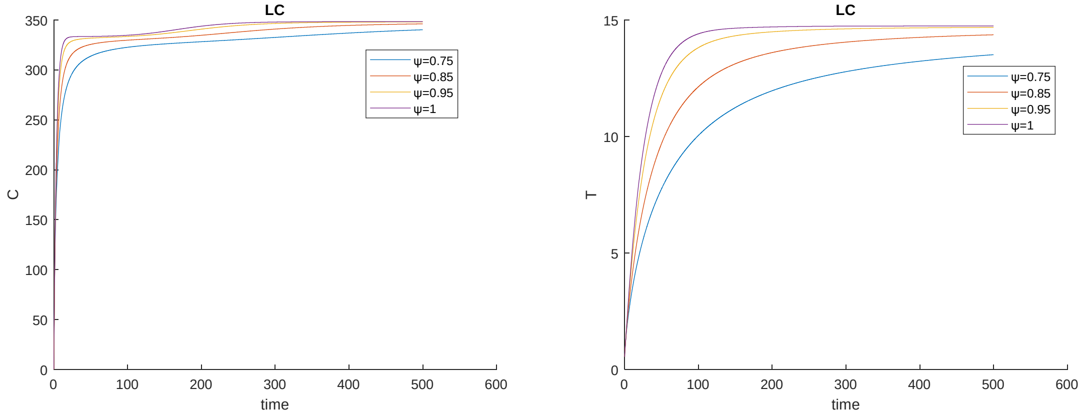


Fig. 4 Comparison graph of $T(t)$ via LC,CF and AB.

Fig. 4 provides variation in the atmospheric temperature by the variable order function $\psi(t)$=15e$^{0.01t}$, where 15 (degree Celsius) initial temperature of the atmosphere and 0.01 represents the growth rate. It is noted that in all instances LC, CF and AB ,the atmospheric temperature rises gradually increases over time.

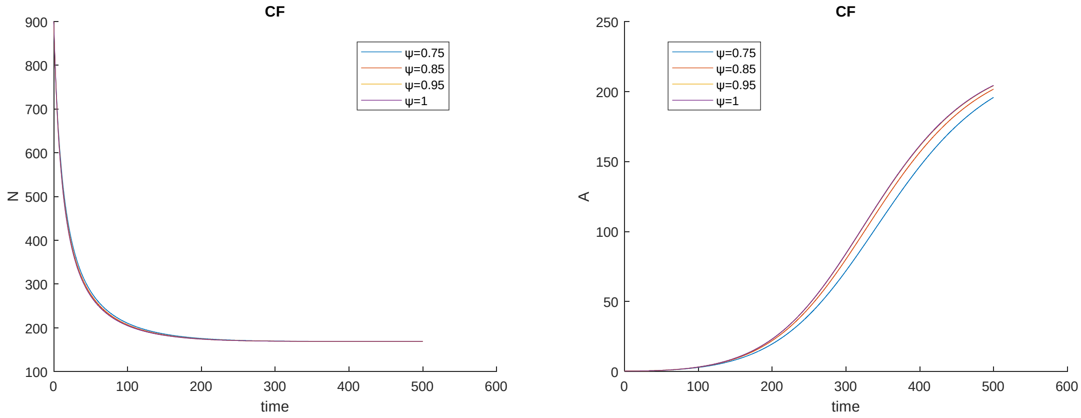


(a) (b)

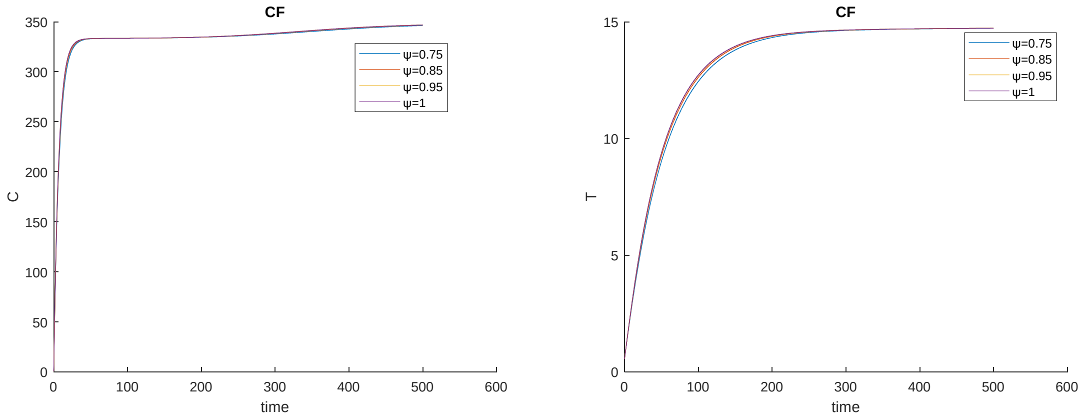


(c) (d)

Fig. 5 Numerical simulations for various order of ψ at 0.75, 0.85, 0.95 and 1 in Liouville-Caputo sense.



