Article

Fractal basins in an ecological model

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Received 28 May 2013; *Accepted 2 July 2013*; *Published online 1 September 2013*

Abstract
Complex dynamics is detected in an ecological model of host-parasitoid interaction. It illustrates fractalization of basins with self-similarity and chaotic attractors. This paper describes these dynamic behaviors, bifurcations, and chaos. Fractals basins are displayed by numerical simulations.

Keywords host-parasitoid model; chaotic behavior; bifurcation; fractal basin.

Computational Ecology and Software
ISSN 2220-721X
URL: http://www.iaees.org/publications/journals/ces/online-version.asp
RSS: http://www.iaees.org/publications/journals/ces/rss.xml
E-mail: ces@iaees.org
Editor-in-Chief: WenJun Zhang
Publisher: International Academy of Ecology and Environmental Sciences

1 Introduction

We deal with two-dimensional noninvertible maps which have contributed greatly to the understanding of complex nonlinear dynamics. Such behaviors have been studied extensively, particularly in the applied dynamics literature, and constitute a central issue in population ecology. A host-parasite model is considered, it is of interest and offers a richness of bifurcations and an interesting set of dynamical phenomena due to the presence of multistable states basins, deformation of basin boundaries and transient chaos.

An attractor is a set towards which a dynamical system evolves over time. It is very common for dynamical systems to have more than one attractor. Such attractors can be static, periodic, quasiperiodic, or chaotic and are contained within a basin of attraction, which is the set of initial conditions that eventually approach the attractor. Studying the attraction basins of such nonlinear maps helps in understanding the ways of multistability formation.

Multistability poses a challenge for studying and investigating the dynamics occurring in various areas of engineering and environmental sciences. Sometimes the basin boundaries are fractal sets, which can make the identification of the final behavior extremely difficult. The fractal structure may be revealed by fractal basin boundaries or by patterns of self-similarity. The analysis of these structures is useful for obtaining information about the future behavior of attractors and their basins, and it provides important knowledge about the relation between them. The concept of chaos and strange attractors was also very attractive in ecological research see (Ivanchikov and Nedorezov, 2011, 2012).

In the present work, we consider an ecological system for proving persistence, the local exponential stability of a positive fixed point. Our approach extensively revolves around and relies on numerical simulations. It concerns one research area, the detection of fractal basin sets and the identification and verification of some properties of attractors, and it intends to give the basic patterns of complex non-uniqueness in the dynamical behavior of this parasitised host population proposed in Kaitala & Heino, 1996, and Kaitala et al., 1999 in a complex nonlinear mathematical expression. The authors present evidence supporting the claim that their basins of attraction are fractals, and the system exhibits multiple attractors with a qualitative dynamics which depends on the initial conditions. The importance of this research is to help us to understand the mechanisms inducing the irregular fluctuations of the parasitised populations.

This paper is organized as follows: Section 2 describes some peculiar properties of such a map, their dependence on the parameters, and stability of the fixed points or attractors. The qualitative behavior and bifurcations of this map are examined by using a qualitative theory and standard bifurcation theory. In section 3, we discuss some cases where bifurcations can lead to creation of fractal basins and can cause qualitative changes in the structure of the domain as parameters are varied. Finally, we give the conclusion.

2 Host-parasite Interaction Model

To investigate the multiple attractors with complex basins in the case of interacting populations, we consider the model T studied by (Holling, 1959; Royama, 1971; Rogers, 1972; Zhao, 2009), for understanding the population dynamics and the mechanisms that induce the population variations.

$$T(x,y)=\begin{cases} x' = x\exp(r(1-x)-\dfrac{100ay}{1+ax}) \\ y' = x(1-\exp(-\dfrac{100ay}{1+ax})) \end{cases} \tag{2.1}$$

where r is the intrinsic growth rate of the host population, a is the instantaneous search rate.The variables x and y are, respectively, the host and parasitoid population sizes.

These authors observed rare features and complex dynamics patterns such as supertransients and chaotic transients, multiple attractors and basins of attraction with fractal properties (patterns of selfsimilarity and fractal basin boundaries).

The host-parasitoid model may produce stable, periodic, quasiperiodic or chaotic dynamics. Here we are interested in sustained coexistence of both species. We consider the dynamics of the host population in terms of constant parameter values of r, and use a as a bifurcation parameter.

Because the analytical intractability of this mathematical model, we validate our results by numerical simulations and discuss their implications. We provide some numerical evidence for the qualitative dynamic behavior of the map.

