Article

The modeling of predator-prey interactions

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Abstract

In this paper, we aim to study the interactions between the territorial animals like foxes and the rabbits. The territories for the foxes are considered to be the simple cells. The interactions between predator and its prey are represented by the chemical reactions which obey the mass action law. In this sense, we apply the mass action law for predator prey models and the quasi chemical approach is applied for the interactions between the predator and its prey to develop the modeled equations for different possible mechanisms of the predator prey interactions.

Keywords predator-prey interactions; reaction diffusion systems; modeling; fox; rabbit.

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1 Introduction

The Lotka Voltera equations are a pair of first order non linear differential equations, and these are also known as the predator prey equations .i.e. when growth rate of one population is decreased and the other increased then these populations are said to be in a predator-prey situation. The Lotka–Volterra equations are frequently used to describe the dynamics of biological system in which two species interact, one as a predator and the other as a prey (Murray, 2003).

The Lotka Volterra predator prey models were originally introduced by Alfred J. Lotka (Lotka, 1920) in the theory of autocatalytic chemical reactions. In 1920, the model to "organic systems", he made an extension while using plant species and an herbivorous animal species. In 1925, he utilized these equations for the possible analysis of predator-prey interactions and arrived at the equations which are well-known now a day.

In 1926, Vito Volterra (Volterra, 1926), made a statistical analysis of fish catches in the Adriatic independently investigated the equations. V. Voltera applied these equations to predator prey interactions; consist of a pair of first order autonomous ordinary differential equations.

Since that time the Lotka Voltera model has been applied to problems in chemical kinetics, population biology, epidemiology and neural networks. These equations also model the dynamic behavior of an arbitrary number of competitors. The Lotka–Volterra system of equations is an example of a Kolmogorov model, which is a more general framework that can model the dynamics of ecological systems with predator-prey interactions, competition, disease, and mutualism.

Lotka Volterra is the most famous predator prey model. According to Volterra if $x(t)$ is the prey population and $y(t)$ that of the predator at time t then Volterra's model is (Murray, 2003),

$$\frac{dx}{dt} = x(a - by); \qquad (1)$$

$$\frac{dy}{dt} = y(cx - d). \qquad (2)$$

where *a, b, c* and *d* are positive constants.

The reaction diffusion system is a mathematical model which clarifies how the concentrations of one or more substances which are distributed in the space, change under the influence of two processes. In the local chemical reactions, the substances are transformed into each other and the diffusion occur which causes the substances to spread out over a surface in space. Such systems are naturally applied in chemistry, however; the system can also explain dynamical processes of non chemical nature. Mathematically, the reaction diffusion system takes the form of the semi-linear parabolic partial differential equation.

2 Foxes and Rabbits Interaction: A Non-Linear System

In the existence of hundreds of different animals on the island, the modeling of the interaction among these animals is not an easy task. However, the interaction between two species is possible taking one as a predator and the other as a prey. The same single population growth model can be used for predators (foxes), if prey (rabbits) is considered as an unlimited food supply to foxes. But Foxes are territorial animals, and in general each fox claims its own territory by marking their territories with signals that other foxes will recognize e. g., by leaving their droppings in prominent positions but they also pair up in winter season. So it will be difficult for the foxes to cover the whole island without defining their own territories.

In the ecosystem there are various species and every one play a significant role in maintaining populations near the carrying capacity and in keeping the system in balance. In the ecosystems Predator and prey assist them through particular adaptations to compete for food resources. Predator and prey populations are directly related and they cannot survive without each other. Here this relationship is illustrated by using foxes and rabbits.

Between the foxes and rabbits populations here is a complicated and natural predator-prey relationship, since rabbits thrive in the absence of foxes and foxes thrive in the presence of rabbits. Foxes like to occupy a combination of forest and open fields. Foxes usually define their territory zones and use the transition zone or "edge" between these habitats as hunting areas. Foxes do not interfere in the defined areas of each other but they pair up in the winter season.

In case when more rabbits are there, the more foxes will prey on them, so that the number of foxes will increase, when more foxes will predate a lot such that the number of rabbits will start to decrease, swiftly it follows by falling fox numbers as there are a small amount of rabbits to sustain them, therefore it is observed that the populations of rabbits and foxes are locked together in an interactive population 'dance.' This system is non-linear in its behavior. In mathematics non-linear simultaneous equations are used to represent these interactions for which there exist an infinite number of solutions.

