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# A dynamical model describing how acid rain affects the growth of plants in a habitat

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**Abstract**
Acid rain formation in the atmosphere is a serious environmental hazard that causes enormous ecological harm. It is formed by the interaction of air pollutants with vapour clouds. It threatens biodiversity, damages ecosystems, and has a negative effect on plant growth. In this paper, we propose a nonlinear mathematical model to examine how acid rain affects the growth of plant biomass in a habitat. The model takes into account five important factors: the density of plant biomass, the density of human population, the cumulative density of sources that release pollutants, the cumulative concentration of pollutants, and the cumulative concentration of acid rain. The model makes the assumptions of logistic growth for both human population density and plant biomass, which is adversely affected by acid rain. The model analysis shows that increasing the concentration of acid rain leads in a considerable drop in plant biomass density. The model analysis, which employs the stability theory of differential equations, shows that a substantial decrease in plant biomass density at equilibrium results from rising acid rain concentrations or human population density. These findings are supported by numerical simulations, which show that acid rain has a considerable effect on plant biomass growth. This study provides a framework for understanding the ecological consequences of acid rain.

**Keywords** mathematical model; stability; plant biomass; population density; acid rain.

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## 1 Introduction

Acid rain is a major environmental problem that arises when air pollutants mix with water vapour or airborne droplets to produce acidic chemicals. Carbon dioxide ($CO_2$), sulphur dioxide ($SO_2$), and nitrogen oxides ($NO_x$) are significant contributors from various activities, including automobile traffic, industrial processes, domestic fuel use, and building. These pollutants undergo chemical changes in the atmosphere, producing acids such as carbonic acid, sulphuric acid, and nitric acid. Following that, these acids are deposited on the surface of the Earth by dry deposition or precipitation. The primary sources of these emissions are fossil fuel combustion in power plants and vehicles, industrial operations, and localized activities like burning fuels in households.

For example, carbon dioxide produces carbonic acid which is very weak compound, sulphur dioxide produces sulphuric acid, and nitrogen oxide forms nitric acid, with their interaction with water vapour cloud (Seinfeld, 1986)

$$CO_2 + H_2O = H_2CO_3 \text{ (Carbonic acid)}$$

$$3NO_2 + H_2O = 2HNO_3 \text{ (Nitric acid) } + NO$$

$$2SO_2 + 2H_2O + O_2 = 2H_2SO_4 \text{ (Sulphuric acid)}$$

The impacts of acid rain are widespread and severe. It lowers the pH of soils and water bodies, disrupting ecosystems, harming aquatic and terrestrial organisms, and damaging vegetation by leaching essential nutrients and eroding plant surfaces. In urban areas, acid rain accelerates the deterioration of infrastructure, corroding buildings, bridges, and monuments. Its effects are not confined to the source region, as atmospheric transport can spread pollutants over vast distances. Addressing acid rain requires coordinated efforts, including reducing emissions, adopting cleaner technologies, and implementing stricter environmental regulations to mitigate its ecological, infrastructural, and public health impacts.

Numerous studies have explored the effects of acid rain on plant growth and vegetation. Pioneering works by (Lal, 2016; Du et al., 2017; Huang et al., 2019; Grennfeltet al., 2020; Pham et al., 2021; Liu et al., 2022; Zhang et al., 2023; Zhong et al., 2024). In this regard, Zhu et al. (2025) have shed light on the correlation between acid rain and its detrimental effects on vegetation. The concentration of traffic emissions on vegetables and plants was found to have a strong association (Amato-Lourenco et al., 2016). Experimental investigations have demonstrated the harmful impact of air pollution on plant structures, including cuticles, stomata, and epidermal cells (Winner and Atkinson, 1986; Gupta, 2016).

During past several years the impact of pollutants has been studied and analyzed using mathematical modelling on crops, plants biomass, forestry resources and also on life of several biological species (Shukla et al., 1996; Dubey et al., 2003; Naresh et al., 2006; Dubey and Naraynan, 2010; Shukla et al., 2013; Naresh et al., 2014; Sundar and Naresh, 2015; Trivedi et al., 2025). The effects of industrialization and pollution on resource biomass have been modelled and analysed, revealing that industrialization and rising pollution levels have a negative impact on resource biomass (Dubey et al., 2003). Naresh et al. (2006) have shown that the main reason for the decrease in growth rate of plant biomass and equilibrium level is the formation of intermediate toxic products inside the plant as a result of toxicant uptake and sap (liquid present in plant body) contact. As the equilibrium density of population and pollution emitting sources rises, the equilibrium density of resource biomass drops (Dubey et al., 2009). Impact of both wood-based and non-wood-based industries on forest resources was mathematically modeled and studied highlighting the direct and indirect harm caused by wood-based industries on forest resources but non-wood-based sectors only drain forest resources indirectly, it was concluded that forest resources are adversely impacted and may go extinct as a result of increased industrialization (Tandon and Jyotsna, 2016).

Despite significant advancements and studies, the nonlinear dynamics of acid rain formation and the effects of cumulative concentration of pollutants, emitted through various pollutant emitting sources, on plant biomass are still less understood. Experimental investigations frequently concentrate on certain contaminants or localized effects while comprehensive modeling approaches to assess the interplay of multiple components are still under explored. This emphasizes the motivation for the present work.

The main aim of this research is to develop and analyze a nonlinear mathematical model to study the formation of acid rain due to multiple pollution sources and its impact on plant biomass in human habitats. The model considers the interplay between plant biomass density, human population density, pollutant-emitting sources, and the cumulative concentration of pollutants and acid rain. The significance of this study lies in its ability to provide a deeper understanding of the complex dynamics of acid rain formation and its ecological consequences, aiding policymakers in designing effective mitigation strategies. The importance of this study stems from its ability to provide a better knowledge of the complicated processes of acid rain generation and its ecological implications, which will help policymakers in developing effective mitigation solutions.

In order to investigate the dynamics of acid rain formation and its effects on plant biomass in human settings, this work introduces a novel nonlinear mathematical model using ordinary differential equations. This study incorporates the intricate relationships between plant biomass density, human population density, pollution-emitting sources, cumulative pollutant concentrations, and the cumulative concentration of acid rain. The model investigates the cascading impacts of pollutants on acid rain and vegetation growth by taking into account human activities' dual roles as contributors to and dependents on ecological balance. The approach incorporates stability criteria, providing new insights into threshold effects and ecological sustainability. This method provides a predictive framework to assist mitigation techniques and policy-making by bridging the gap between theoretical modeling and environmental impacts.

In the modeling process, the following assumptions are made,

(i) The density of plant biomass is assumed to be governed by logistic equation, adversely affected by acid rain and human population.

(ii) The density of human population also follows a logistic growth model, with the rate of growth increasing as plant biomass increases.

(iii) The cumulative density of polluting sources is proportional to the density of the human population.

(iv) The cumulative concentration of pollutants increases from both natural and human population density dependent sources.

(v) Acid rain occurs when contaminants mix with atmospheric water vapour clouds, whose density is supposed to remain constant.

## 2 Mathematical Model

Acid rain, which is caused by pollutants released from a variety of sources like automobile traffic, industrial operations, and other man-made projects, poses a serious threat to the health of plants and vegetables in a human habitat where they are essential for sustaining life and ecological balance. Sulfur dioxide ($SO_2$) and nitrogen oxides ($NO_x$) are released into the atmosphere by these pollutant emitting sources, and when they combine with moisture, they form acid rain. Acid rain impairs growth, destroys plant tissues, and alters nutrient availability in soil and vegetation, all of which have an impact on the environment and food supply. These emissions release sulfur dioxide ($SO_2$) and nitrogen oxides ($NO_x$) into the atmosphere, which react with moisture to form acidic compounds. When acid rain falls on soil and vegetation, it disrupts nutrient availability, damages plant tissues, and hampers growth, ultimately affecting food production and the environment. This demonstrates the interdependence of human activity, environmental pollution, and ecological health.

To describe the dynamics of acid rain formed by emissions from polluting sources, its impacts on plants, and the human population, the following dependent variables have been considered:

(i) The plant biomass density $B(t)$

(ii) The human population density $N(t)$

(iii) The cumulative density of pollutants emitting sources $S(t)$.

(iv) The cumulative concentrations $C_a(t)$ of pollutants forming acid rain

(v) The cumulative concentrations of acid rain in the atmosphere $A_r(t)$

The density of plant biomass ($B$) is governed by a logistic model in equation (1), with $s$ and $L$ representing its intrinsic growth rate and carrying capacity respectively, $s_1$ is the plant biomass depletion rate coefficient due to its use by human population. Since the plant biomass is adversely affected by acid rain, it is anticipated that acid rain causes a drop in density of plant biomass with a rate coefficient $s_2$.

