Article

Projective scaling method for single objective linear optimization problems based on projection operations

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Abstract
An interior point algorithm to solve single objective linear programming problems has been proposed in this paper. The method uses the projection operation of the gradient of the objective function onto the null space of the feasible region in order to generate, at each iterate, an interior search direction. It can be taken an interior step from the current iterate to the next one along this feasible direction. During the execution of the algorithm, a sequence of interior points will be generated. It has been proved that this sequence converges to an ε-optimal solution, where ε is a predetermined error tolerance known a priori. Numerical single objective linear optimization problems of different kinds, feasible, infeasible and unbounded are illustrated using this algorithm.

Keywords linear programming; linear optimization; projection operation; Interior point method; scaling algorithm.

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1 Introduction

After the seminal algorithm of Karmarkar (1984) for solving linear optimization problems in polynomial time $O(n^{3.5}L)$ arithmetic operations, where n is the number of unknown variables including slack or surplus variables and L is the length of the input data (total number of bits used in the description of the problem data), a great number of the so-called interior point methods for linear optimization have been proposed. These methods can be classified in two main categories:

The first category is the extensions and variants of Karmarkar's algorithm which can be divided also into two subgroups:

1. The projective algorithms as: (Karmarkar, 1984; de Ghellinck and Vial, 1986, 1987; Anstreicher, 1986; Todd and Burell, 1986; Darvay, 2003; Yu and Sun, 2009; Wang and Luo, 2015; Tlas, 2024, 2025).
2. The “affine” methods as: (Barnes, 1986; Gay, 1987; Gill et al., 1986; Vanderbei et al., 1986).

The second category is the path following approaches as: (Gonzaga, 1989; Renegar, 1988).

The methods in the second group are polynomially bounded and require $O(n^{0.5}L)$ iterations. The overall complexity is $O(n^3L)$. The projective methods in the first group are also polynomially bounded. They require $O(nL)$ iterations and $O(n^{3.5}L)$ operations.

Following these proposals, a new projective scaling algorithm is proposed for solving single objective linear optimization problems. This algorithm is mainly based on the projection operation of the gradient of the objective function onto the null space of the feasible region of the problem.

Then, the main idea focuses on the projection operation onto the null space of the feasible region which computes a feasible direction (line search) on every iteration in at most $O(nm^2)$ arithmetic operations, where m is the number of constraints ($m < n$).

It can be taken an interior step from the current iterate to the next one along this feasible direction. During the execution of the algorithm, a sequence of interior points will be generated.

It has been proved that this sequence converges to an $\varepsilon-$optimal solution, where ε is a predetermined error tolerance known a priori.

Also, it is proven that the number of iterations required for the algorithm to converge to a good solution is bounded and estimated to be no more than $O(nL)$ iterations.

Consequently the complexity of the algorithm is computed to be at most $O(n^2m^2L)$ arithmetic operations.

Different kinds of numerical single objective linear optimization problems, feasible, infeasible or unbounded are solved using this algorithm.

2 Statement of the Single Objective Linear Optimization Problem

Consider the linear programming problem given in standard form through:

$$\begin{aligned} &maximize \quad z = c_0 + \sum_{j=1}^{n} c_j x_j \\ &subject\ to \quad \sum_{j=1}^{n} a_{ij} x_j = b_i \quad (i = 1,\dots,m) \\ &\qquad\qquad x_j \geq 0 \quad (j = 1,\dots,n) \end{aligned} \tag{1}$$

where, $c, x \in \mathbf{R}^n$, $b \in \mathbf{R}^m$, A is $m \times n$ matrix, n is the number of unknown (decision) variables including slack and surplus variables, m is the number of linear constraints so that $m < n$, and $c_0 \in \mathbf{R}$.

Assuming that the feasible set: $X = \left\{ x \in \mathbf{R}^n / Ax = b \; and \; x \geq 0 \right\}$ is compact and convex in the nonnegative orthant of $\mathbf{R}^n$.

It is easy to transform the canonical feasibility problem (1) into the following equivalent affine feasibility problem:

$$\begin{aligned} & maximize \quad \tilde{z} = c_0 x_0 + \sum_{j=1}^{n} c_j x_j - M x_{n+1} \\ & subject\ to \quad \sum_{j=1}^{n} a_{ij} x_j + \left(b_i - \sum_{j=1}^{n} a_{ij} \right) x_{n+1} = b_i \quad (i = 1, \ldots, m) \\ & \qquad x_j \geq 0 \quad (j = 1, \ldots, n+1) \end{aligned} \tag{2}$$

where x_0 and x_{n+1} are new decision variables introduced and $M > 0$ is a real constant arbitrary chosen to be large enough.