The cell-jump models may be considered as the proper diffusion models by themselves, the territorial animal like fox is given a simple cell as its territory. The sense in which the discrete equations for cells converge to the partial differential equations of diffusion is that the cell models give the semi-discrete approximation of the partial differential equations for diffusion which result in a system of ordinary differential equations in cells. There are jumps of concentrations on the boundary of cells. The system of semi discrete models for cells with “no-flux” boundary conditions has all the nice properties of the chemical kinetic equations for closed systems. Under the proper relations between coefficients, like complex balance or detailed balance, this system demonstrated globally stable dynamics. Here we specially refer this modeling by Gorban et al. (Gorban et al., 2011) who developed the idea and modeled diffusion equations for different mechanisms and proved the dissipation inequalities.

Let us suppose that our space is divided on two territories, a system represented as a chain of territories each with homogeneous composition and elementary acts of transfers on the boundary (for us there are only two territories). On each territory we have some concentrations (number of foxes and rabbits) for these processes. Let F^I (fox/ foxes in first territory) and R^I (rabbit/ rabbits in first territory) is the vector concentration in the first territory *I* and F^{II} (fox/ foxes in second territory) and R^{II} (rabbit/ rabbits in second territory) is the vector of concentration in the second territory *II.* Then in our study case the following mechanisms are possible between the predator (foxes) and its prey (rabbits).

3 Mechanism of Circulation

3.1 Foxes case

As the foxes are the territorial animals and they define their own territories to prey for, but they also pair up in winter seasons, so the following mechanism may or may not be possible in real, in study case of foxes, but theoretically this type of mutually inverted and mutually inverse processes can be written as,

$$\begin{aligned} &F^I \to F^{II}, \\ &F^{II} \to F^I. \end{aligned} \tag{3}$$

Let us suppose that c^I and c^{II} be the concentrations of foxes in the first and the second cell respectively, γ_F^I and γ_F^{II} be the stoichiometric vectors in the respective cells, w_r^I and w_r^{II} be the reactions rates in the respective cells and J_F be the total flux.

The mass action law is applicable here for the above mechanisms and we will calculate the stoichiometric vectors, reaction rates, vector of total flux and their diffusion equations as,

For mechanism, $F^I \to F^{II}$,

$$\gamma_{1F}^I = -1,\ w_r^I = kc^I.$$

For second mechanism, $F^{II} \to F^I$,

$$\gamma_{2F}^I = 1,\ w_r^{II} = kc^{II}.$$

Total flux will be calculated as, $J_F = w_r^{II} - w_r^I$,

$$J_F = k\left[c^{II} - c^I\right] = \frac{kl\left[c^{II} - c^I\right]}{l} = D\nabla c. \tag{4}$$

In this approximation, $kl = D$, where l is the cell size.

Then its modeled equation will be, $\frac{\partial c}{\partial t} = -div\left(J_F\right) = -D\Delta c.$ (5)

3.2 Rabbit's case

As the rabbits do not have moving restrictions like the foxes, so this mechanism for the case of rabbits will be written as,

$$\begin{aligned} &R^I \rightarrow R^{II}, \\ &R^{II} \rightarrow R^I. \end{aligned} \tag{6}$$

Now let d^I and d^{II} be the concentrations of rabbits in the first and the second cell respectively, γ_R^I and γ_R^{II} be the stoichiometric vectors in the respective cells, w_r^I and w_r^{II} are the reactions rates in the respective cells and J_R be the total flux. Then,