In equation (2), $r$ is the intrinsic growth rate of human population density $N$ with carrying capacity $K$. Due to natural dependence of human population on plant biomass, it is believed that, the growth rate of human population would increase at a rate of $r_1$ with increase in density of plant biomass. The cumulative density $S$ of pollution emitting sources is assumed to grow at a rate proportional to the density of the human population, with $\lambda > 0$ being the growth rate coefficient with depletion rate coefficient $\lambda_0 > 0$, due to inefficient source operation.

The cumulative concentration of pollutants forming acid rain $C_a$ is modeled in equation (4), where $Q_a$ represents the constant emission rate of pollutants forming acid rain in the atmosphere by natural sources and $\delta_{a0} > 0$ represents depletion rate coefficient of pollutants forming acid rain due to some natural factors. Since the pollutants are formed into acid rain by their reaction with water vapour cloud hence its density $w_0$ is considered to be fixed, with rate $\delta_{a1} w_0 > 0$, consequently the cumulative concentration $C_a$ falls, this is considered to be the increase rate of acid rain's cumulative concentration $A_r$ in the atmosphere. Cumulative concentration $A_r$ of acid rain is modeled in equation (5), where $\delta_{r0} > 0$ is the acid rain depletion rate coefficient owing to natural causes, and $\delta_{r1} > 0$ is the acid rain depletion rate coefficient due to plant biomass.

Given the foregoing, the proposed model, in terms of nonlinear differential equations, is proposed as follows;

$$\frac{dB}{dt} = s\left(B - \frac{B^2}{L}\right) - s_1 BN - s_2 BA_r \quad (1)$$

$$\frac{dN}{dt} = r\left(N - \frac{N^2}{K}\right) + r_1 BN \quad (2)$$

$$\frac{dS}{dt} = \lambda N - \lambda_0 S \quad (3)$$

$$\frac{dC_a}{dt} = Q_a + \delta_a S - \delta_{a0} C_a - \delta_{a1} w_0 C_a \quad (4)$$

$$\frac{dA_r}{dt} = \delta_{a1} w_0 C_a - \delta_{r0} A_r - \delta_{r1} A_r B \quad (5)$$

Where, $B(0) \geq 0$ $N(0) \geq 0$, $S(0) \geq 0$, $C_a(0) \geq 0$, $A_r(0) \geq 0$

**Remark** From equation (1) of the model system (1) – (5), we note that $s - s_1 N - s_2 A_r > 0$ for all time $t > 0$. This condition ensures the feasibility and sustainability of biomass density growth within the model system. Here, if the combined negative impacts of human population density and acid rain concentration outweigh the intrinsic growth rate $(s)$, the net growth rate of plant biomass becomes negative, leading to a continuous decline in forest or vegetation cover, which is ecologically unsustainable. Therefore, maintaining $s - s_1 N - s_2 A_r > 0$ ensures that the plant biomass density remains positive and capable of regenerating, thereby supporting ecosystem stability and the long-term validity of the model.

**3 Equilibrium Analysis**

There are four non-zero of equilibria of the model (1) – (5) as given below,

(i) $E_0(0,0,0,C_{a0}, A_{r0})$, where $C_{a0} = \dfrac{Q_a}{\delta_{a0} + \delta_{a1} w_0}$, $A_{r0} = \dfrac{\delta_{a1} w_0 Q_a}{\delta_{r0}\left(\delta_{a0} + \delta_{a1} w_0\right)}$

The existence of above equilibrium obvious because in the absence of; human population density, plant biomass resources density, and various pollution emitting sources density, both pollutant and acid rain cumulative concentrations will remain at their natural equilibrium levels $C_{a0}$ and $A_{r0}$ respectively.

(ii) $\tilde{E}(0, K, \dfrac{\lambda}{\lambda_0} K, \tilde{C}_a, \tilde{A}_r)$, where $\tilde{C}_a = \dfrac{Q_a + \dfrac{\delta_a \lambda}{\lambda_0} K}{\delta_{a0} + \delta_{a1} w_0}$ and $\tilde{A}_r = \dfrac{\delta_{a1} w_0 \tilde{C}_a}{\delta_{r0}}$

This equilibrium point represents a degraded environmental state where biomass density is completely depleted (zero), while the human population density reaches its maximum carrying capacity $K$, and pollutant-emitting sources are active at a steady level proportional to the population. The cumulative concentration of pollutants that cause acid rain and the cumulative concentration of acid rain remain at their respective equilibrium values $\tilde{C}_a$ and $\tilde{A}_r$. This scenario illustrates an unsustainable ecosystem dominated by human activity and pollution, where environmental degradation has led to the total loss of biomass, reflecting ecological collapse driven by excessive anthropogenic pressure.

(iii) $\hat{E}(\hat{B},0,0,\hat{C}_a,\hat{A}_r)$ where $\hat{B}$ is the quadratic equation's only positive root

$\frac{s}{L}\delta_{r1}B^2+s\left(\frac{\delta_{r0}}{L}-\delta_{r1}\right)B-\delta_{r0}\left(s-s_2\frac{\delta_{a1}w_0\hat{C}_a}{\delta_{r0}}\right)=0$, where $\hat{C}_a=\frac{Q_a}{\delta_{a0}+\delta_{a1}w_0}$ and $\hat{A}_r=\frac{\delta_{a1}w_0\hat{C}_a}{\delta_{r0}+\delta_{r1}\hat{B}}$

The equilibrium point $\hat{E}(\hat{B},0,0,\hat{C}_a,\hat{A}_r)$ represents a steady-state condition of the system in which the biomass density stabilizes in the absence of both the human population density and pollutant-emitting sources, indicating a scenario free from anthropogenic pressures. In this state, cumulative concentration of pollutant stabilize at its equilibrium level, which is due to natural factors. Similarly, the acid rain concentration stabilizes at its equilibrium level. This equilibrium reflects a largely natural or undisturbed ecosystem where biomass can sustain itself without the negative impacts of human activity or pollution sources. This equilibrium reflects a stable environment with no ongoing human or pollution pressure, where biomass thrives despite the presence of natural emissions.

(iv) The interior equilibrium $E^*(B^*,N^*,S^*,C_a^*,A_r^*)$

The equilibrium point $E^*(B^*,N^*,S^*,C_a^*,A_r^*)$ represents a balanced and coexisting state of the system where biomass density, human population density, and pollutant-emitting sources all stabilize at positive equilibrium levels. This indicates that while human activity and pollution are present, they are maintained at levels that allow the biomass to survive and regenerate. The pollutant concentration and acid rain concentration also stabilize, suggesting that the environmental system has reached a steady state where ecological degradation is controlled or mitigated to a degree that supports the continued existence of natural vegetation. This equilibrium reflects a sustainable coexistence between human development and environmental preservation.

### 3.1 Existence of the equilibrium $E^*(B^*,N^*,S^*,C_a^*,A_r^*)$

The algebraic equations that determine the equilibrium values of various variables in model system (1) - (5) are as follows:

$$s\left(1-\frac{B}{L}\right)-s_1N-s_2A_r=0 \quad (6)$$

$$r\left(1-\frac{N}{K}\right)+r_1B=0 \quad (7)$$

$$\lambda N-\lambda_0 S=0 \quad (8)$$

$$Q_a+\delta_a S-\delta_{a0}C_a-\delta_{a1}w_0C_a=0 \quad (9)$$

$$\delta_{a1}w_0C_a-\delta_{r0}A_r-\delta_{r1}A_rB=0 \quad (10)$$

From the equations (7) – (10) we get,

$$N = \frac{K}{r}(r + r_1 B) \quad (11)$$

$$S = \frac{\lambda K}{\lambda_0 r}(r + r_1 B) \quad (12)$$

$$C_a = \frac{Q_a + \frac{\delta_a \lambda K}{\lambda_0 r}(r + r_1 B)}{\delta_{a0} + \delta_{a1} w_0} \quad (13)$$

$$A_r = \frac{\delta_{a1} w_0 \left( Q_a + \frac{\delta_a \lambda K}{\lambda_0 r}(r + r_1 B) \right)}{(\delta_{a0} + \delta_{a1} w_0)(\delta_{r0} + \delta_{r1} B)} \quad (14)$$

Using all the values from equations (11) and (14) in equation (6) we get

$$F(B) = s\left(1 - \frac{B}{L}\right) - \frac{s_1 K}{r}(r + r_1 B) - \frac{s_2 \delta_{a1} w_0 \left( Q_a + \frac{\delta_a \lambda K}{\lambda_0 r}(r + r_1 B) \right)}{(\delta_{a0} + \delta_{a1} w_0)(\delta_{r0} + \delta_{r1} B)} = 0 \quad (15)$$