It is easy to find that the point $e^T = (1, \ldots, 1) \in R^{n+2}$ belongs to the feasible set of problem (2).

Now, by introducing the following variables to problem (2):

$$\begin{aligned} & \tilde{c}^T = (c_0, c_1, \ldots, c_n, -M) \in R^{n+2} \\ & \tilde{a}_{i0} = -b_i \quad (i = 1, \ldots, m) \\ & \tilde{a}_{ij} = a_{ij} \quad (i = 1, \ldots, m, \quad j = 1, \ldots, n) \\ & \tilde{a}_{in+1} = b_i - \sum_{j=1}^{n} a_{ij} \quad (i = 1, \ldots, m) \\ & d^T = (1, 0, \ldots, 0) \in R^{n+2} \end{aligned}$$

And after making some mathematical arrangements, problem (2) can be written as follows:

$$\begin{aligned} & maximize \quad \tilde{z} = \sum_{j=0}^{n+1} \tilde{c}_j x_j \\ & subject\ to \quad \sum_{j=0}^{n+1} \tilde{a}_{ij} x_j = 0 \quad (i = 1, \ldots, m) \\ & \qquad \sum_{j=0}^{n+1} d_j x_j = 1 \\ & \qquad x_j \geq 0 \quad (j = 0, 1, \ldots, n+1) \end{aligned} \tag{3}$$

Below we present a recursive algorithm designed to solve the single objective linear mathematical program (1) and operating within the feasible region, where in this algorithm the transformed and equivalent problem (3) is solved by solving, on every iterate, the next problem (4):

$$
\begin{aligned}
&maximize \quad \tilde{z}' = \sum_{j=0}^{n+1} \tilde{c}_j x_j^k y_j \\
&subject\ to \quad \sum_{j=0}^{n+1} \tilde{a}_{ij} x_j^k y_j = 0 \quad (i = 1,...,m) \\
&\qquad \sum_{j=0}^{n+1} d_j x_j^k y_j = 1 \\
&\qquad y_j \geq 0 \quad (j = 0,1,...,n+1)
\end{aligned}
\tag{4}
$$

After using the following affine linear transformation: $y = \left(X^k\right)^{-1} x$, on every iteration, applied to problem (3).

2.1 Algorithm for solving single objective linear optimization problems

Step 1: *Initialization*. Let $\varepsilon > 0$ be a tolerance level. Let be $\left(x^0\right)^T = e^T = (1,...,1) \in R^{n+2}$, and $k = 0$ the iteration counter.

Step 2: *Change*. Let $B^k = \begin{pmatrix} d^t \\ \tilde{A} \end{pmatrix} X^k$ be an expanded matrix with dimension $(m+1)\times(n+2)$ where $X^k = diag\left(x_j^k\right)_{j=0,..,n+1}$ is a diagonal matrix.

Step 3: *Projection.* Find $p^k \in \mathbf{R}^{n+2}$ and $u^k \in \mathbf{R}^{m+1}$ which solve the linear system of equations

$$
\left.\begin{aligned} p + \left(B^k\right)^T u = X^k \tilde{c} \\ B^k p = 0 \end{aligned}\right\} (p_k)
$$

where, p^k is the projection of $X^k \tilde{c}$ onto the null space of B^k and u^k is the dual variable.

Step 4: *Termination tests*.

1. (Feasibility): if $\left\|p^k\right\| < \varepsilon$ and $x_{n+1}^k < \varepsilon$, then stop, the point $x_j^k (j = 1,...,n)$ is an optimal solution of problem (1) and the value of the objective function z is bounded
2. (Infeasibility): if $\left\|p^k\right\| < \varepsilon$ and $x_{n+1}^k > \varepsilon$, then there is not any feasible solution of problem, the feasible set of problem is empty (infeasible problem)
3. (Unboundedness): if $\left\|p^k\right\| > \varepsilon$ and $x_{n+1}^k < \varepsilon$, then the feasible set of the problem is unbounded and consequently the value of the objective function z is also unbounded

