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Stability analysis in generalist predator-prey dynamics with predator harvesting

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Abstract

This paper seeks to explore the stability assessment of a generalist predator-prey system with predator harvesting. The study presents a model examining the dynamics between primary prey and predator, considering the existence of a generalist predator. Prey growth follows a logistic rate, while predator consumption is modeled with a cyrtoid functional response. In the absence of primary prey, the predator population adopts a generalist strategy, akin to the Beverton-Holt model. The model also incorporates harvesting on the predator population. We analyze the model's equilibrium, stability, positivity, and boundedness, and use numerical simulations to explore its predictions. This study enhances our understanding of ecological interactions and supports the development of effective conservation and management strategies.

Keywords equilibrium point; generalist predator; cyrtoid functional response; local and global stability; harvesting.

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1 Introduction

The dynamics of prey-predator relationships play a crucial role in shaping species composition within an ecosystem and the dynamics of those ecosystems. Specifically, interactions between prey and predators serve as mechanisms leading to the emergence of population cycles. Since Lotka (1920) and Volterra's (1926) pioneering work, mathematical models of prey-predator interactions have continually enhanced our understanding of ecological dynamics. Harvesting is widely acknowledged as a critical factor in the study of population dynamics for biological resources, including fisheries, forestry, and wildlife management. In dynamical systems theory, stability analysis, persistence, bifurcation, and related concepts present significant challenges. In this study, we examine the stability analysis of generalist predator-prey dynamics with predator harvesting. Predators can be divided into two categories: generalist predators and specialist predators. The

population dynamics of generalist predators in prey-predator models have been relatively overlooked compared to those of specialist predators. A specialist predator relies exclusively on a specific prey species, while a generalist predator can exploit a range of food sources beyond its primary prey species. The former faces extinction without its primary prey species, whereas the latter can flourish depending on the availability of alternative food sources. This paper seeks to explore the stability assessment of a generalist predator-prey system with predator harvesting. The study presents a model examining the dynamics between primary prey and predator, considering the existence of a generalist predator. Prey growth follows a logistic rate, while predator population adopts a generalist strategy, akin to the Beverton-Holt model. The model also incorporates harvesting of the predator population. We analyze the model's equilibrium, stability, positivity, and boundedness, and use numerical simulations to explore its predictions. This study enhances our understanding of ecological interactions and supports the development of effective conservation and management strategies.

The forthcoming model of ordinary differential equations can illustrate a predator-prey scenario, featuring either a generalist or a specialist predator (Erbach et al., 2013).

$$\frac{dP_1}{dT} = F^*(P_1)P_1 - G^*(P_1,P_2)P_2$$
(1)
$$\frac{dP_2}{dT} = YG^*(P_1,P_2)P_2 + H^*(P_2)P_2$$

Where P_1 and P_2 depict the densities among prey and predator populations respectively. In the lack of the other prey, functions F^{*} and H^{*} depict the dynamics of individual species. For instance, F^{*} may represent logistic growth, and when H^{*} < 0, it indicates a specialist predator. Conversely, for a generalist predator, we find that $G^*(0, P_2)+H^*(P_2) > 0$. The function G^{*}, also known as the functional response, describes predation as the per capita consumption rate of prey by the predator. Additionally, Y represents the efficiency of converting prey biomass into predator biomass. The functional response typically relies solely on the prey species (Holling, 1959); however, it may also rely on both prey and predator populations (Arditi and Ginzburg, 1989; Beddington, 1975; Cantrell and Cosner, 2001; Kuang and Beretta, 1998).

This paper is structured as outline below. Segment 2 outlines the mathematical model of a two-species population consisting of one prey and one predator. Segment 3 analyzes the system's positivity and boundedness. Segment 4 deals with the analysis of local stability in the nonlinear system. Segment 5 assesses global stability at the internal equilibrium point. Segment 6 presents numerical simulations, and the final segment provides the discussion and conclusions.