From above equation (15) we get

(i) $$F(0) = s - s_1 K - \frac{s_2 \delta_{a1} w_0 \left( Q_a + \frac{\delta_a \lambda K}{\lambda_0} \right)}{\delta_{r0}(\delta_{a0} + \delta_{a1} w_0)} > 0$$

(ii) $$F(L) = -s_1 \frac{K}{r}(r + r_1 L) - \frac{s_2 \delta_{a1} w_0 \left( Q_a + \frac{\delta_a \lambda K}{\lambda_0 r}(r + r_1 L) \right)}{(\delta_{a0} + \delta_{a1} w_0)(\delta_{r0} + \delta_{r1} L)}$$

(iii) $$F'(B) = -\frac{s}{L} - s_1 \frac{K}{r} r_1 - \frac{s_2 \delta_{a1} w_0 \left( \frac{\delta_{r0} \delta_a \lambda K r_1}{\lambda_0 r} - \left( Q_a + \frac{\delta_a \lambda K}{\lambda_0} \right) \delta_{r1} \right)}{(\delta_{ro} + \delta_{r1} B)^2 (\delta_{a0} + \delta_{a1} w_0)} < 0$$

provided

$$\frac{\delta_{ro} \delta_a \lambda K r_1}{\lambda_0 r} \geq \left( Q_a + \frac{\delta_a \lambda K}{\lambda_0} \right) \delta_{r1} \quad (16)$$

This means that $F(B) = 0$ has a single positive root (say $B = B^*$) in ($0 \leq B \leq L$), provided the condition (16) is satisfied. Using $B = B^*$, the values of $N^*, S^*, C_a^*$ and $A_r^*$ can be obtained from equations (11) – (14) respectively.

**3.2 Variation of different variables with relevant parameters**

3.2.1 Variation of $B$ with $r_1$

In view of equation (16), it is noted from equation (15) that

$$\frac{dB}{dr_1}=-\frac{B(\delta_{ro}+\delta_{r1}B)\left(\frac{s_1K}{r}(\delta_{a0}+\delta_{a1}w_0)(\delta_{r0}+\delta_{r1}B)+\frac{s_2\delta_{a1}w_0\delta_a\lambda K}{\lambda_0 r}\right)}{\left(\frac{s}{L}+\frac{s_1Kr_1}{r}\right)(\delta_{a0}+\delta_{a1}w_0)(\delta_{r0}+\delta_{r1}B)^2+s_2\delta_{a1}w_0\left(\frac{\delta_{r0}\delta_a\lambda Kr_1}{\lambda_0 r}-\left(Q_a+\frac{\delta_a\lambda K}{\lambda_0}\right)\delta_{r1}\right)}<0 \quad (17)$$

This implies that an increase in human population growth driven by the availability of biomass leads to a decline in plant biomass density. This relationship reflects the pressure exerted by a growing population that depends on biomass for resources such as food, fuel, or land. As biomass availability boosts population growth, the increased consumption, exploitation, and land-use change associated with a larger population result in the depletion of biomass over time.

3.2.2 Variation of $B$ with $s_2$

Similarly, it can also be shown that $\frac{dB}{ds_2}<0$ which signifies that as the harmful effect of acid rain on biomass becomes more severe (i.e. $s_2$ increases), the overall biomass density declines. This reflects the damaging impact of acid rain on vegetation, such as leaf injury, soil nutrient imbalance, and root damage, which collectively reduce the health, growth, and survival of plant species.

## 4 Stability Analysis

To study the local stability behaviour of equilibriums $E_0$, $\tilde{E}$ and $\hat{E}$ ,the Jacobian matrix $M(E)$ is computed for the model system (1) – (5) as shown below,

$$M=\begin{bmatrix} s-2\frac{sB}{L}-s_1N-s_2A_r & -s_1B & 0 & 0 & -s_2B \\ r_1N & r-2\frac{rN}{K}+r_1B & 0 & 0 & 0 \\ 0 & \lambda & -\lambda_0 & 0 & 0 \\ 0 & 0 & \delta_a & -(\delta_{a0}+\delta_{a1}w_0) & 0 \\ -\delta_{r1}A_r & 0 & 0 & \delta_a w_0 & -(\delta_{r0}+\delta_{r1}B) \end{bmatrix}$$

We may see from the preceding matrix that the equilibrium,

(i) The equilibrium point $E_0$ is unstable as two eigenvalues $s-s_2A_{ro}>0$ and $r>0$ of the Jacobian matrix $M(E_0)$ evaluated at this point are positive, indicating that small disturbances or deviations from this state will grow over time rather than return to equilibrium.

(ii) The equilibrium point $\tilde{E}$ is unstable because one of the eigenvalues $s-s_1K-s_2\tilde{A}_r$ of the Jacobian matrix $M(\tilde{E})$ at this point is positive, indicating that any small disturbance in the corresponding direction will grow over time. This implies that the system cannot return to or remain at $\tilde{E}$ when slightly perturbed.

(iii) The equilibrium point $\hat{E}$ is unstable because one of the eigenvalues $r + r_1\hat{B}$ of the Jacobian matrix $M(\hat{E})$ evaluated at this point is positive, which implies that small disturbances in at least one direction will lead the system away from this state over time. In this equilibrium, biomass exists in the absence of human population and pollutant-emitting sources, with pollutant and acid rain concentrations at certain levels.

**Theorem 4.1** The equilibrium $E^*$ is locally asymptotically stable provided the following conditions are satisfied,

$$\frac{2sB^*}{L}(\delta_{r0} + \delta_{r1}B^*) - (s_2B^* + \delta_{r1}A_r^*)^2 > 0 \quad (18)$$

$$2s_1 r\lambda_0^2 B^*(\delta_{r0} + \delta_{r1}B^*)(\delta_{a0} + \delta_a w_0)^2 - \delta_{a1}^2 w_0^2 r_1 \delta_a^2 \lambda^2 K > 0 \quad (19)$$

**Proof** Let we consider positive definite function $V$ as defined below,

$$V = \frac{1}{2}l_1B_1^2 + \frac{1}{2}l_2N_1^2 + \frac{1}{2}l_3S_1^2 + \frac{1}{2}l_4C_{a1}^2 + \frac{1}{2}l_5A_{r1}^2 \quad (20)$$

where $B_1$ $N_1, S_1, C_{a1}, A_{r1}$ are the small perturbations about $E^*(B^*N^*, S^*, C_a^*, A_r^*)$ as described below,

$$B = B^* + B_1,\ \ N = N^* + N_1, S = S^* + S_1, C_a = C_a^* + C_{a1},\ \ A_r = A_r^* + A_{r1}$$

and $l_i\ (i = 1,2,3,4,5)$ are positive constants to be chosen appropriately.

By equation (20) we get,

$$\frac{dV}{dt} = l_1B_1\frac{dB_1}{dt} + l_2N_1\frac{dN_1}{dt} + l_3S_1\frac{dS}{dt} + l_4C_{a1}\frac{dC_{a1}}{dt} + l_5A_{r1}\frac{dA_{r1}}{dt} \quad (21)$$

The linearized system corresponding to $E^*(B^*, N^*, S^*, C_a^*, A_r^*)$ is given as,

$$\begin{bmatrix} \dot{B}_1 \\ \dot{N}_1 \\ \dot{S}_1 \\ \dot{C}_{a1} \\ \dot{A}_{r1} \end{bmatrix} = \begin{bmatrix} -\frac{sB^*}{L} & -s_1B^* & 0 & 0 & -s_2B^* \\ r_1N^* & -\frac{rN^*}{K} & 0 & 0 & 0 \\ 0 & \lambda & -\lambda_0 & 0 & 0 \\ 0 & 0 & \delta_a & -\left(\delta_{a0} + \delta_a w_0\right) & 0 \\ -\delta_{r1}A_r^* & 0 & 0 & \delta_{a1} & -\left(\delta_{r0} + \delta_{r1}B^*\right) \end{bmatrix} \begin{bmatrix} B_1 \\ N_1 \\ S_1 \\ C_{a1} \\ A_{r1} \end{bmatrix}$$

Using above linearized system in equation (21) and on simplification we have,

$$\frac{dV}{dt} = -l_1\frac{sB^*}{L}B_1^2 - l_2\frac{rN^*}{K}N_1^2 - l_3\lambda_0S_1^2 - l_4(\delta_{a0} + \delta_{a1}w_0)C_{a1}^2 - l_5(\delta_{r0} + \delta_{r1}B^*)A_{r1}^2$$

$$+ (-l_1s_1B^* + l_2r_1N^*)B_1N_1 - (l_1s_2B^* + l_5\delta_{r1}A_{r1}^*)B_1A_{r1}$$

$$+ l_3\lambda N_1S_1 + l_4\delta_aS_1C_{a1} + +l_5\delta_{r1}w_0C_{a1}A_{r1} \quad (22)$$