Step 5: *Normalization.* Define $q^k = \dfrac{p^k}{\left\|p^k\right\|}$

Step 6: *Line search step*. Find $0 \leq \alpha^k \leq 1$ which satisfies the following inequalities:

$$\begin{cases} \alpha^k \geq 0 \\ 1+\alpha^k q^k \geq 0 \\ e^t q^k - \dfrac{\alpha^k}{2} + \dfrac{\left(\alpha^k\right)^2}{3\left(1-\alpha^k\right)} \leq 0 \end{cases}$$

A possible choice for α^k which enforces these conditions is $\alpha^k \in \left(0,\ 0.6\right]$

Step 7: *New iterate*. Let

$$y^{k+1} = e + \alpha^k q^k$$

$$x^{k+1} = X^k y^{k+1}$$

Set $k = k+1$ (increment the iteration counter) and return to *step 2*.

Remark: p_k is the set of so-called normal equations whose solution p^k is unique and is the projection of the vector $X^k \tilde{c}$ onto the null space of the matrix B^k. This problem is purely linear and can be solved in $O(nm^2)$ arithmetic operations.

Lemma: When $\left\|p^k\right\| \xrightarrow[k\to\infty]{} 0$, then $e^T p^k \leq 0 \quad (\forall k \geq 0)$.

Proof: Taking into consideration the Holder inequality $\left\|p^k\right\| \geq \dfrac{1}{\sqrt{n+2}} \sum_{i=0}^{n+1} \left|p_i^k\right|$ and the condition $\left\|p^k\right\| \xrightarrow[k\to\infty]{} 0$, it can be concluded that:

$$\sum_{i=0}^{n+1} p_i^k \xrightarrow[k\to\infty]{} 0 \qquad \text{(a)}$$

From step 3 of the previous algorithm, we can write the following:

$\sum_{i=0}^{n+1} p_i^k = \sum_{i=0}^{n+1} \tilde{c}_i x_i^k$, and $\sum_{i=0}^{n+1} p_i^{k+1} = \sum_{i=0}^{n+1} \tilde{c}_i x_i^{k+1}$.

From step 7 we can find that $x_i^{k+1} - x_i^k = \alpha^k x_i^k q_i^k \quad (i = 0,\dots,n+1)$.

So $\sum_{i=0}^{n+1} p_i^{k+1} - \sum_{i=0}^{n+1} p_i^k = \sum_{i=0}^{n+1} \tilde{c}_i \left(x_i^{k+1} - x_i^k\right) = \alpha^k \sum_{i=0}^{n+1} \tilde{c}_i x_i^k q_i^k = \alpha^k \left\|p^k\right\| \geq 0$, then

$$\sum_{i=0}^{n+1} p_i^{k+1} \geq \sum_{i=0}^{n+1} p_i^k \qquad (k = 0,1,\dots)$$

(b)

From (a) and (b), it can be concluded that $\sum_{i=0}^{n+1} p_i^k \leq 0 \quad (\forall k = 0,1,...)$, and the proof of the lemma is completed.

2.2 Complexity calculation

First, we propose some propositions needed for calculation the complexity of the algorithm.

Definition: If A is $n \times n$ nonsingular matrix, $b \in \mathbf{R}^n$ and $T_A : \mathbf{R}^n \to \mathbf{R}^n$ so that $T_A(x) = Ax + b$, then T_A is called an affine transformation. Affine transformations have several important priorities. One of them is the affine transformations preserve set inclusion if:

$$T_A(W) = \{\xi \in \mathbf{R}^n : \xi = Ax + b, x \in W\}$$

Proposition 1: If $W \subseteq W' \subseteq \mathbf{R}^n$, then $T_A(W) \subseteq T_A(W') \subseteq \mathbf{R}^n$. So volumes are changed by a constant factor and the relative volumes are preserved.

Proposition 2: If $W \subseteq \mathbf{R}^n$ is full dimensional and convex, then $vol(T_A(W)) = |\det A| \times vol(W)$.

Considering the propositions 1 and 2, it can be concluded that for any ellipsoid there is an affine transformation which gives a sphere centered at the point $e^T = (1,1,...,1)$ when it is mapping on the given ellipsoid.

Suppose that E^k is an ellipsoid centered at x^k in the iteration number k and E^{k+1} is an ellipsoid centered at x^{k+1} in the iteration number $k+1$.