2 Specific Model

We regard a model featuring prey growth with a logistic rate and predator consumption following a cyrtoid functional response. In the absence of primary prey, we assume the predator to be a generalist with procreation governed by a Beverton-Holt-like function (Erbach et al., 2013), and the predator is the sole harvesting species. Hence, the model system is

$$\frac{\mathrm{dP}_1}{\mathrm{dT}} = \mathrm{rP}_1 \left(1 - \frac{\mathrm{P}_1}{\mathrm{k}} \right) - \frac{\alpha \mathrm{P}_1 \mathrm{P}_2}{1 + \mathrm{hP}_1}$$

(2)

$$\frac{\mathrm{d}P_2}{\mathrm{d}T} = \Upsilon \frac{\alpha P_1 P_2}{1 + \mathrm{h}P_1} + \frac{\delta P_2}{1 + \mathrm{s}P_2} - \eta P_2 - \mu P_2$$

Here,

r -Internal growth rate of prey.

k-The environment's carrying capacity for prey.

 $\frac{\alpha}{h}$ - Maximum hunting rate.

h- Manipulation time.

 Υ -Efficiency of converting prey biomass into predator biomass.

 δ -Procreation rate per capita of the predator.

s -Density dependence strength for predator.

 $\boldsymbol{\eta}$ - The predator's natural death rate.

μ- The predator's harvesting activity.

All parameters are considered to be positive. Note that the term $rP_1\left(1-\frac{P_1}{k}\right)$ represents prey growth with a logistic rate, and the growth rate of the generalist predator, represented by the Bevorton-Holt-like function $\frac{\delta P_2}{1+sP_2}$, depends upon the availability of alternative food resources. Additionally, we assume that $\eta + \mu < \delta$ to ensure that the predator with harvesting can carry on when their primary prey does not exist.

To reduce parameter complexity, we non-dimensionalize system (2) by changing variables such that $m_1 = \frac{P_1}{k}$, $m_2 = \frac{\alpha P_2}{r}$ and t = rT. This provides

$$\frac{dm_1}{dt} = m_1(1 - m_1) - \frac{m_1m_2}{1 + am_1}$$
(3)
$$\frac{dm_2}{dt} = \frac{bm_1m_2}{1 + am_1} + \frac{cm_2}{1 + em_2} - fm_2 - gm_2$$

Given $m_1(0) > 0$ and $m_2(0) > 0$ as initial conditions.

Where a = hk, $b = \frac{\alpha\gamma k}{r}$, $c = \frac{\delta}{r}$, $e = \frac{sr}{\alpha}$, $f = \frac{\eta}{r}$ and $g = \frac{\mu}{r}$. In this scenario, the relationship f + g < c holds true, since $\eta + \mu < \delta$.

3 The System's Positivity and Its State of Being Bounded

In this segment, our aim is to define the criteria for achieving solutions that are both positive and bounded within the system.

Theorem 3.1: If $m_1(0)$ and $m_2(0)$ are consistently positive, then all solutions in the equations (3) remain positive.

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Proof: Take into consideration the initial equation in the model (3)

$$\frac{dm_1}{m_1} = \left[(1 - m_1) - \frac{m_2}{1 + am_1} \right] dt$$

$$\frac{\mathrm{d}\mathbf{m}_1}{\mathbf{m}_1} = \ _1(\mathbf{m}_1,\mathbf{m}_2)\mathrm{d}\mathbf{t}$$

Where $_{1}(m_{1}, m_{2}) = \left[(1 - m_{1}) - \frac{m_{2}}{1 + am_{1}} \right]$

Taking the integral of both sides over the interval [0, t], we get

$$m_1(t) = m_1(0) \exp \int {}_1(m_1, m_2) dt$$

Since $m_1(0) > 0$, $m_1(t) > 0$ for all t. Next, take into consideration the secondary equation in the model (3)

$$\frac{dm_2}{m_2} = \left[\frac{bm_1}{1+am_1} + \frac{c}{1+em_2} - f - g\right] dt$$

$$\frac{dm_2}{m_2} = _2(m_1, m_2)dt$$

Where $_{2}(m_{1}, m_{2}) = \left[\frac{bm_{1}}{1+am_{1}} + \frac{c}{1+em_{2}} - f - g\right]$

Taking the integral of both sides over the interval [0, t], we get

$$m_2(t) = m_2(0) \exp \int_2 (m_1, m_2) dt$$

Since $m_2(0) > 0$, $m_2(t) > 0$ for all t. Therefore, all solutions of the equations (3) remain positive.