(a) (b)



(c) (d)

Fig. 6 Numerical simulations for various order of ψ at 0.75, 0.85, 0.95 and 1 in Caputo-Fabrizio sense.

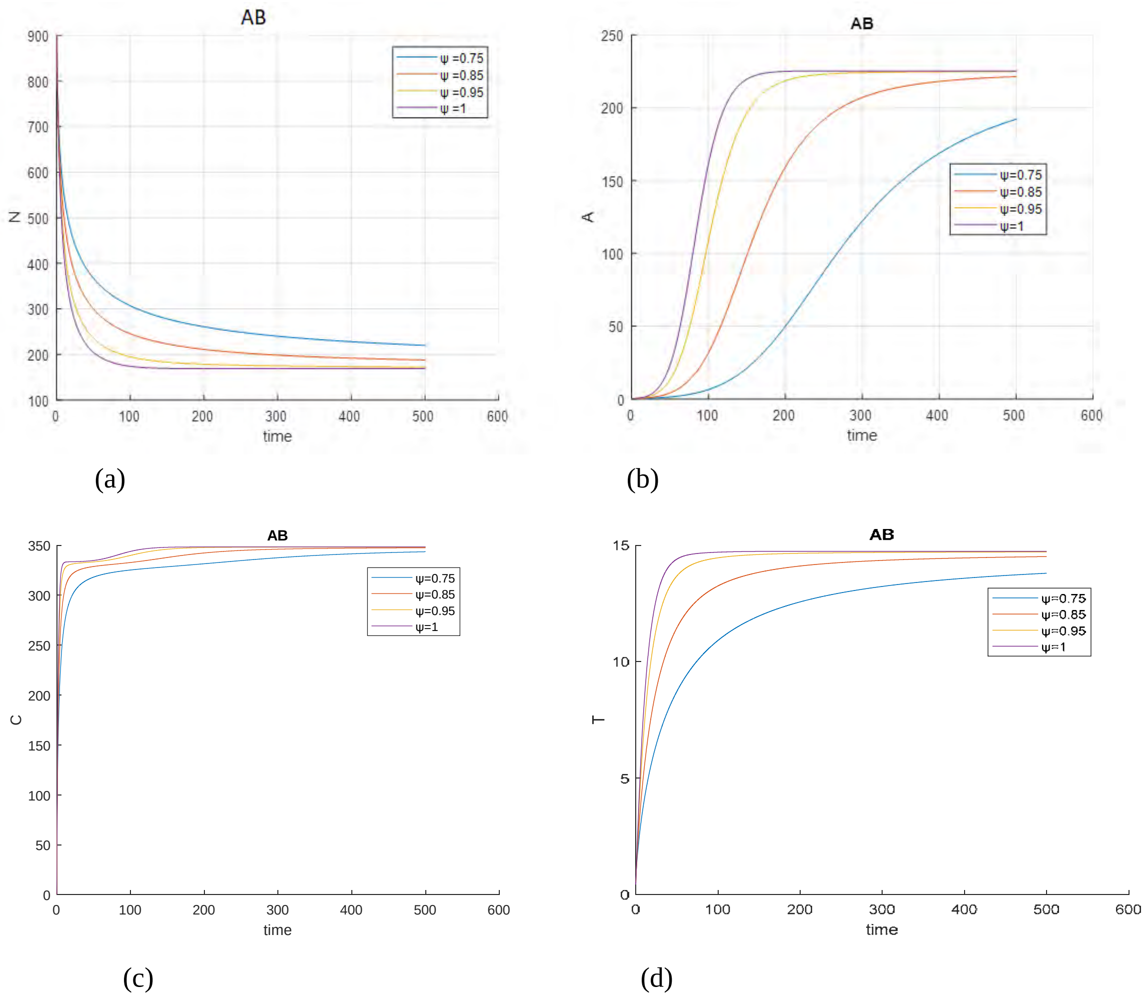


Fig. 7 Numerical simulations for various order of ψ at 0.75, 0.85, 0.95 and 1 in Atangana-Baleanu sense.

Case 2 (Fractional-order case): Figs 5,6 and 7 depicts the numerical simulation of $N(t)$, $A(t)$, $C(t)$ and $T(t)$ for various fractional values 0.75, 0.85, 0.95 and 1 of Liouville-Caputo,Caputo-Fabrizio and Atangana-Baleanu models respectively.

In the context of Liouville-Caputo interpretation, Fig. 5 illustrates a sudden decline in the human population density, reaching a saturation point. Conversely, in the case of CF Fig. 6, the quantity N(t) promptly diminishes and remains constant for all considered values of ψ. When considering Figs 5 and 7, it is evident that the overall density of human impacts on the environment, denoted as $A(t)$, increases linearly as ψ grows, particularly seen in LC and AB scenarios. Conversely, as shown in Fig. 6, the simulation of CF remains consistent across all ψ values, showcasing a steady pattern in human impacts on the environment. Figs 5 and 7 demonstrate that as ψ increases, the concentration of CO_2 also increases and remains stable in the atmosphere for LC and AB. However, in the case of CF, the CO_2 concentration increment remains consistent across all ψ values. From Figs 5 and 7 it is observed that, when ψ increases for LC,AB there is a gradual rise in atmospheric temperature. Conversely, in Fig. 6 (CF), the temperature $T(t)$ spikes instantly and remains stable across all ψ values.