The inequalities $\left(\prod_{j=0}^{n+1} v_j\right)^{\frac{1}{n+2}} \leq \frac{1}{n+2}\sum_{j=0}^{n+1} v_j \quad and\ v_j \geq 0\,(j = 0,...,n+1)$, express the relation between the geometric mean and the arithmetic mean on the non negative orthant of $\mathbf{R}^{n+2}$.

In view of the previous inequality it followed that:

$$\left(\prod_{j=0}^{n+1}\left(1+\alpha^k q_j^k\right)\right)^{\frac{1}{n+2}} \leq \frac{1}{n+2}\sum_{j=0}^{n+1}\left(1+\alpha^k q_j^k\right) = 1 + \frac{\alpha^k}{n+2}\sum_{j=0}^{n+1} q_j^k$$

As $\sum_{j=0}^{n+1} p_j^k \leq 0$, then $\sum_{j=0}^{n+1} q_j^k \leq 0$ the last inequality becomes

$$\left(\prod_{j=0}^{n+1}\left(1+\alpha^k q_j^k\right)\right)^{\frac{1}{n+2}} \leq 1 + \frac{\alpha^k}{n+2}\sum_{j=0}^{n+1} q_j^k \leq 1$$

By applying the proposition 2, it can be concluded that:

$\frac{vol(B^k)}{vol(B^{k+1})} = \frac{\frac{1}{\prod_{j=0}^{n+1} x_j^k}}{\frac{1}{\prod_{j=0}^{n+1} x_j^{k+1}}} \frac{vol(E^k)}{vol(E^{k+1})}$, where B^k and B^{k+1} are spheres centered at $e^T = (1,1,...,1)$ with radius $r^k = \alpha^k q^k$ and $r^{k+1} = \alpha^{k+1} q^{k+1}$ $\left(\left\|r^{k+1}\right\| \le \left\|r^k\right\|\right)$, then

$\frac{vol(E^{k+1})}{vol(E^k)} = \frac{\prod_{j=0}^{n+1} x_j^{k+1}}{\prod_{j=0}^{n+1} x_j^k} \frac{vol(B^{k+1})}{vol(B^k)} \le \prod_{j=0}^{n+1}\left(1+\alpha^k q_j^k\right) \le 1$. So

It follows that $vol(E^{k+1}) \le vol(E^k)$, then $vol(E^k) \xrightarrow[k\to\infty]{} 0$.

Using the development of Taylor for the function $\log(1+\lambda)$ around $\lambda = 0$ with $|\lambda| < 1$, it can be seen that: $\log(1+\lambda) = \lambda - \frac{\lambda^2}{2} + \xi(\lambda)$, where $\xi(\lambda) = \sum_{j=3}^{\infty} \frac{\lambda^j (-1)^{j+1}}{j}$.

So $|\xi(\lambda)| = \sum_{j=3}^{\infty} \left|\frac{\lambda^j (-1)^{j+1}}{j}\right| = \sum_{j=3}^{\infty} \frac{|\lambda^j|}{j} |-1|^{j+1} = \sum_{j=3}^{\infty} \frac{|\lambda^j|}{j}$

Since $j \ge 3$, then $\frac{1}{j} \le \frac{1}{3}$, it follows that $|\xi(\lambda)| \le \sum_{j=3}^{\infty} \frac{|\lambda|^j}{3} = \frac{|\lambda|^3}{3(1-|\lambda|)}$.

So $\log(1+\lambda) \le \lambda - \frac{\lambda^2}{2} + \frac{|\lambda|^3}{3(1-|\lambda|)}$

Let $\lambda = \alpha^k q_j^k$, then $\log(1+\alpha^k q_j^k) \le \alpha^k q_j^k - \frac{\left(\alpha^k\right)^2}{2}\left(q_j^k\right)^2 + \frac{\left|\alpha^k q_j^k\right|^3}{3\left(1-\left|\alpha^k q_j^k\right|\right)}$.

It follows that $\sum_{j=0}^{n+1} \log(1+\alpha^k q_j^k) \le \alpha^k \sum_{j=0}^{n+1} q_j^k - \frac{\left(\alpha^k\right)^2}{2} \sum_{j=0}^{n+1}\left(q_j^k\right)^2 + \sum_{j=0}^{n+1} \frac{\left|\alpha^k q_j^k\right|^3}{3\left(1-\left|\alpha^k q_j^k\right|\right)}$.