Choosing $l_1 = 1$ and $l_2 = \frac{s_1 B^*}{r_1 N^*}$, equation (22) reduces to

$$\frac{dV}{dt} = -\frac{sB^*}{L}B_1^2 - \frac{rs_1 B^*}{Kr_1}N_1^2 - l_3\lambda_0 S_1^2 - l_4(\delta_{a0}+\delta_{a1}w_0)C_{a1}^2 - l_5(\delta_{r0}+\delta_{r1}B^*)A_{r1}^2$$

$$-(s_2 B^* + l_5\delta_{r1}A_r^*)B_1 A_{r1} + l_3\lambda N_1 S_1 + l_4\delta_a S_1 C_{a1} + l_5\delta_{a1}w_0 C_{a1}A_{r1} \quad (23)$$

If the following conditions are met, $\frac{dV}{dt}$ will be negative definite,

$$(s_2 B^* + l_5\delta_{r1}A_r^*)^2 < 2l_5\frac{sB^*}{L}(\delta_{r0}+\delta_{r1}B^*) \quad (24)$$

$$l_3\lambda^2 < \frac{2rs_1 B^*}{Kr_1}\lambda_0 \quad (25)$$

$$l_4\delta_a^2 < l_3\lambda_0(\delta_{a0}+\delta_{a1}w_0) \quad (26)$$

$$l_5\delta_{a1}^2 w_0^2 < l_4(\delta_{a0}+\delta_{a1}w_0)(\delta_{r0}+\delta_{r1}B^*) \quad (27)$$

After performing some algebraic calculations and choosing,

$$l_5 = 1,\ l_3 < \frac{2rs_1\lambda_0 B^*}{K\lambda^2 r_1},\ l_4 < \frac{2rs_1\lambda_0^2 B^*(\delta_{a0}+\delta_{a1}w_0)}{K\lambda^2\delta_a^2 r_1}$$

$\frac{dV}{dt}$ is a negative definite provided the conditions (18) and (19) are satisfied, hence $V$ is Lyapunov function. This proves the theorem.

**Lemma 4.1** The model system (1) – (5), which attracts all solutions beginning in the interior of the positive octant, has the following zone of attraction,

$$\Omega = \{(B,N,S,C_a,A_r)\in R^5 : 0\le B\le L, 0\le N\le N_m, 0\le S\le S_m, \frac{Q_a}{\delta_{a0}+\delta_{a1}w_0}\le C_a\le C_{am}, 0\le A_r\le A_{rm})\}$$

where, $N_m = K\left(1+\frac{r_1 L}{r}\right)$, $S_m = \frac{\lambda K}{\lambda_0}\left(1+\frac{r_1 L}{r}\right)$, $C_{am} = \frac{Q_a+\delta_a S_m}{\delta_{a0}+\delta_{a1}w_0}$, $A_{rm} = \frac{\delta_{a1}w_0}{\delta_{r0}}C_{am}$

**Proof** The equation (1) gives

$$\frac{dB}{dt} \le s\left(1-\frac{B}{L}\right)B$$

implying that $\limsup_{t\to\infty} B(t) = L$.

Equation (2) provides,

$$\frac{dN}{dt} \le r\left(N - \frac{N^2}{K}\right) + r_1 BN$$

implying that $\underset{t\to\infty}{Lim}\sup N(t) = K\left(1+\frac{r_1 L}{r}\right) = N_m$ (say)

From equation (3) we have,

$$\frac{dS}{dt} = \lambda N - \lambda_0 S$$

$$\leq \lambda K\left(1+\frac{r_1 L}{r}\right) - \lambda_0 S$$

again implying that, $\underset{t\to\infty}{Lim}\sup S(t) = \frac{\lambda K}{\lambda_0}\left(1+\frac{r_1 L}{r}\right) = S_m$ (say)

Similarly from equations (4) and (5) we can show that,

$$\underset{t\to\infty}{Lim}\sup C_a(t) = C_{am}, \quad \underset{t\to\infty}{Lim}\sup A_r(t) = A_{rm}$$

where $C_{am} = \frac{Q_a + \delta_a S_m}{\delta_{a0} + \delta_{a1} w_0}$ and $A_{rm} = \frac{\delta_{a1} w_0}{\delta_{r0}} C_{am}$

Hence proof.

**Theorem 4.2** The equilibrium $E^*$ is nonlinearly asymptotically stable provided the following conditions are satisfied in Ω,

$$S_1 = \frac{2s}{L}\delta_{r0} - (s_2 + \delta_{r1} A_r^*)^2 > 0 \qquad (28)$$

$$S_2 = 2 s_1 r \lambda_0^2 \delta_{r0} (\delta_{a0} + \delta_{a1} w_0)^2 - \delta_{a1}^2 w_0^2 r_1 \delta_a^2 \lambda^2 K > 0 \qquad (29)$$

**Proof** Let we consider positive definite function $W$ as defined below,

$$W = p_1\left(B - B^* - B^* \log\frac{B}{B^*}\right) + p_2\left(N - N^* - N^* \log\frac{N}{N^*}\right) + \frac{1}{2} p_3 (S - S^*)^2$$

$$\frac{1}{2} p_4 (C_a - C_a^*)^2 + \frac{1}{2} p_5 (A_r - A_r^*)^2 \qquad (30)$$

where and $p_i\ (i = 1,2,3,4,5)$ are positive constants to be chosen appropriately.

Differentiating above equation with respect to '$t$' we get,

$$\frac{dW}{dt} = p_1 \frac{1}{B}\frac{dB}{dt}(B - B^*) + p_2 \frac{1}{N}\frac{dN}{dt}(N - N^*) + p_3 (S - S^*)\frac{dS}{dt}$$

$$+ p_4 (C_a - C_a^*)\frac{dC_a}{dt} + p_5 (A_r - A_r^*)\frac{dA_r}{dt} \qquad (31)$$

Putting the values of the derivatives from the model system in equation (31) and simplifying, we get

$$\frac{dW}{dt} = -p_1 \frac{s}{L}(B - B^*)^2 - p_2 \frac{r}{K}(N - N^*)^2 - p_3 \lambda_0 (S - S^*)^2 - p_4 (\delta_{a0} + \delta_{a1} w_0)(C_a - C_a^*)^2$$

$$- p_5 (\delta_{r0} + B\delta_{r1})(A_r - A_r^*)^2 - (p_1 s_1 - p_2 r_1)(B - B^*)(N - N^*)$$

$$-(p_1 s_2 + p_5\delta_{r1}A_r^*)(B-B^*)(A_r-A_r^*) + p_3\lambda(N-N^*)(S-S^*)$$

$$+ p_4\delta_a(S-S^*)(C_a-C_a^*) + p_5\delta_{a1}w_0(C_a-C_a^*)(A_r-A_r^*) \quad (32)$$

Choosing $p_1 = 1$ and $p_2 = \frac{s_1}{r_1}$, equation (4.32) reduces to

$$\frac{dW}{dt} = -\frac{s}{L}(B-B^*)^2 - p_2\frac{rs_1}{Kr_1}(N-N^*)^2 - p_3\lambda_0(S-S^*)^2 - p_4(\delta_{a0}+\delta_{a1}w_0)(C_a-C_a^*)^2$$

$$-p_5(\delta_{r0}+B\delta_{r1})(A_r-A_r^*)^2 - (s_2 + p_5\delta_{r1}C_2^*)(B-B^*)(A_r-A_r^*)$$

$$+p_3\lambda(N-N^*)(S-S^*) + p_4\delta_a(S-S^*)(C_a-C_a^*) + p_5\delta_{a1}w_0(C_a-C_a^*)(C_2-C_2^*)$$

Now, $\frac{dW}{dt}$ will be negative if the following conditions are satisfied,

$$(s_2 + p_5\delta_{r1}A_r^*)^2 < 2p_5\frac{s}{L}(\delta_{r0}+B\delta_{r1}) \quad (33)$$

$$p_3\lambda^2 < 2\frac{rs_1}{Kr_1}\lambda_0 \quad (34)$$

$$p_4\delta_a^2 < p_3\lambda_0(\delta_{a0}+\delta_{r1}w_0) \quad (35)$$

$$p_5\delta_{a1}^2 w_0^2 < p_4(\delta_{a0}+\delta_{a1}w_0)(\delta_{r0}+B\delta_{r1}) \quad (36)$$

Maximising left hand side and minimizing right hand side and choosing,

$$p_5 = 1,\ p_3 < \frac{2rs_1\lambda_0}{K\lambda^2 r_1},\quad p_4 < \frac{2rs_1\lambda_0^2(\delta_{a0}+\delta_{a1}w_0)}{K\lambda^2\delta_a^2 r_1}$$

$\frac{dW}{dt}$ will be negative definite inside the region of attraction $\Omega$ provided the conditions (28) - (29) are satisfied showing that $W$ is Lyapunov function. This proves the theorem.