For mechanism: $R^I \rightarrow R^{II}$,

$$\gamma_{1R}^I = -1,\ w_r^I = kd^I.$$

For mechanism: $R^{II} \rightarrow R^I$,

$$\gamma_{2R}^I = 1,\ w_r^{II} = kd^{II}.$$

The flux in this case will be,

$$J_R = \frac{kl\left[d^{II} - d^I\right]}{l} = D\nabla d. \tag{7}$$

And modeled equation for this case will be,

$$\frac{\partial d}{\partial t} = -div\left(J_R\right) = -D\Delta d \tag{8}$$

4 Mechanism of Sharing Place

4.1 Foxes case

The mechanism of sharing place with interaction of *n* different foxes i.e. when a fox F_i^I is in territory one and another fox F_j^{II} is in territory two and then fox F_i^I moves from territory one to territory two and vise versa. As the foxes are the territorial animals and they define their own territories to prey but they also pair up in winter seasons, so the above stated mechanism will have both the possibilities, that it may or may not be happens in real but in a multi component system it is given by stoichiometric equations of the form:

$$\begin{aligned} &F_i^I + F_j^{II} \to F_j^I + F_i^{II}, \\ &F_i^{II} + F_j^I \to F_i^I + F_j^{II}. \end{aligned} \tag{9}$$

For mechanism: $F_i^I + F_j^{II} \to F_j^I + F_i^{II}$,

$$\gamma_{F_i}^I = -1,\ \gamma_{F_j}^I = 1,\ w_r^I = kc_i^I c_j^{II}.$$

For mechanism: $F_i^{II} + F_j^I \to F_i^I + F_j^{II}$.

$$\gamma_{F_i}^I = 1,\ \gamma_{F_j}^I = -1,\ \ w_r^{II} = kc_i^{II} c_j^I.$$

Total flux will be calculated as,

$$J_F = k\left[c_i^{II} c_j^I - c_i^I c_j^{II}\right],\ J_F = k\left[\left(c_i^I + l\nabla c_i\right)c_j^I - c_i^I\left(c_j^I + l\nabla c_j\right)\right] = kl\left[c_j^I \nabla c_i - c_i^I \nabla c_j\right],$$

$$J_F = D\left[c_j^I \nabla c_i - c_i^I \nabla c_j\right]. \tag{10}$$

The modeled equation in this case will be,

$$\frac{\partial c}{\partial t} = -D\left[c_j^I \Delta c_i - c_i^I \Delta c_j\right]. \tag{11}$$

4.2 Rabbit's case

In the case study of rabbits the above stated mechanism may or may not be possible because rabbit's movement is naturally free without any restrictions. In this case the possible mechanisms will be as under,

$$R_i^I + R_j^{II} \to R_j^I + R_i^{II},$$
$$R_i^{II} + R_j^I \to R_i^I + R_j^{II}. \tag{12}$$

For the mechanism: $R_i^I + R_j^{II} \to R_j^I + R_i^{II}$,

$$\gamma_{R_i}^I = -1, \ \gamma_{R_j}^I = 1, \ w_r^I = kd_i^I d_j^{II}.$$

For the mechanism: $R_i^{II} + R_j^I \to R_i^I + R_j^{II}$,

$$\gamma_{R_i}^I = 1, \ \gamma_{R_j}^I = -1, \ \ w_r^{II} = kd_i^{II} d_j^I.$$

$$J_R = D\left[d_j^I \nabla d_i - d_i^I \nabla d_j\right]. \tag{13}$$

Its modeled equation will be,

$$\frac{\partial d}{\partial t} = -D\left[d_j^I \Delta d_i - d_i^I \Delta d_j\right]. \tag{14}$$

4.3 Mixed situation

Now in the case of mix situations i.e. by taking foxes and rabbits case together. When a fox is in territory one and a rabbit is in territory two, as foxes are territorial naturally but rabbits movement is free, so if a rabbit that moves from territory two to territory one (or from territory one to territory two), the fox in that area will prey that rabbit (territory change is possible only between (among) the foxes but with respect to any other animal (here rabbits) change of territory for foxes will not be possible). In this case the possible mechanisms will be,

$$F_i^I + R_j^{II} \to F_i^I + R_j^I \to F_i^I,$$
$$F_i^{II} + R_j^I \to F_i^{II} + R_j^{II} \to F_i^{II}. \tag{15}$$

In mechanism: $F_i^I + R_j^{II} \to F_i^I + R_j^I \to F_i^I$, If we take the first possibility. i.e. $F_i^I + R_j^{II} \to F_i^I + R_j^I$,

Then $\gamma_F^I = 0$, $w_r^I = kc^I d^{II}$, $\gamma_R^I = 1$, $w_r^{II} = kc^{II} d^I$..