As $\sum_{j=0}^{n+1}\left(q_j^k\right)^2 = 1$, $\sum_{j=0}^{n+1}\left|\alpha^k q_j^k\right|^3 \le \left\|\alpha^k q^k\right\|^3$, and $\left|\alpha^k q_j^k\right| \le \left\|\alpha^k q^k\right\|$ $(j = 0,...,n+1)$, then the last inequality becomes: $\sum_{j=0}^{n+1} \log(1+\alpha^k q_j^k) \le \alpha^k \sum_{j=0}^{n+1} q_j^k - \frac{\left(\alpha^k\right)^2}{2} + \frac{\left\|\alpha^k q^k\right\|^3}{3\left(1-\left\|\alpha^k q^k\right\|\right)}$

Hence, $\sum_{j=0}^{n+1}\log(1+\alpha^k q_j^k) \le \alpha^k \sum_{j=0}^{n+1} q_j^k - \frac{(\alpha^k)^2}{2} + \frac{(\alpha^k)^3}{3(1-\alpha^k)}$

The value of α^k should be chosen so that $\sum_{j=0}^{n+1} q_j^k - \frac{\alpha^k}{2} + \frac{(\alpha^k)^2}{3(1-\alpha^k)} < 0$, then it can be written that:

$\sum_{j=0}^{n+1}\log(1+\alpha^k q_j^k) \le -\frac{\beta}{n}, \beta > 0$. So $\prod_{j=0}^{n+1}(1+\alpha^k q_j^k) \le e^{-\frac{\beta}{n}}, \beta > 0$

From this relation and since $vol(E^{k+1}) \le vol(E^k) \times \prod_{j=0}^{n+1}\left(1+\alpha^k q_j^k\right)$, it follows that

$vol(E^{k+1}) \le vol(E^k) \times e^{-\frac{\beta}{n}} \le ... \le vol(E^0) \times e^{-\beta\frac{k+1}{n}}$, that gives $vol(E^{k+1}) \le V^0 \times e^{-\beta\frac{k+1}{n}}$.

Choosing $\varepsilon = e^{-dL}$, where d is a positive number and L is the length of the input data (total number of bits used in the description of the problem data), then $vol(E^{k+1}) \le \varepsilon$ after K iterations, where $K = \left\lfloor -1 + \frac{n}{\beta}\log\left(\frac{\varepsilon}{V^0}\right)^{-1} \right\rfloor + 1 = O(nL)$, and $\lfloor u \rfloor$ denotes the integer part of $u \ge 0$.

From step 3 of the algorithm, the problem P_k is a set of the so-called normal equations which has a unique solution. It is known that this problem is purely linear and can be solved in $O(nm^2)$ arithmetic operations. The proposed algorithm stops in no more than $O(nL)$ iterations, then it can be seen that the complexity of the algorithm will be at most $O(n^2m^2L)$. Consequently, the optimal solution of the linear problem will be reached in polynomial time.

2.3 Convergence analysis

From the proposed algorithm, we have $x^k \in X$ so that $p^k = 0$. Suppose that there is another point $x^* \in X$ so that $\tilde{c}^t x^* > \tilde{c}^t x^k$. This implies $\tilde{c}^t(x^* - x^k) > 0$ which means that the vector $d^* = x^* - x^k$ is an improvement direction. Since $x^{k+1} = x^k + \alpha^k X^k q^k \in X$, then let $w = x^k + \alpha d^*$ where $0 < \alpha \le 1$. If $d^* \ne X^k q^k$ (matrix $\tilde{A}$ is full rank), then $\tilde{A}d^* \ne \tilde{A}X^k q^k = 0$, this implies $\tilde{A}d^* \ne 0$ which means the point w is not feasible.

For, $\alpha = 1$ we get $x^* = w$ and that conflicts with the fact that x^* is a feasible point. Then it must be $d^* = X^k q^k$ which means clearly $x^{k+1} = w$.

For, $\alpha = 1$ we have, $x^{k+1} = x^k = w$, and this means that x^k is an accumulation point in X , the condition of optimality of Karush-Kuhn-Tuker (*KKT*) in accumulation point x^k are given as follows:

$\exists u^k \in \mathbf{R}^{m+1}, \lambda \in \mathbf{R}_+^{n+2}$:

$$\tilde{c} - \begin{pmatrix} d^t \\ \tilde{A} \end{pmatrix}^t u^k - I\lambda = 0$$

$$\lambda_i x_i^k = 0 \quad (i = 0,...,n+1)$$

To demonstrate the verification of these conditions, from step 3 of the algorithm, we can write the following equation (at $p^k = 0$):

$$\left(B^k\right)^t u^k = X^k \tilde{c} \quad \text{Where } B^k = \begin{pmatrix} d^t \\ \tilde{A} \end{pmatrix} X^k \text{, then}$$

$$\left(\begin{pmatrix} d^t \\ \tilde{A} \end{pmatrix} X^k\right)^t u^k = X^k \tilde{c} \Rightarrow X^k \begin{pmatrix} d^t \\ \tilde{A} \end{pmatrix}^t u^k = X^k \tilde{c} \Rightarrow X^k \tilde{c} - X^k \begin{pmatrix} d^t \\ \tilde{A} \end{pmatrix}^t u^k = 0$$

As the proposed algorithm creates a sequence of points $\{x^k\}_{k=0,1,...}$ contained in X set with $x_j^k > 0\ (j = 0,...,n+1)$ and $k \geq 0$, then it can be concluded that:

$$\tilde{c} - \begin{pmatrix} d^t \\ \tilde{A} \end{pmatrix}^t u^k = 0$$

As a result of taking $\lambda_j = 0\ (j = 0,...,n+1)$, the conditions of optimality of *KKT* in point x^k are satisfied. Sequence $\{x^k\}_{k=0,1,...}$ converges to a solution that satisfies the conditions of optimality of *KKT* of problem (3). Consequently, this succession creates a sequence of points $\{x^k\}_{k=1,...}$ contained in X and converges to an optimal solution of problem (1).

3 Conclusions

A new projective scaling algorithm has been proposed for solving single objective linear programming problems. This algorithm is mainly based on the projection operation of the gradient of the objective function onto the null space of the feasible region of the problem in order to generate, at each iterate, an interior search direction. It can be taken an interior step from the current iterate to the next one along this feasible direction. During the execution of the algorithm, a sequence of interior points will be generated. It has been proved that this sequence converges to an $\varepsilon -$ optimal solution, where ε is a predetermined error tolerance known a priori. Numerical single objective linear optimization problems of different kinds, feasible, infeasible and unbounded are illustrated using this algorithm.

4 Illustrative Examples

The practical demonstration of the proposed algorithm will be done through the following numerical examples where three kinds of single objective linear optimization problems, feasible, infeasible and unbounded are solved by the algorithm using MATLAB computer code.

Example 1: (first feasible problem). The given single objective linear optimization problem is:

$$
\begin{aligned}
&\max z = 7x_1 + 2x_2 \\
&\quad -x_1 + 2x_2 + x_3 = 4 \\
&\quad 5x_1 + x_2 + x_4 = 20 \\
&\quad 2x_1 + 2x_2 - x_5 = 7 \\
&\quad x_1, x_2, x_3, x_4, x_5 \geq 0
\end{aligned}
$$

Input data of the primal problem, form (1):

$$
c^T = \begin{pmatrix} 7 & 2 & 0 & 0 & 0 \end{pmatrix}
$$

$$
A = \begin{pmatrix} -1 & 2 & 1 & 0 & 0 \\ 5 & 1 & 0 & 1 & 0 \\ 2 & 2 & 0 & 0 & -1 \end{pmatrix}
$$

$$
b^T = \begin{pmatrix} 4 & 20 & 7 \end{pmatrix}
$$

Input data of the modified problem, form (3):

$$
\tilde{c}^T = \begin{pmatrix} 0 & 7 & 2 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}
$$

$$
\tilde{A} = \begin{pmatrix} -4 & -1 & 2 & 1 & 0 & 0 & 2 \\ -20 & 5 & 1 & 0 & 1 & 0 & 13 \\ -7 & 2 & 2 & 0 & 0 & -1 & 4 \end{pmatrix}
$$

$$
d^T = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}
$$

Output data: The optimal solution x^*, the optimal value of the objective function z^* and the termination criterions p^k and x_{n+1}^k.

$$
x^* = \begin{pmatrix} 3.2727 & 3.6364 & 0.0001 & 0.0001 & 6.8182 \end{pmatrix}^T
$$

$$
z^* = 30.1818
$$

$$
\left\| p^k \right\| \xrightarrow[k \to \infty]{} 0
$$

$$
x_{n+1}^k \xrightarrow[k \to \infty]{} 0
$$

Example 2: (second feasible problem). The given single objective linear optimization problem is:

$$\max z = x_1 + 3x_2$$
$$x_1 + x_2 + x_3 = 14$$
$$-2x_1 + 3x_2 + x_4 = 12$$
$$2x_1 - x_2 + x_5 = 12$$
$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