Theorem 3.2: The system (3) exhibits bounded trajectories.

Proof: Define the operation $v(t) = m_1 + m_2$ and differentiate v(t) with respect to t, we get

$$\frac{\mathrm{d}\mathbf{v}(t)}{\mathrm{d}t} = \frac{\mathrm{d}\mathbf{m}_1}{\mathrm{d}t} + \frac{\mathrm{d}\mathbf{m}_2}{\mathrm{d}t}$$
$$\frac{\mathrm{d}\mathbf{v}(t)}{\mathrm{d}t} = \mathbf{m}_1(1-\mathbf{m}_1) + \frac{\mathrm{c}\mathbf{m}_2}{1+\mathrm{e}\mathbf{m}_2} - \mathrm{f}\mathbf{m}_2 - \mathrm{g}\mathbf{m}_2$$

Now,

$$\frac{dv(t)}{dt} + \check{v}(t) = m_1(1 - m_1) + \frac{cm_2}{1 + em_2} - fm_2 - gm_2 + \check{m}_1 + \check{m}_2$$

Where is positive constant for +1 > 0, c > f + g > > 0, given > 0 there exists t_0 such that $t \ge t_0$ we get

$$\frac{\mathrm{d}v(t)}{\mathrm{d}t} + \check{v}(t) \leq \acute{\chi} + , \ \acute{\chi} = \min\left\{(\check{} + 1), \left(\frac{\mathrm{c} - (\mathrm{f} + \mathrm{g}) + \check{}}{\mathrm{e}((\mathrm{f} + \mathrm{g}) - \check{})}\right)\right\}$$

Lemma 3.3: Let 🛛 be an absolutely-continuous function satisfying the differential inequality

 $\frac{d(t)}{dt} + \alpha_1(t) \le \alpha_2, t \ge 0, \text{where } (\alpha_1, \alpha_2) \in R_+^2, \ \alpha_1 \neq 0.$ Then,

$$\forall t \geq \widetilde{T} \geq 0, (t) \leq \frac{\alpha_2}{\alpha_1} - \left(\frac{\alpha_2}{\alpha_1} - (\widetilde{T})\right) e^{-\alpha_1(t-\widetilde{T})}.$$

By aforementioned lemma (Aziz-Alaoui and Daher Okiye, 2003), we get

$$\Rightarrow v(t) \le v(t_0) e^{-(t-t_0)} + \left(\frac{\dot{\chi} +}{\tilde{z}}\right) \left(1 - e^{-(t-t_0)}\right)$$

Letting the limit as $t \to \infty$, and letting $\to 0$

$$\lim_{t\to\infty}(\sup v(t))\leq \frac{\dot{\chi}}{z}$$

Hence, the model (3) of equations is bounded under the initial conditions.

4 Analysis of Local Stability In Nonlinear Systems

In this segment, we first verify the presence of equilibrium points in the model (3), then proceed to assess their local stability by calculating the eigenvalues of the Jacobian matrix at each equilibrium point.

4.1 Equilibrium points

The system's equilibrium points are crucial for examining the local stability properties of the predator-prey system. There exist four equilibrium points within the system (Table 1).

Table 1 Explanation of equilibrium points.

Equilibrium points	Explanation
E0 {m ₁ =0, m ₂ = 0}	Trivial
$E1\{m_1=1, m_2=0\}$	Predator washed out
E2 {m ₁ = 0, m ₂ = $\left[\frac{c-(f+g)}{e(f+g)}\right]$ }	Prey washed out
E3 { $m_1 = m_1^*$, $m_2 = m_2^*$ }	Coexistence

4.2 Assessment of the presence and local stability of the equilibrium points

Given model (3) at the equilibrium point $E(m_1, m_2)$, the Jacobian matrix is

$$J = \begin{bmatrix} 1 - 2m_1 - \frac{m_2}{(1 + am_1)^2} & \frac{-m_1}{1 + am_1} \\ \frac{bm_2}{(1 + am_1)^2} & \frac{bm_1}{1 + am_1} + \frac{c}{(1 + em_2)^2} - f - g \end{bmatrix}$$

Theorem 4.2.1: The equilibrium point $E0{m_1=0,m_2=0}$ in the system (3) exhibits instability.