Equation of flux will be,

$$J = k\left[c^{II} d^I - c^I d^{II}\right] = k\left[\left(c^I + l\nabla c\right)d^I - c^I\left(d^I + l\nabla d\right)\right], \tag{16}$$

Modeled equation for foxes in this case will be, $\frac{\partial c}{\partial t} = -D\left[d^I \Delta c - c^I \Delta d\right].$ (17)

Now if we take the second possibility i.e. $F_i^I + R_j^I \to F_i^I$,

Then, $\gamma_F^I = 0$, $w_r^I = kc^I d^I$, $\gamma_R^I = -1$, $w_r^{II} = kc^{II} d^{II}$.

Equation of flux will be, $J = k\left[c^{II} d^{II} - c^I d^I\right] = k\left[\left(c^I + l\nabla c\right)\left(d^I + l\nabla d\right) - c^I d^I\right]$,

$$J = kl\left[c^I \nabla d + d^I \nabla c + l\nabla c \nabla d\right] = D\left[c^I \nabla d + d^I \nabla c + l\nabla c \nabla d\right]. \tag{18}$$

Modeled equation for foxes in this case will be,

$$\frac{\partial c}{\partial t} = -D\left[c^I \nabla d + d^I \nabla c + l\nabla c \nabla d\right]. \tag{19}$$

5 Mechanism of Attraction

5.1 Foxes case

Foxes are solitary and necessitate quite huge hunting areas. A fox constantly patrols its territory looking for food, using its urine to mark places it has completed searching. Foxes are territorial and fight other foxes that they find on their territory but they also pair up in winter seasons so the mechanism of attraction in the case study of foxes will take place only in winter seasons, so in a multi component system it is given by stoichiometric equations of the form:

Attraction:

$$\begin{aligned} &2F^I + F^{II} \to 3F^I, \\ &F^I + 2F^{II} \to 3F^{II}. \end{aligned} \tag{20}$$

For mechanism: $2F^{I} + F^{II} \rightarrow 3F^{I}$,

$$\gamma_F^I = 1,\ w_r^I = k\left(c^I\right)^2 c^{II}.$$

For mechanism: $F^{I} + 2F^{II} \rightarrow 3F^{II}$,

$$\gamma_F^I = -1,\ w_r^{II} = kc^I\left(c^{II}\right)^2.$$

Flux from cell one to cell two will be,

$$J_F = kc^I c^{II}\left(c^{II} - c^I\right) = klc^I c^{II}\frac{\left(c^{II} - c^I\right)}{l} = klc^I\left(c^I + l\nabla c\right)\nabla c,$$

To the first order we have that,

$$J_F = klc^2\nabla c = \frac{1}{3}kl\left(3c^2\nabla c\right) = \frac{1}{3}kl\nabla c^3. \tag{21}$$

And its modeled equation will be,

$$\frac{\partial c}{\partial t} = -kldiv\left(c^2\nabla c\right) = \frac{\partial c}{\partial t} = -kl\frac{1}{3}\Delta c^3. \tag{22}$$

5.2 Rabbit's case

In the study case of rabbits the mechanisms of attraction may or may not be possible in real because rabbits move freely in nature without any restrictions. If two rabbits are there in territory one and one rabbit is in territory two and vise versa, if they all move in the same territory then these are mechanisms of attraction, so in a multi component system it is given by stoichiometric equations of the form:

Attraction:

$$\begin{aligned} &2R^I + R^{II} \rightarrow 3R^I, \\ &R^I + 2R^{II} \rightarrow 3R^{II}. \end{aligned} \tag{23}$$

For mechanism: $2R^I + R^{II} \rightarrow 3R^I$,

$$\gamma_R^I = 1,\ w_r^I = k\left(d^I\right)^2 d^{II}.$$

For mechanism: $R^I + 2R^{II} \rightarrow 3R^{II}$,

$$\gamma_R^I = -1,\ w_r^{II} = kd^I\left(d^{II}\right)^2.$$

Equation of flux to the first order will be,

$$J_R = kld^2\nabla d = \frac{1}{3}kl\left(3d^2\nabla d\right) = \frac{1}{3}kl\nabla d^3. \tag{24}$$