**Remark** From the above theorems, it is observed that when $s_2$ (the depletion rate of plant biomass due to acid rain) and $\delta_a$ (the growth rate of pollutant concentration from emitting sources) are very small, the system is more likely to satisfy the stability conditions (18) – (19) and (28) – (29). This suggests that higher values of these parameters tend to violate the stability criteria, leading to destabilization of the model system. In other words, increased acid rain impact on biomass or faster accumulation of pollutants due to emissions disrupts the ecological balance, pushing the system away from equilibrium. The destabilizing influence of $s_2$ and $\delta_a$ on the system's stability is clearly illustrated in Figs 10 and 11, respectively, where changes in these parameters are shown to shift the system from a stable to an unstable state.

**5 Numerical Simulation and Discussion**

Some numerical simulation have been carried out in this section to examine the practicality of analytical findings regarding local and nonlinear stability behaviour of interior equilibria and feasibility of the model system (1) – (5) using MAPLE 18 by selecting the following set of parameter values,

$s = 1.5$ , $L = 15000$ , $s_1 = 0.000005$ , $s_2 = 0.000001$ , $r = 1$ , $K = 10000$ , $r_1 = 0.000005$ , $\lambda = 0.1$ ,

$\lambda_0 = 0.25$, $Q_a = 4$, $\delta_a = 0.002$, $\delta_{a0} = 0.02$, $\delta_{a1} = 0.02$, $w_0 = 1$, $\delta_{r0} = 0.1$, $\delta_{r1} = 0.0000001$

The equilibrium values of different variables in $E^*(B^*, N^*, S^*, C_a^*, A_r^*)$ corresponding to the above data are,

$B^* = 14463.22199$, $N^* = 10723.16110$, $S^* = 4289.26444$, $C_a^* = 314.46322$, $A_r^* = 61.99598$

The eigenvalues of the Jacobean matrix that corresponds to $E^*(B^*, N^*, S^*, C_a^*, A_r^*)$ for the model system (1) - (5) are $-1.4356$, $-1.0829$, $-0.2500$, $-0.0400$, $-0.1014$.

Since all of the eigenvalues are negative. the internal equilibrium $E^*(B^*, N^*, S^*, C_a^*, A_r^*)$ is locally asymptotically stable

The solution trajectories of the model system are illustrated in Fig.1 with varied beginning starts to demonstrate the nonlinear stability of $E^*$ in $B - N - S$ space. It is obvious from the graphic that all solution trajectories starting within the region of attraction approach the equilibrium value, demonstrating its nonlinear behaviour.

Figs 2 - 9 depict the variation of various variables for various values of relevant parameters. Human population is highly dependent on plant biomass in number of ways in order to fulfil their needs, consequently decreasing the growth of plant biomass. These occurrences are depicted in Figs 2 and 3. It is clear from the Fig. 2, as the growth rate coefficient owing to plant biomass increases, so does human population density, and plant biomass density declines (Fig. 3).

In Fig. 4, the variation in plant biomass density is plotted over time $t$ for different values of $s_1$, illustrating the reduction in plant biomass density as $s_1$ increases.

The cumulative concentration of pollutants, and hence acid rain, is determined by the sources of pollution which are human population density dependent. Thus, the variation of pollutants causing acid rain with the growth rate coefficient of pollutants emitting sources is shown in Fig. 5, in which the variation of cumulative concentration $C_a$ of pollutants, generating acid rain over time $t$ is graphed for different values of $\lambda$. It is noticed that the cumulative concentration of pollutants, generating acid rain grows in the atmosphere when the rate of pollution emitting sources increases owing to the growth of the human population.

For various values of $\lambda$, the growth rate coefficient of pollutants emitting sources due to human population, the variation of cumulative concentrations of acid rain $A_r$ over time $t$ is graphed in Fig. 6. It is noticed that the cumulative concentration of acid rain in the atmosphere grows as the growth rate of polluting sources owing to human population increases. Further, Figure 7 depicts the variation in cumulative concentration of acid rain over time $t$ for various values of the growth rate coefficient $\delta_a$ of pollutants. As seen in the graph the cumulative concentration of acid rain grows as the growth rate coefficient $\delta_a$ increases having negative impacts on plant biomass. Figs 8 and 9 depict the detrimental impact of acid rain on plant biomass. For varying values of the rate of generation $\delta_{a1}w_0$ of acid rain in the atmosphere, the fluctuation of plant biomass density is displayed over time $t$ in Fig.8. For different values of the interaction rate coefficient $s_2$ of the plant biomass with acid rain, variation in plant biomass density is displayed over time $t$ in Fig. 9.

Figs. 10 and 11 examine the nonlinear stability conditions outlined in Theorem 4, confirming that $s_2$ (the depletion rate coefficient of plant biomass density due to acid rain) and $\delta_a$ (the growth rate coefficient of acid-forming gases) both have a destabilizing effect on the stability of the system.

Table 1 shows the variation in cumulative concentrations of gases that cause acid rain and cumulative concentrations of acid rain in the atmosphere over time $t$ for various values of $\lambda$. It indicates that the concentration of gases that create acid rain, as well as the concentration of acid rain, increases when the growth rate coefficient of pollution emitting sources $\lambda$ increases due to human population.

Table 2 shows the variation in the cumulative concentrations of acid rain in the atmosphere over time $t$ for different values of $\delta_a$. It indicates that as the growth rate coefficient of pollutants that cause acid rain increases, the amount of acid rain in the atmosphere increases as well.

Table 3 shows the variation in plant biomass density and human population density over time $t$ for various values of $\delta_{a1}w_0$. It shows that as the rate $\delta_{a1}w_0$ of acid rain in the atmosphere increases, plant biomass density drops and human population decreases.