Proof: Given system (3) at E0 $\{m_1=0,m_2=0\}$, the Jacobian matrix is

$$J0 = \begin{bmatrix} 1 & 0 \\ 0 & c - (f+g) \end{bmatrix}$$

The eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = c - (f + g)$.

Since c > f + g, both eigenvalues are positive.

Hence, the equilibrium point E0 { $m_1=0,m_2=0$ } in the system (3) exhibits instability.

Theorem 4.2.2: The equilibrium point $E1\{m_1=1,m_2=0\}$ in the system (3) is a saddle point.

Proof: Given model (3) at $E1\{m_1=1,m_2=0\}$, the Jacobian matrix is

$$J1 = \begin{bmatrix} -1 & \frac{-1}{1+a} \\ 0 & \frac{b}{1+a} + c - (f+g) \end{bmatrix}$$

The eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = \frac{b}{1+a} + c - (f + g)$.

Since c > f + g, both eigenvalues have opposite signs.

Hence, the equilibrium point $E1\{m_1=1,m_2=0\}$ in the system (3) is a saddle point.

Theorem 4.2.3: If $1 < \left[\frac{c-(f+g)}{e(f+g)}\right]$, then the equilibrium point E2 { $m_1 = 0, m_2 = \left[\frac{c-(f+g)}{e(f+g)}\right]$ } in the system (3) exhibits local asymptotic stability.

Proof: Given system (3) at E2 { $m_1 = 0, m_2 = \left[\frac{c-(f+g)}{e(f+g)}\right]$ }, the Jacobian matrix is

$$J2 = \begin{bmatrix} 1 - \left[\frac{c - (f + g)}{e(f + g)}\right] & 0\\ b\left[\frac{c - (f + g)}{e(f + g)}\right] & \frac{-(f + g)}{c}\left[c - (f + g)\right] \end{bmatrix}$$

The eigenvalues are $\lambda_1 = 1 - \left[\frac{c - (f+g)}{e(f+g)}\right]$ and $\lambda_2 = \frac{-(f+g)}{c}[c - (f+g)]$.

Therefore, if $1 < \left[\frac{c-(f+g)}{e(f+g)}\right]$, then the equilibrium point E2 { $m_1 = 0, m_2 = \left[\frac{c-(f+g)}{e(f+g)}\right]$ } in the system (3) exhibits local asymptotic stability.

Theorem 4.2.4: If $tr(J_3) < 0$ and $det(J_3) > 0$, then the internal equilibrium point E3 { $m_1 = m_1^*, m_2 = m_2^*$ } in the system (3) exhibits local asymptotic stability.

Proof: Given system (3) at the internal equilibrium point E3 $\{m_1 = m_1^*, m_2 = m_2^*\}$, the Jacobian matrix is

$$J3 = \begin{bmatrix} 1 - 2m_1^* - \frac{m_2^*}{(1 + am_1^*)^2} & \frac{-m_1^*}{1 + am_1^*} \\ \frac{bm_2^*}{(1 + am_1^*)^2} & \frac{bm_1^*}{1 + am_1^*} + \frac{c}{(1 + em_2^*)^2} - f - g \end{bmatrix}$$

The Jacobian matrix J_3 has a characteristic equation of

$$F(\lambda) = \lambda^2 - tr(J_3)\lambda + det(J_3) = 0.$$

Where trace and determinant of the Jacobian matrix J3 is denoted by $tr(J_3)$ and $det(J_3)$.