From Fig. 1, we can draw the following conclusion: For $r>0$ and $a>0$, in the white area, the host and the parasitoid can coexist. For $r=3.5$ and $a=0.037$, a period-3 cycle can coexist with a closed invariant curve. For $r=2.2$ and $a=0.04$,a period-5 cycle exists and for $r=1.93$ and $a=0.053$, we have a period-6 cycle.

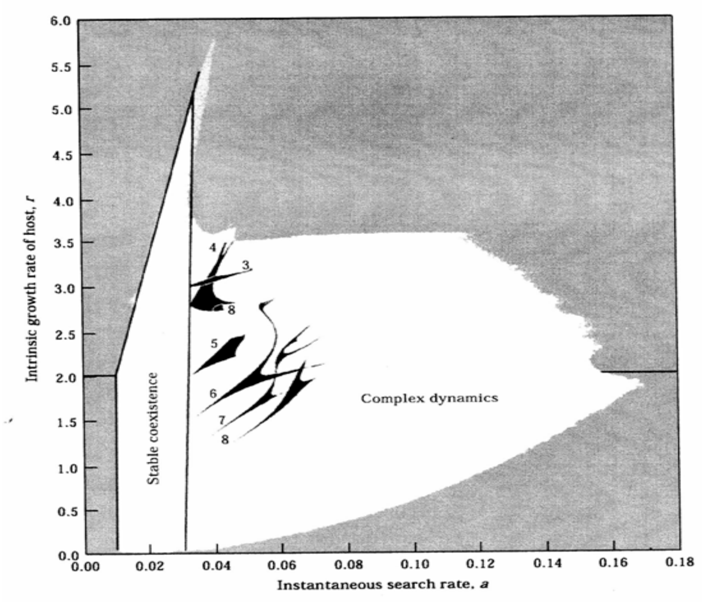


Fig. 1 Bifurcation diagram for the host-parasitoid system (Kaitala et al., 1999).

The fixed points of T in Eq. (2.1) are solutions obtained by a trivial manipulation of (2.1) with $x' = x$ and $y' = y$. Besides the trivial solution $(0,0)$ which always exists, we observe that two additional fixed points exist if $a \geq 0$.

We have to focus attention on bifurcations playing an important role in the dynamics, those happening for $a \geq 0$ and $r > 0$. We can state the following propositions:

Proposition 1 : *If $a = 0$, then $x^* = 1$ is the unique fixed point of the unidimensional reduced map $x' = x\exp(r(1-x))$. If $a > 0$, then two more fixed points $P_1 = (1,0)$ and P_2 can exist such that $P_2 = (x, x(1-\exp(r(1-x)))$.*

Lemma 2 : *The system (2.1) admits P_2 as a positive fixed point if $a > 0.01$.*

Proof: The fixed point $P_2 = (x, x(1-\exp(r(1-x)))$ has positive components because x is the intersection of two curves $r(1-x)(1+ax) = 100ax(1-\exp(r(1-x)))$ and $0 < x < 1$, we can not obtain the solutions explicitly

Let us investigate the qualitative behaviors of the system (2.1). As usual, the local dynamics of map (2.1) in the neighborhood of a fixed point is dependent on the Jacobian matrix. The Jacobian is evaluated at the fixed point, which we denote by $J = \det DT(x,y)$.

Let

$$J = \begin{pmatrix} \exp^{(r(1-x)-\frac{100ay}{1+ax})}(1 - x(r - \frac{100a^2 y}{(1+ax)^2})) & \exp^{(r(1-x)-\frac{100ay}{1+ax})}(-\frac{100ax}{1+ax}) \\ (1-\exp^{-\frac{100ay}{1+ax}}) - \frac{100a^2 xy}{(1+ax)^2}\exp^{-\frac{100ay}{1+ax}} & \frac{100ax}{1+ax}\exp^{\frac{-100ay}{1+ax}} \end{pmatrix}$$

be the Jacobian matrix of T at the state variable (x,y).

We consider now the conditions for local stability of the fixed point $P_1 = (1,0)$ in terms of the parameters in Eq. (2.1). The study of the stability and attractivity of the positive fixed point P_2 gets difficult. The Jacobian matrix $J_{(1,0)} = \begin{pmatrix} 1-r & \frac{-100a}{1+a} \\ 0 & \frac{100a}{1+a} \end{pmatrix}$ of T in $P_1=(1,0)$ has two multipliers 1-r; 100a/(1+a).