Input data of the primal problem, form (1):

$$c^T = \begin{pmatrix} 1 & 3 & 0 & 0 & 0 \end{pmatrix}$$
$$A = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ -2 & 3 & 0 & 1 & 0 \\ 2 & -1 & 0 & 0 & 1 \end{pmatrix}$$
$$b^T = \begin{pmatrix} 14 & 12 & 12 \end{pmatrix}$$

Input data of the modified problem, form (3):

$$\tilde{c}^T = \begin{pmatrix} 0 & 1 & 3 & 0 & 0 & 0 & -100 \end{pmatrix}$$
$$\tilde{A} = \begin{pmatrix} -14 & 1 & 1 & 1 & 0 & 0 & 11 \\ -12 & -2 & 3 & 0 & 1 & 0 & 10 \\ -12 & 2 & -1 & 0 & 0 & 1 & 10 \end{pmatrix}$$
$$d^T = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Output data: The optimal solution x^*, the optimal value of the objective function z^* and the termination criterions p^k and x_{n+1}^k.

$$x^* = \begin{pmatrix} 6 & 8 & 0 & 0 & 8 \end{pmatrix}^T$$
$$z^* = 30$$
$$\left\| p^k \right\| \xrightarrow[k \longrightarrow \infty]{} 0$$
$$x_{n+1}^k \xrightarrow[k \longrightarrow \infty]{} 0$$

Example 3: (unbounded problem). The given single objective linear programming problem is:

$$\max z = 5x_1 + 7x_2$$
$$x_1 + x_2 - x_3 = 6$$
$$x_1 - x_4 = 4$$
$$x_2 + x_5 = 3$$
$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

Input data of the primal problem, form (1):

$$c^T = \begin{pmatrix} 5 & 7 & 0 & 0 & 0 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 1 & -1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

$$b^T = \begin{pmatrix} 6 & 4 & 3 \end{pmatrix}$$

Input data of the modified problem, form (3):

$$\tilde{c}^T = \begin{pmatrix} 0 & 5 & 7 & 0 & 0 & 0 & -100 \end{pmatrix}$$

$$\tilde{A} = \begin{pmatrix} -6 & 1 & 1 & -1 & 0 & 0 & 5 \\ -4 & 1 & 0 & 0 & -1 & 0 & 4 \\ -3 & 0 & 1 & 0 & 0 & 1 & 1 \end{pmatrix}$$

$$d^T = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Output data: the value of the objective function z^* and the termination criterions p^k and x_{n+1}^k.

$$\left\| p^k \right\| \xrightarrow[k \longrightarrow \infty]{not} 0$$

$$x_{n+1}^k \xrightarrow[k \longrightarrow \infty]{} 0$$

$$z \xrightarrow[k \longrightarrow \infty]{} \infty$$

Example 4: (infeasible problem). The given single objective linear programming problem is:

$$\begin{aligned} \max z &= x_1 - x_2 \\ & 2x_1 - x_2 - x_3 = -2 \\ & x_1 - 2x_2 + x_4 = -8 \\ & x_1 + x_2 + x_5 = 5 \\ & x_1, x_2, x_3, x_4, x_5 \geq 0 \end{aligned}$$

Input data of the primal problem, form (1):

$$c^T = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \end{pmatrix}$$

$$A = \begin{pmatrix} 2 & -1 & -1 & 0 & 0 \\ 1 & -2 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{pmatrix}$$

$$b^T = \begin{pmatrix} -2 & -8 & 5 \end{pmatrix}$$

Input data of the modified problem, form (3):

$$\tilde{c}^T = \begin{pmatrix} 0 & 1 & -1 & 0 & 0 & 0 & -100 \end{pmatrix}$$

$$\tilde{A} = \begin{pmatrix} 2 & 2 & -1 & -1 & 0 & 0 & -2 \\ 8 & 1 & -2 & 0 & 1 & 0 & -8 \\ -5 & 1 & 1 & 0 & 0 & 1 & 2 \end{pmatrix}$$

$$d^T = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Output data: the termination criterions p^k and x_{n+1}^k.

$$\|p^k\| \xrightarrow[k \to \infty]{} 0$$

$$x_{n+1}^k \xrightarrow[k \to \infty]{not} 0$$

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