Where
$$\operatorname{tr}(J_3) = 1 - 2m_1^* - \frac{m_2^*}{(1 + am_1^*)^2} + \frac{bm_1^*}{1 + am_1^*} + \frac{c}{(1 + em_2^*)^2} - f - g$$

and

$$\det(J_3) = \left[1 - 2m_1^* - \frac{m_2}{(1 + am_1^*)^2}\right] \left[\frac{bm_1}{1 + am_1^*} + \frac{c}{(1 + em_2^*)^2} - f - g\right] + \left[\frac{bm_1}{(1 + am_1^*)^3}\right].$$

When the following Routh-Hurwitz criteria are fulfilled, the internal equilibrium point $E3(m_1 = m_1^*, m_2 = m_2^*)$ in the system (3) exhibits local asymptotic stability (Erbach et al., 2013; Manna and Banerjee, 2022;

Manna and Banerjee, 2024):

$$tr(J_3) = 1 - 2m_1^* - \frac{m_2^*}{(1 + am_1^*)^2} + \frac{bm_1^*}{1 + am_1^*} + \frac{c}{(1 + em_2^*)^2} - f - g < 0$$

and

$$det(J_3) = \left[1 - 2m_1^* - \frac{m_2^*}{(1 + am_1^*)^2}\right] \left[\frac{bm_1^*}{1 + am_1^*} + \frac{c}{(1 + em_2^*)^2} - f - g\right] + \left[\frac{bm_1^*m_2^*}{(1 + am_1^*)^3}\right] > 0.$$

5 Global Stability Analysis

In this segment, we explore the global asymptotic stability of the internal equilibrium point E3 of the system (3) through the Bendixson-Dulac criteria.

Theorem 5.1: The internal equilibrium point E3 in the system (3) exhibits global asymptotic stability

Proof: Let $H(m_1, m_2) = \frac{1+am_1}{m_1m_2}$

Clearly $H(m_1, m_2)$ is positive for both $m_1 > 0$, $m_2 > 0$

Let
$$\beta_1(m_1, m_2) = m_1(1 - m_1) - \frac{m_1m_2}{1 + am_1}$$

 $\beta_2(m_1, m_2) = \frac{bm_1m_2}{1+am_1} + \frac{cm_2}{1+em_2} - fm_2 - gm_2$

Then $\Delta(m_1, m_2) = \frac{\partial}{\partial m_1} (\beta_1 H) + \frac{\partial}{\partial m_2} (\beta_2 H)$

$$= -\left[\frac{1+2am_1-a}{m_2}\right] - \left[\frac{ce(1+am_1)}{m_1(1+em_2)^2}\right]$$

Using the Bendixson-Dulac criterion (Hsu and Huang, 1995), we observe that $\Delta(m_1, m_2)$ maintains a consistent sign and is not identically zero within the inside of the positive quadrant of the $m_1 - m_2$ plane. Hence, the internal equilibrium point E3 in the system (3) exhibits global asymptotic stability

6 Numerical Simulation

For the numerical simulation, consider the model (3).

By defining the system's parameters as $\tau = (a = 10, b = 3.35, c = 1.4, e = 0.9, f = 0.2, g = 0.3)$, the initial conditions (m₁ = 0, m₂ = 15) are satisfied, representing an assessment of the predator's stability incorporating predator harvesting (see Fig. 1), and allowing for the creation of a phase diagram depicting prey and predator populations based on the same initial conditions (see Fig. 2).

Defining the system's parameters, as mentioned above, allows for an assessment of prey and predator

interaction stability incorporating predator harvesting based on satisfying the initial conditions ($m_1 = 10, m_2 = 5$) (see Fig. 3). It also enables the creation of a phase diagram depicting prey and predator populations (see Fig. 4).



Fig. 1 Assessment of the predator's stability incorporating predator harvesting.



Fig. 2 The phase diagram depicting prey and predator populations at $(m_1 = 0, m_2 = 15)$.



Fig. 3 Assessment of prey and predator interaction stability incorporating predator harvesting.



Fig. 4 The phase diagram depicting prey and predator populations at $(m_1 = 10, m_2 = 5)$.

7 Discussions and Conclusions

For ecological communities, understanding prey-predator interactions is crucial. Researchers have examined various mathematical models and analyzed their dynamics for this purpose. Our contribution involved proposing a predator-prey model that included a generalist predator. In this model, we used a mathematical expression to describe the dynamics of several prey populations, specifically incorporating the cyrtoid functional response for the generalist predator. Our paper presented a model that investigated the stability

assessment of generalist predator-prey dynamics with predator harvesting, utilizing the cyrtoid functional response. Unlike many studies that focused on generalist predators and opted for a sigmoidal functional response, often for predators switching prey or targeting refuge-seeking prey, we chose the cyrtoid response. This choice was motivated by our long-term aspiration to explicitly include multiple prey populations as sources for the generalist predator.