For a>0, and by a simple computation, it is straightforward to obtain the following result:

- If 0<r<2 and a<0.01: the fixed point $P_1=(1,0)$ is a stable and P_2 is a saddle with a negative second component.
- If r>2 and a>0.01 : $P_1=(1,0)$ which always exists, is unstable and P_2 is a stable point. Both of these fixed points undergo bifurcations in parameter space for positive values of a. We can see that for a=0.010, one of the multipliers of $P_1=(1,0)$ is 1 and the other is not one with modulus. Thus the transcritical bifurcation occurs with a=(1/99) .

Proposition 3 : *If r=2, the map (1) undergoes a flip bifurcation at the fixed point $P_1=(1,0)$.*

Proof: By simple computation, we can prove this proposition.

The population dynamics will be fully investigated through numerical simulations. In Figure 2a, for a=0.01 the transcritical bifurcation takes place with $P_1=P_2$ with the extinction of the Y species, the saddle P_1 with the two multipliers S_1=-1,$S_2\le 1$ gives rise to a stable period-2 cycle located on the x-axis.

In Figure 2b, for a=0.0097 the stable fixed point and interior is P_1 appears and its attraction basin is illustrated by the red color, P_2 has negative components and it is a saddle located on the fourth quadrant.

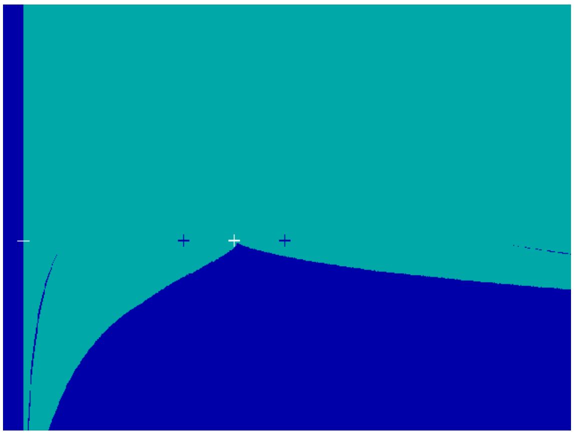


Fig. 2a Coexistence of two merged fixed points, (0,0) and (1,0) and period-2 cycle (blue crosses) for $r = 2.01$, $a = 0.01$.

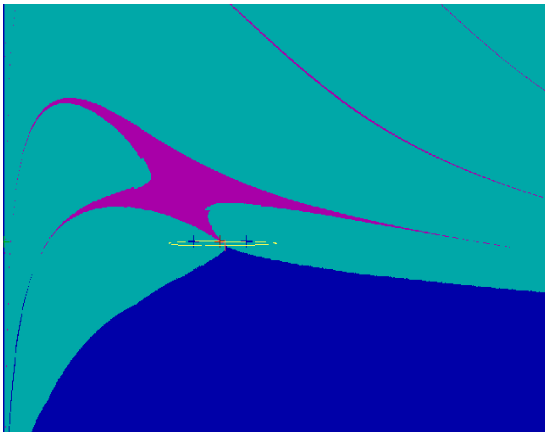


Fig. 2b Coexistence of three fixed points (0,0), P_1 and P_2 and period-2 cycle (blue crosses) for $r = 2.01$, $a = 0.0097$.

Proposition 4 : *If a=(10/3), the system (2.1) undergoes a Neimark-Sacker bifurcation at the stable interior fixed point P_2. Moreover an attracting invariant closed curve bifurcates from the fixed point.*

For $r = 2.00$, the fixed point (1,0) undergoes a flip bifurcation. In this case the parasitoid becomes extinction and the host pass through period doubling bifurcation. As r increases, the interior fixed point P_2 becomes unstable through a discrete Neimark-Sacker bifurcation and the behavior of host-parasitoid model becomes chaotic (see Fig. 4) .

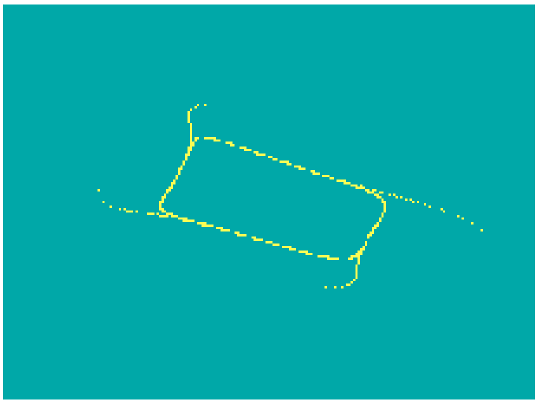


Fig. 3 Neimark-Sacker Bifurcation at P_2, $r = 2.82$, $a = 0.033$.