As from the literature (Misra and Verma, 2013; Tsai, 2019; Onozaki, 2009), it seems that for variable order case the results of CF and AB have almost the same variation when compared to LC. For fractional–order case ,it is observed that AB have better memory effect than LC and CF. Since LC reaches a saturation point in the study of human population density, it's not feasible. Additionally, for all CF cases, it's observed that no variation occurs for any values of ψ, which violates our core consideration.

7 Conclusions

The research investigates the impact of human-induced carbon dioxide emissions on global warming using a novel fractional variable-order model. It examines the relationship between CO_2 emissions and atmospheric temperature through numerical simulations employing different fractional derivatives. The study emphasizes the changes in greenhouse gas balance since the Industrial Revolution and their effects on Earth's climate. The results demonstrate differences in population density, human activities, CO_2 concentration, and temperature across different parameters and fractional derivatives. The study highlights the significance of fractional calculus in understanding climate dynamics and suggests implications for mitigation strategies.

References

Atangana A.2015.On the stability and convergence of the time-fractionalvariable-order telegraph equation. Journal of Computational Physics, 293: 104-114

Atangana A, Alqahtani RT. 2016. Stability analysis of nonlinear thin viscous fluid sheet flow equation with local fractional variable-order derivative. Journal of computational and theoretical Nanoscience, 13(5): 2710-2717

Alkahtani BST, Koca I, Atangana A.2016. A novel approach of variable order derivative: theory and methods. Journal of Nonlinear Sciences and Applications, 9(6):4867–76.

Coronel-Escamilla A, Gómez-Aguilar JF, Torres L, Escobar-Jiméne RF, Valtierra Rodríguez M. 2017. Synchronization of chaotic systems involving fractional operators of Liouville-Caputo type with variable-order. Physica A, 487: 1-21

Dubey B, Upadhyay RK, Hussain J. 2003. Effects of industrialization and pollution on resource biomass: a mathematical model. Ecological Modelling, 167: 83-95

Dubey B, Narayanan AS. 2010. Modelling effects of industrialization, population and pollution on a renewable resource. Nonlinear Analysis: Real World Applications, 11: 2833-2848

Engr JOM, Thomas CG. 2015. Automotive exhaust emissions and its implications for environment sustainability. International Journal of Advanced Academic Research, 1(2): 1-11

Guo YM, Zhao Y, Zhou YM, Xiao ZB, Yang XJ. 2017. On the local fractional LWR model in fractal traffic flows in the entropy condition. Mathematical Models and Methods in Applied Sciences, 40(17): 6127-6132

Misra AK, Verma M .2013.A mathematical model to study the dynamics of carbon dioxide gas in the atmosphere. Applied Mathematics and Computation, 219(16): 8595-8609

Misra AK, Verma M, Venturino E. 2015. Modeling the control of atmospheric carbon dioxide through reforestation: effect of time delay. Modeling Earth System and Environment, 1(3): 1-17

Nikol'skii MS .2010.A controlled model of carbon circulation between the atmosphere and the ocean. Computational Mathematics and Modeling, 21: 414-424

Onozaki K .2009.Population is a critical factor for global carbon dioxide increase. Health Science Journal 55(1): 125-127

Sundar S, Naresh R. 2015. Modeling and analysis of the survival of a biological species in a polluted environment: Effect of environmental tax. Computational Ecology and Software, 5(2): 201-221

Shukla JB, Verma M, Misra AK .2017. Effect of global warming on sea level rise: a modeling study.

Ecological Complexity, 32: 99-110

Solís-Pérez, J.E., Gómez-Aguilar, J.F., Atangana. 2018. A. Novel numerical method for solving variable-order fractional differential equations with power, exponential and Mittag-Lefflerlaws. Chaos, Solitons and Fractals, 114: 175-185

Sundar S, Mishra AK, Naresh R, Shukla JB. 2022. Impact of carbon dioxide emissions caused by human activities on atmospheric temperature: A mathematical model. Computational Ecology and Software, 12(2): 23-37

Tsai WH. 2019. Modeling and simulation of carbon emission-related issues. Energies, 12(13): 2531

Verma M, Gautam C. 2022. Optimal mitigation of atmospheric carbon dioxide through forest management programs: a modeling study. Computational and Applied Mathematics, 41(7): 320

Verma M, Misra AK. 2018. Optimal control of anthropogenic carbon dioxide emissions through technological options: a modeling study. Computational and Applied Mathematics, 37(1): 605-626