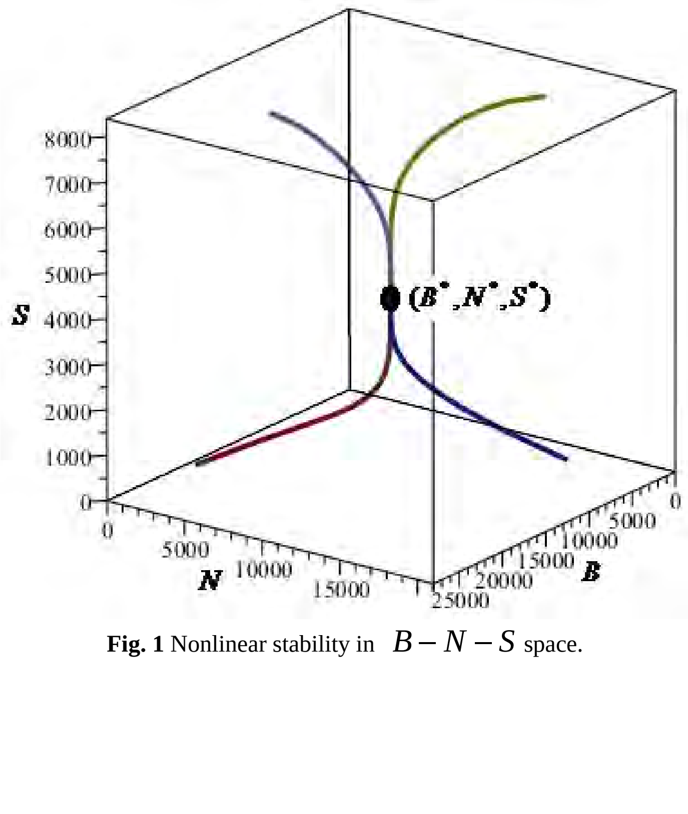


**Fig. 1** Nonlinear stability in $B-N-S$ space.

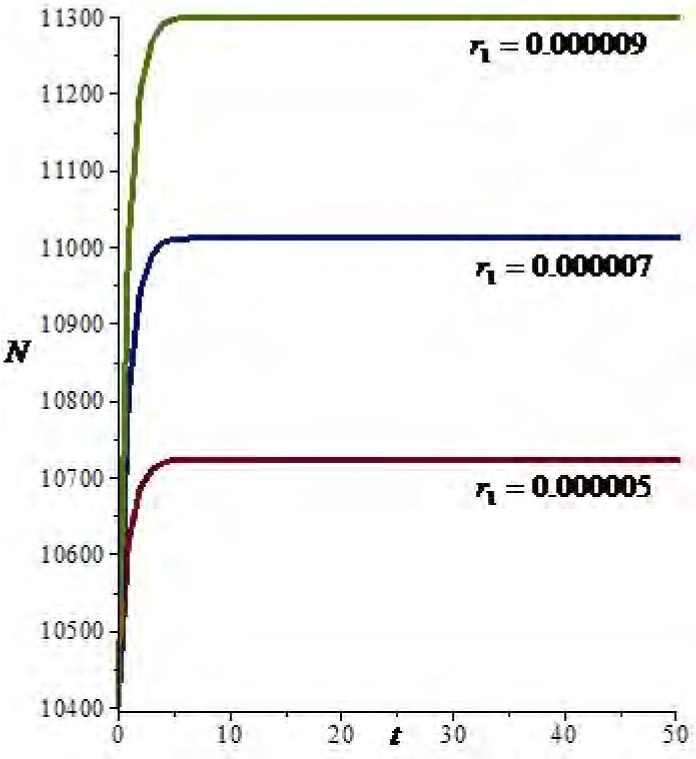


**Fig. 2** Time-dependent variation of $N$ for various values of $r_1$.

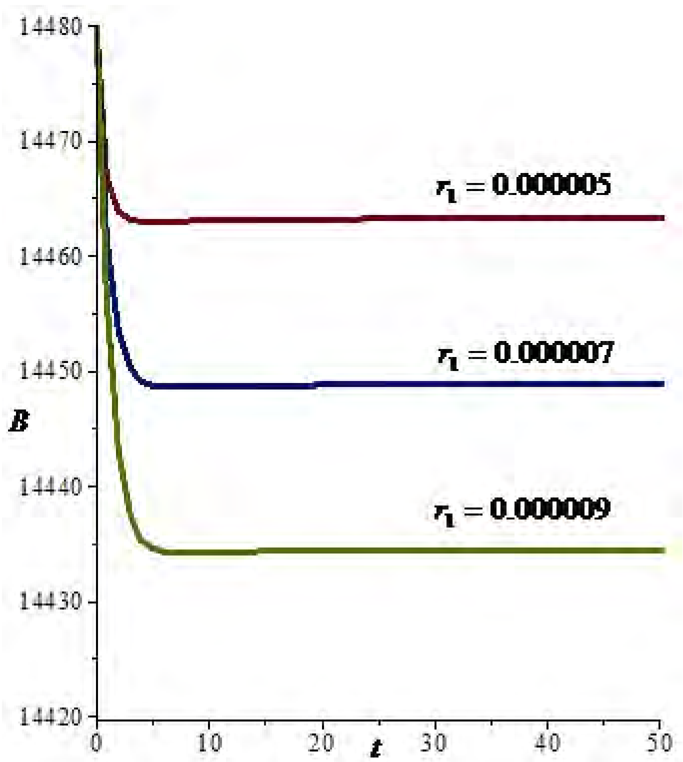


**Fig. 3** Time-dependent variation of $B$ for various values of $r_1$

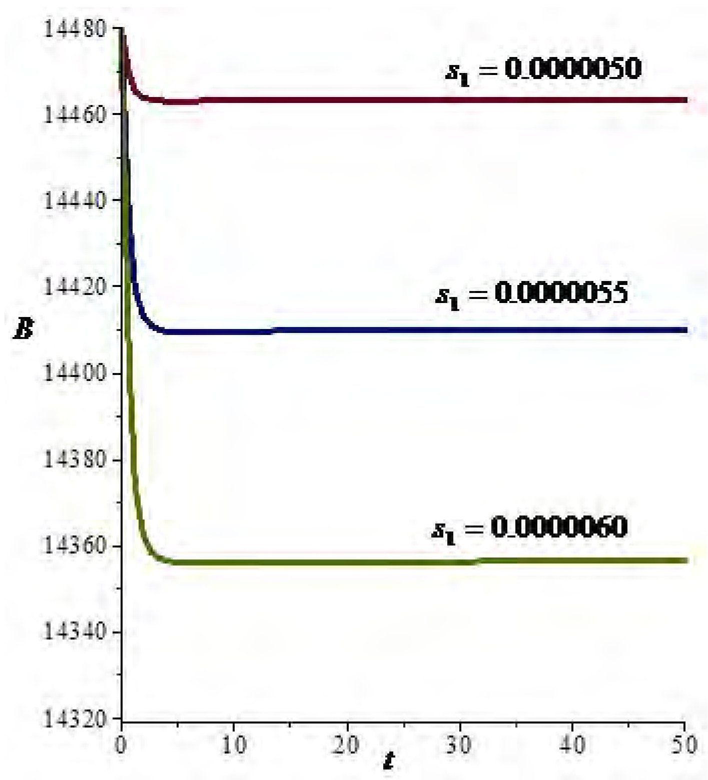


**Fig. 4** Time-dependent variation of $B$ for various values of $s_1$

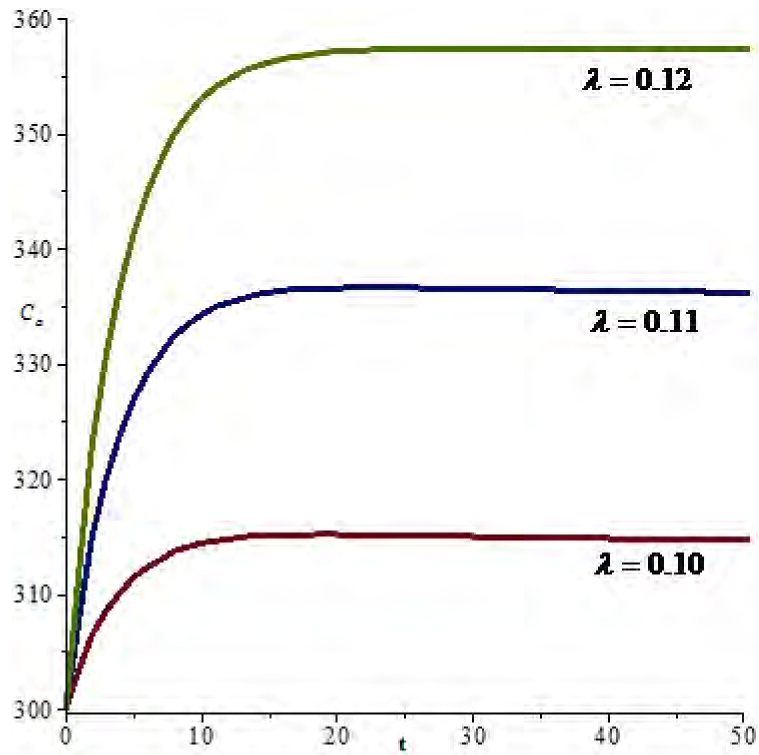


**Fig. 5** Time-dependent variation of $C_a$ for various values of $\lambda$

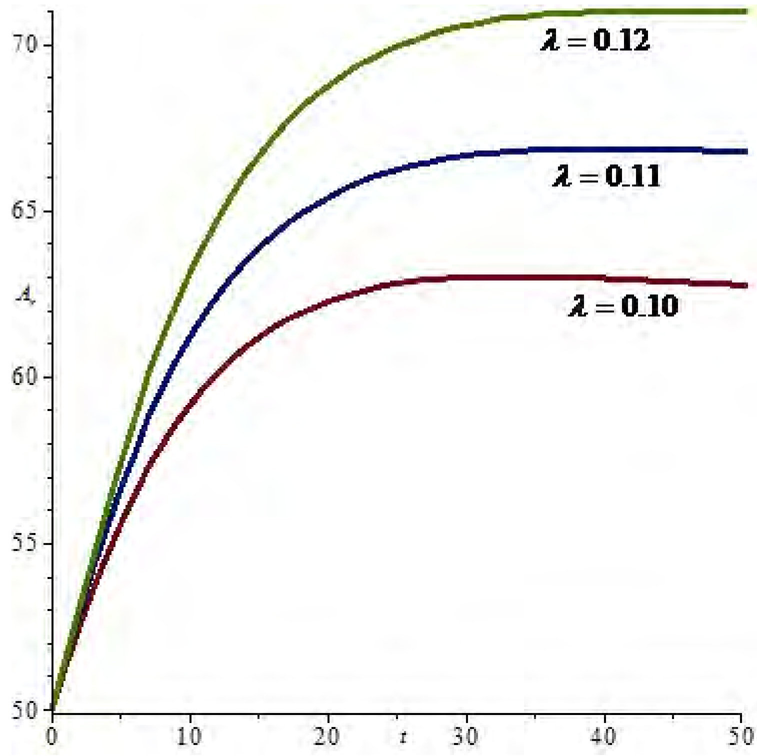


**Fig. 6** Time-dependent variation of $A_r$ for various values of $\lambda$

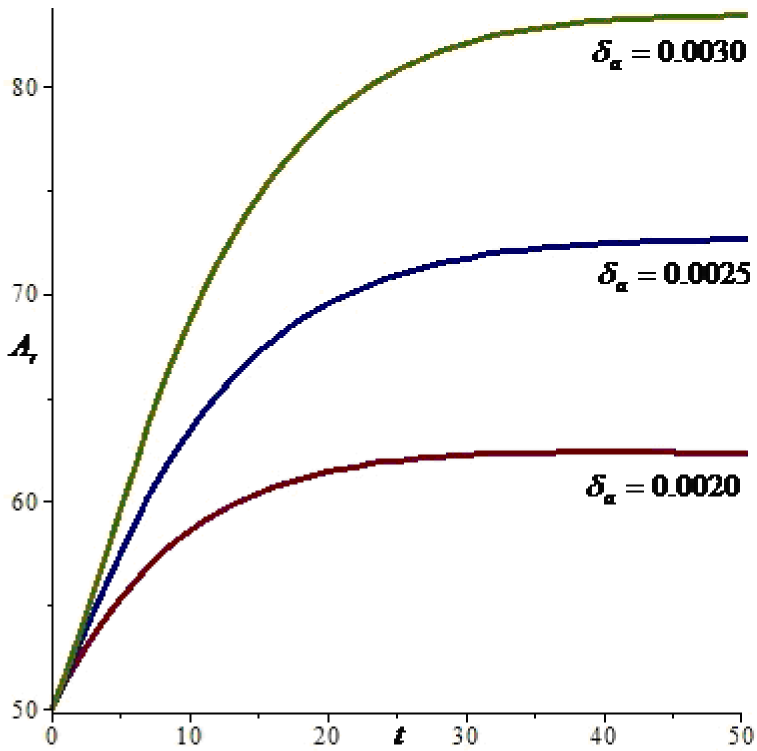


**Fig. 7** Time-dependent variation of $A_r$ for various values of $\delta_a$

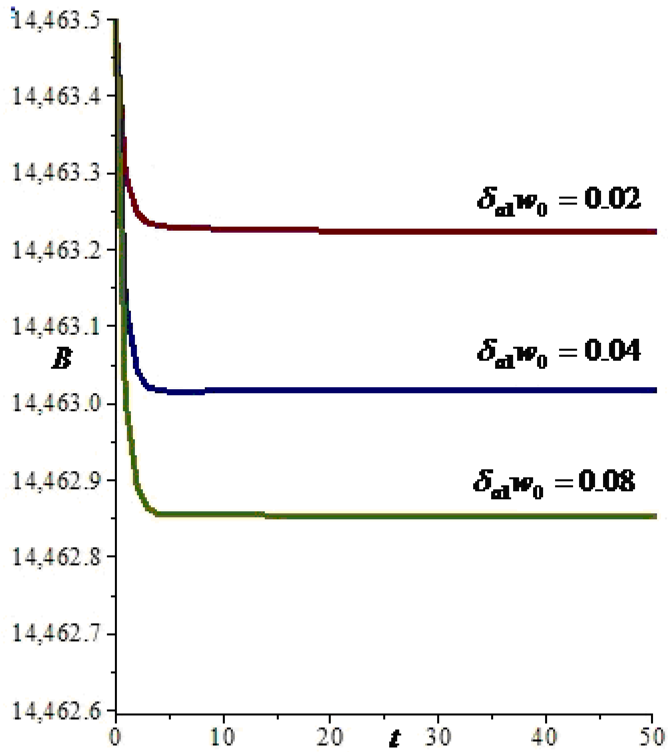


**Fig. 8** Time-dependent variation of $B$ for various values of $\delta_{a1}w_0$

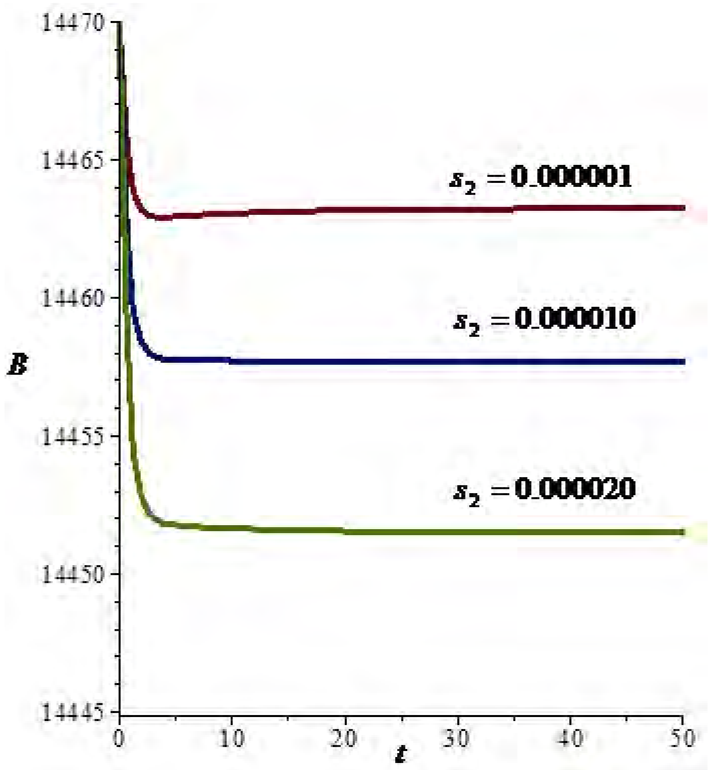


**Fig. 9** Time-dependent variation of $B$ for various values of $s_2$

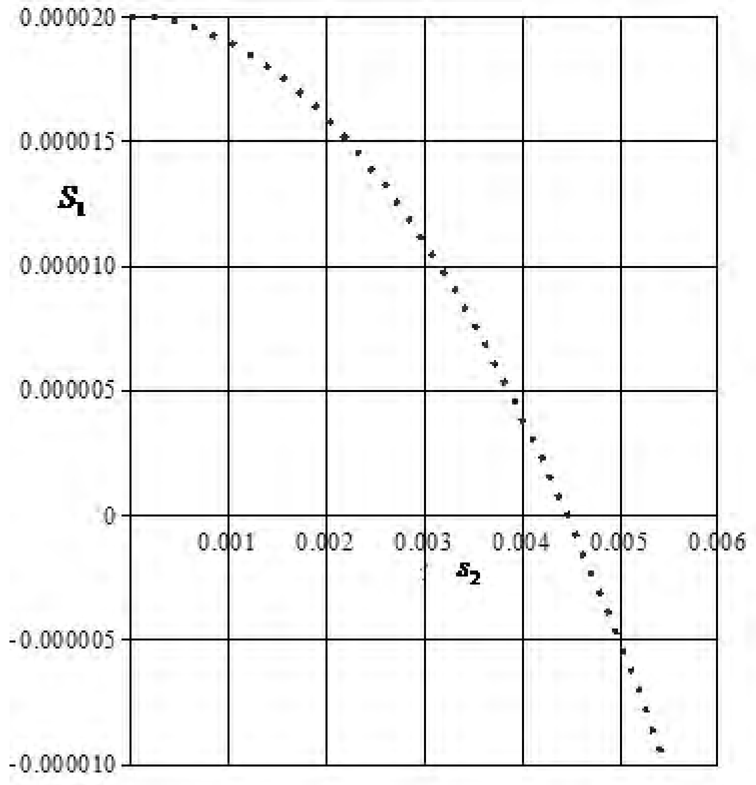


**Fig. 10** variation of stability condition $S_1$ with $s_2$

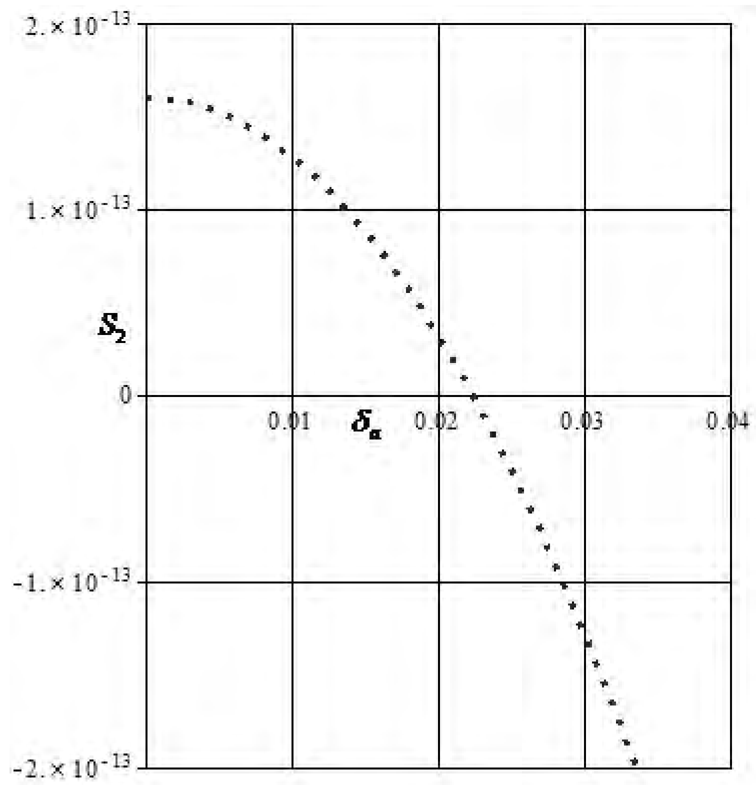


**Fig. 11** variation of stability condition $S_2$ with $\delta_a$

**Table 1** variation of $C_a$ and $A_r$ with $\lambda$.

| $\lambda$ | 0.100 | 0.105 | 0.120 | 0.125 | 0.130 |
|---|---|---|---|---|---|
| $C_a$ | 314.4632 | 325.1863 | 357.3557 | 368.0788 | 378.8020 |
| $A_r$ | 61.9959 | 64.1100 | 70.4521 | 72.5662 | 74.6802 |

**Table 2** variation of $A_r$ with $\delta_a$

| $\delta_a$ | 0.00200 | 0.00225 | 0.0025 | 0.00275 | 0.00300 |
|---|---|---|---|---|---|
| $A_r$ | 61.9959 | 67.2811 | 72.5662 | 77.8513 | 83.1364 |

**Table 3** variation of $B$ and $N$ with $\delta_{a1}w_0$

| $\delta_{a1}w_0$ | 0.02 | 0.03 | 0.04 | 0.05 | 0.06 |
|---|---|---|---|---|---|
| $B$ | 14463.2219 | 14463.0983 | 14463.0158 | 14462.9569 | 14462.9127 |
| $N$ | 10723.1611 | 10723.1549 | 10723.1507 | 10723.1478 | 10723.1456 |

## 6 Result and Discussion

The mathematical analysis and numerical simulations of the model system (1)–(5) reveal significant insights into the interplay between plant biomass, human population, pollutant-emitting sources, and acid rain. The Jacobian matrix evaluated at the internal equilibrium point $E^*$ has all eigenvalues negative, confirming that $E^*$ is locally asymptotically stable. This implies that, under suitable parameter values, the system returns to equilibrium after small disturbances. Fig. 1 illustrates this nonlinear stability, showing converging trajectories from different initial conditions in the *B–N–S* (biomass–population–pollution source) space. Figs 2 and 3 highlight the inverse relationship between plant biomass and human population with respect to growth rate of human population due to biomass. As the growth rate of human population due to biomass increases, population density rises (Fig. 2), leading to a decline in biomass density (Fig. 3), demonstrating the pressure human demand places on natural vegetation. In Fig. 4, increasing the parameter $s_1$, the depletion rate of biomass due to human interaction, results in a decline in biomass