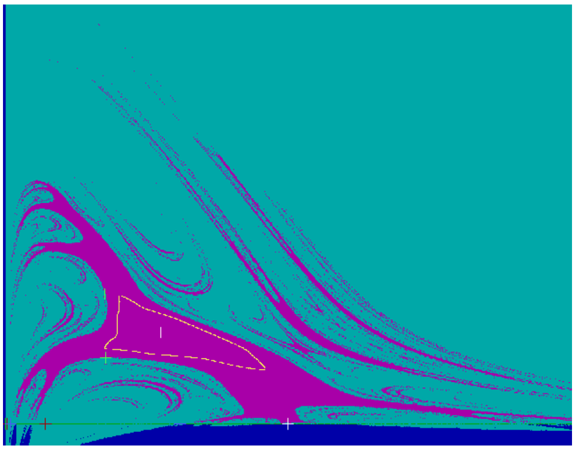
3 Fractal Basins

A plot of the fractal basins associated with a dynamical system provides a qualitative indication of the difficulty in predicting its future evolution. Since the relation between fractality and nonlinear dynamics has been established, we use a numerical technique to characterize the fractal nature of the basins. We follow the evolution of the phase plane as we vary the parameters r and a. We made a survey in the phase plane, the system displays mostly the dynamics of two coexisting attractors.

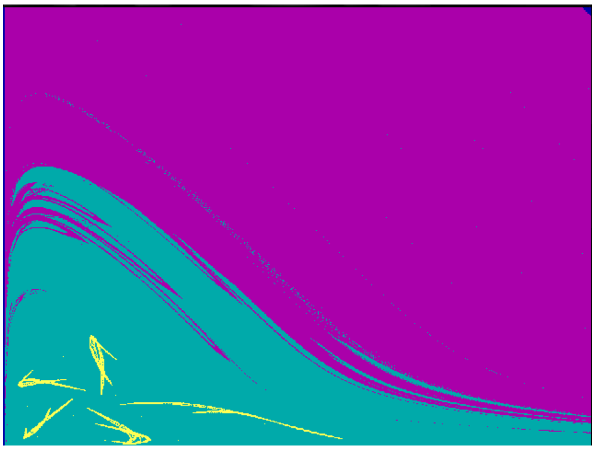
We assume that in the range $1.93 < r < 3.5$, stable coexistence becomes possible for the range where $0.037 < a < 0.053$ as a consequence of successive saddle-node bifurcations followed by a greater number of coexisting regular attractors in the phase space (see Fig. 4(a-c)).

By analyzing the figures, we show how the particular feature is involved in explaining some properties of the dynamical behaviors of the map associated with the basin boundaries and their attracting sets. When $r = 2$ increases, the saddle P_1 located on the frontier (with its two multipliers $S_1 = -1$, and $S_2 < 1$) gives rise to a doubling period cascade of period 2^i cycles on the boundary initially saddles. From this incorporation of more and more sequences of infinitely many instable cycles of increasing order then the route to fractalization of the boundary in the fourth quadrant occurs (see Fig. 4c).

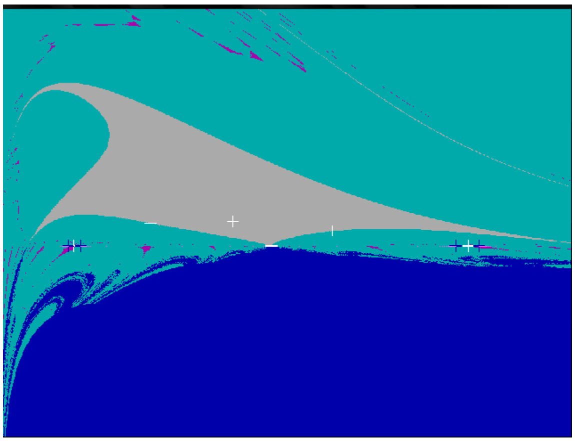
For the case $r = 2.82$, and varying the parameter a, there are two stable attractors, each with an unconnected basin. We observe that for $a = 0.043$ the basins of attraction for the two alternative attractors, the grey and green areas are the basins of attraction for the period-4 chaotic attractor and period-16 cycle respectively (blue crosses) are shown in Fig. 5. The patterns of self-similarity and fractal basin boundaries are also visible in Fig. 6 (Kaitala et al., 1999) revealed this intriguing result (see their work for more details).