This model analyzed the stability and following conclusions were drawn.

The dynamics of the stability assessment for a prey-predator model that includes predator harvesting are shown in Figures 1 and 3.

Thus, it is observed that our assumption that c > f + g is important for the persistence of generalist predators with harvesting, even in the absence of primary prey.

References

- Arditi R, Ginzburg LR. 1989. Coupling in predator-prey dynamics: ratio-dependence. Journal of Theoretical Biology, 139(3): 311-326
- Aziz-Alaoui MA, Daher Okiye M. 2003. Boundedness and global stability for a predator-prey model with modified leslie-gower and holling-type II schemes. Applied mathematics letters, 16(7): 1069-1075
- Beddington JR. 1975. Mutual interference between parasites or predators and its effect on searching efficiency. The Journal of Animal Ecology, 44: 331-340
- Cantrell RS, Cosner C. 2001. On the dynamics of predator-prey models with the Beddington-deangelis functional response. Journal of Mathematical Analysis and Applications, 257(1): 206-222
- Das E, Paul P, Kar TK. 2022. Transient indicator of exploited communities at equilibrium in generalist predator-prey models. The European Physical Journal Plus, 137:1221
- Erbach A, Lutscher F, Seo G. 2013. Bistability and limit cycles in generalist predator-prey dynamics. Ecological Complexity, 14: 48-55
- Holling CS. 1959. The components of predation as revealed by a study of small-mammal predation of the european pine sawfly. The Canadian Entomologist, 91(5): 293-320
- Hsu SB, Huang TW. 1995. Global stability for a class of predator-prey system. Society for Industrial and Applied Mathematics Journal on Applied Mathematics, 55(3): 763-783
- Kuang Y, Beretta E. 1998. Global qualitative analysis of a ratio-dependent predator-prey system. Journal of Mathematical Biology, 36(14): 389-406
- Lotka AJ. 1920. Undamped oscillations derived from the law of mass action. Journal of the American Chemical Society, 42: 1595-1599
- Madhusudanan V, Vijaya S. 2015. Impact of quadratic harvesting on prey-predator model with holling type II functional response. International Journal of Applied Engineering Research, 10 (24): 43953-43959
- Manna K, Banerjee M. 2022. Spatiotemporal pattern formation in a prey-predator model with generalist predator, Mathematical Modelling of Natural Phenomena, 17: 6
- Manna K, Banerjee M. 2024. Dynamics of a prey-predator model with reproductive allee-effect for prey and generalist predator. Nonlinear Dynamics, 112(3)
- Roy J, Banerjee M. 2023. Global stability of a predator-prey model with generalist predator. Applied Mathematics Letters, 142 (1-2)
- Saranya S, Vijaya V, Rekha E. 2022. Susceptible-infected-Treatment model with disease infection in prey population recovered by treatment. Computational Ecology and Software, 12(1): 12-22
- Sen D, Ghorai S, Sharma S, et al. 2021. Allee effect in prey's growth reduces the dynamical complexity in

prey-predator model with generalist predator. Applied Mathematical Modelling, 91: 768-790

- Vijaya S, Ganga S. 2024. Stability analysis in generalist predator-prey dynamics. International Journal of All Research Education and Scientific Methods, 2687-2696
- Vijayalakshmi GM, Gunasekaran M, Vijaya S. 2014. Bifurcation Analysis of Prey-Predator Model with Harvested Predator. International Journal of Engineering Research and Development, 10: 42-51
- Vijaya S, Jayamalsingh J, Rekha E. 2017. Eco-epidemiological prey-predator model for susceptible-infected species. Bangmod International Journal of Mathematics Computational science, 3: 72-95
- Volterra V. 1926. Variazioni e fluttuazioni del numero d' individui in specie animali conviventi. Memorie dell'Accademia Nazionale dei Lincei (Rema), 2(31-113)