over time, reaffirming the damaging impact of human exploitation on vegetation. Pollution dynamics are driven by population-induced emissions. Figure 5 shows that with higher values of $\lambda$, the growth rate coefficient of pollution-emitting sources, the cumulative concentration of pollutants ($C_a$) in the atmosphere increases over time. Fig. 6 further demonstrates that higher $\lambda$ values also accelerate the accumulation of acid rain ($A_r$). Similarly, Fig. 7 shows that increasing $\delta_a$, the growth rate coefficient of acid rain generating pollutants, results in greater acid rain concentration, highlighting its ecological threat. The detrimental effects of acid rain on biomass are further depicted in Figs 8 and 9. Increasing the rate of acid rain formation leads to decreased plant biomass (Fig. 8). Additionally, greater values of the interaction rate $s_2$ between biomass and acid rain lead to sharper biomass depletion (Fig. 9), showing the compounding impact of acid rain on vegetation health.

Figs 10 and 11 illustrate the nonlinear stability conditions $S_1$ and $S_2$ discussed in Theorem 4 [equations (28) and (29) respectively], confirming that both $s_2$ (the depletion rate coefficient of plant biomass density due to acid rain) and $\delta_a$ (the growth rate coefficient of acid-forming gases) exert a destabilizing influence on the stability of the system. Specifically, Fig. 10 shows that the stability condition $S_1$ remains positive only for $s_2 < 0.00446$ and becomes negative beyond this threshold, indicating that increased sensitivity of biomass to acid rain reduces system stability. Similarly, Fig. 11 reveals that the stability condition $S_2$ remains positive for $\delta_a < 0.0223$ and turns negative when $\delta_a > 0.0223$ highlighting that faster accumulation of acid-forming gases undermines the model's equilibrium. These results demonstrate that controlling these key parameters is crucial for maintaining ecological balance and preventing long-term environmental degradation. Thus, the results underscore that human-driven emissions and acid rain are major destabilizing forces in ecological systems.

The quantitative assessment of how environmental pollutants and acid rain concentrations evolve in response to key growth rate parameters is provided in Tables 1 and 2. Specifically, Table 1 shows that an increase in $\lambda$, the growth rate coefficient of pollutant emitting sources due to human activities, leads to a substantial rise in the cumulative concentration of pollutants over time. Similarly, Table 2 demonstrates that an increase in $\delta_a$, the growth rate coefficient of acid-forming gases, results in a significant escalation in the cumulative concentration of acid rain in the atmosphere. These findings underscore the intensifying impact of anthropogenic activities on environmental degradation. Table 3 complements this analysis by illustrating the ecological consequences of such degradation. It reveals that as the rate of acid rain formation increases, both plant biomass density and human population density decline over time. This demonstrates a clear negative feedback mechanism in which environmental deterioration, driven by unchecked pollutant emissions, not only impairs vegetation but also jeopardizes human survival by reducing the natural resources essential for life.