(a)



(b)



(c)

Fig. 4 In (a) for $r = 3.5$ and $a = 0.037$, a period-3 cycle can coexist with the closed invariant curve. It should be noted that the boundary of the attraction basin of the invariant closed curve has a fractal structure. In (b) for $r = 2.32$ and $a = 0.053$, a period-5 chaotic attractor exists and for $r = 2.53$ and $a = 0.014$ we have period 2^i unstable cycles, $i = 1,2,...,$ and period-3 unstable cycle on the X-axis in (c).

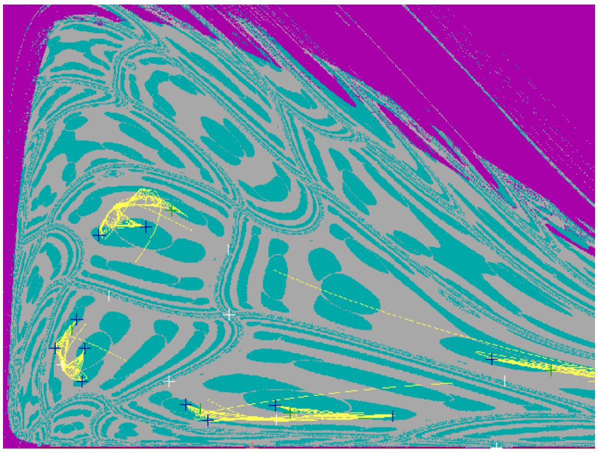


Fig. 5 Fractal Basin for $r = 2.82$, $a = 0.043$.

Basins constitute an interesting object of study by themselves. The strong dependence on parameters generates a rich variety of complex patterns on the plane and gives rise to different mechanisms of basin fractalization as a consequence, such as contact bifurcations between an attractor and its basin boundary.

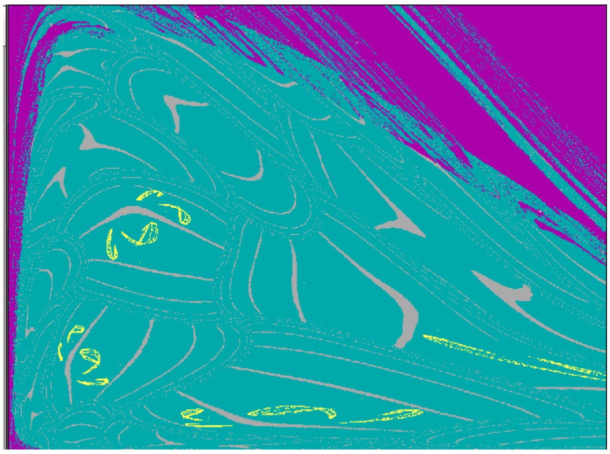


Fig. 6 Fractal basin.

For $r = 1.99$ and $a = 0.0588$, we have two attractors: the first is period-7 closed curves generated from a Neimark-Sacker bifurcation of a period-7 cycle and the second is a chaotic attractor created from a period-6 cycle whose immediate basins are multiply connected (see Mira et al., 1996 for more details). The right and left branches of this unstable manifold of P_1 converge toward the chaotic attractor and are dense inside it. We can see in Fig.7 that the immediate basin of the first attractor (period-7 closed curves) has a sequence of preimages converging toward the unstable focus P_2.

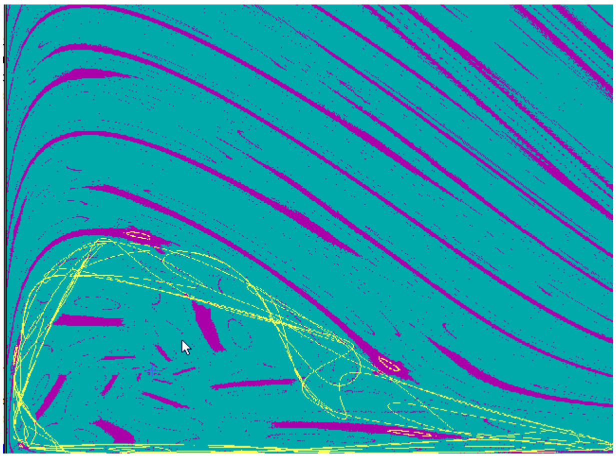


Fig. 7 Representation of basins of two attractors.

4 Conclusion

In ecological research we detect chaos and complex dynamics. As a result, the model considered here has a complex attracting basin structure, mostly multiply connected, producing an unpredictability qualitatively greater than the classical sensitive dependence on initial conditions within a single chaotic attractor. This work underlines the fact this model can exhibit very different dynamics. We clarify the concept of fractality by evaluating and plotting basins computationally.

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