The result of the analysis is that the system representing the interaction between plant biomass, human population, pollutant-emitting sources, pollutant concentration, and acid rain is highly sensitive to certain key parameters. Specifically, the growth rate of pollutant emitting sources due to human population ($\lambda$), the growth rate of acid-forming gases ($\delta_a$), and the depletion rate of plant biomass due to acid rain ($s_2$) have strong destabilizing effects on the model system. As these parameters increase, the system exhibits reduced plant biomass density and a decline in human population over time, primarily due to the negative environmental impacts caused by acid rain due to elevated level of pollutant concentration. Stability analysis through eigenvalues and nonlinear stability conditions further confirms that higher values of $\lambda$, $\delta_a$, and $s_2$

tend to push the system away from equilibrium, making it unstable. Thus, the environmental degradation driven by human induced pollution not only harms plant biomass density but also threatens human sustainability.

**7 Conclusions**

This paper introduces a nonlinear mathematical model to analyze the impact of acid rain, resulting from pollutant emissions, on plant biomass and vegetation within an ecosystem. The model incorporates five key variables: plant biomass density (influenced by human population and acid rain), human population density, density of pollution-emitting sources, cumulative pollutant concentration, and cumulative acid rain concentration. It is assumed that the cumulative concentration of acid rain causing pollutants is proportional to the density of human population and pollution sources within the habitat. The model, grounded in stability theory and formulated through ordinary differential equations, establishes the existence of interior equilibrium and investigates its local and nonlinear stability. Analytical and numerical findings reveal that increasing acid rain concentration leads to a significant decline in plant biomass density, and a growing human population further exacerbates this reduction.

The study holds substantial importance in understanding the intricate dynamics between anthropogenic activities, pollutant emissions, and environmental degradation, emphasizing the urgent need to mitigate pollutant emissions to safeguard plant biomass and ecosystem health. This framework not only enhances our comprehension of these interactions but also provides a predictive tool for policymakers and environmental scientists. Future research could expand the model by incorporating climate change factors, feedback mechanisms between pollutants and atmospheric processes, and mitigation strategies such as technological advancements in emission control. These extensions would further refine the model's applicability and contribute to developing sustainable approaches to managing acid rain and its ecological consequences.

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