And modeled equation in this case will be,

$$\frac{\partial d}{\partial t} = -kl\frac{1}{3}\Delta d^{3}.$$

(25)

6 Mechanism of Repulsion

6.1 Foxes case

As it is stated in the early lines that foxes are solitary and necessitate quite huge hunting areas. Foxes are territorial and fight other foxes that they find on their territory but they also pair up in the winter seasons, so mechanism of attraction in the study case of foxes will take place only in winter seasons while in other months there will be only repulsion so in a multi component system it is given by stoichiometric equations of the form:

Repulsion:

$$3F^{I} \to 2F^{I} + F^{II},$$
$$3F^{II} \to F^{I} + 2F^{II}.$$

(26)

For mechanism: $3F^{I} \to 2F^{I} + F^{II}$,

$$\gamma_F^I = -1,\ w_r^I = k\left(c^I\right)^3.$$

For mechanism: $3F^{II} \to F^{I} + 2F^{II}$,

$$\gamma_F^I = 1,\ w_r^{II} = k\left(c^{II}\right)^3.$$

Equation of flux for this mechanism will be calculated as,

$$J_F = k\left[\left(c^{II}\right)^3 - \left(c^I\right)^3\right],$$

$$J_F = k\left[\left(c^{II} - c^I\right)\left(c^I + l\nabla c\right)^2 + \left(c^I\right)^2\left(c^I + l\nabla c\right)c^I\right],$$

$$J_F = k\left[\left(c^{II} - c^I\right)\left(3\left(c^I\right)^2 + l^2\left(\nabla c\right)^2 + 2l\nabla c + lc^I\nabla c\right)\right],$$

$$J_F = k\left[\left(c^{II} - c^I\right)\left(3\left(c^I\right)^2 + l\nabla c\left(l\nabla c + c^I + 2\right)\right)\right],$$

$$J_F = kl\left[\left(\frac{c^{II} - c^I}{l}\right)\left(3\left(c^I\right)^2 + l\nabla c\left(l\nabla c + c^I + 2\right)\right)\right],$$

$$J_F = kl\left[\left(\nabla c\right)\left(3\left(c^I\right)^2 + l\nabla c\left(l\nabla c + c^I + 2\right)\right)\right],$$

So to the first order we will have,

$$J_F = kl\nabla c.3\left(c^I\right)^2 = kl.c^2\nabla c = kl\nabla c^3. \qquad (27)$$

And its modeled equation will be,

$$\frac{\partial c}{\partial t} = -kldiv\left(c^2 \nabla c\right) = -kl\Delta c^3. \tag{28}$$

6.2 Rabbit's case

In this study case of rabbits the above stated mechanism of repulsion may or may not be possible because rabbits move freely in nature without any restrictions. If three rabbits are there in the same territory and one of them changes his territory then this will be a case of repulsion so in a multi component system it is given by stoichiometric equations of the form:

Repulsion:

$$\begin{aligned} &3R^I \to 2R^I + R^{II}, \\ &3R^{II} \to R^I + 2R^{II}. \end{aligned} \tag{29}$$

For mechanism: $3R^I \to 2R^I + R^{II}$,

$$\gamma_R^I = -1,\ w_r^I = k\left(d^I\right)^3.$$

For mechanism: $3R^{II} \to R^I + 2R^{II}$,

$$\gamma_R^I = 1,\ w_r^{II} = k\left(d^{II}\right)^3.$$

Vector of total flux will be,

$$J_R = kl\nabla d.3\left(d^I\right)^2 = kl.d^2\nabla d = kl\nabla d^3. \tag{30}$$

And its modeled equation will be,

$$\frac{\partial d}{\partial t} = -kldiv\left(d^2 \nabla d\right) = -kl\Delta d^3. \tag{31}$$

7 The Observations

The sense in which the discrete equations for cells converge to the partial differential equations is that the cell models give the semi-discrete approximation of the partial differential equations. They result in a system of ordinary differential equations in cells. Such approximation appears often in finite element methods and cells are discontinuous finite elements. There are jumps of concentrations on the boundary of cells. The Taylor expansion of the right hand sides of the discrete system of ordinary differential equations for cells produces the second order in the cell size approximation to the continuous diffusion equation (the standard result for the central differences).

Significant difference from the classical finite elements is in construction of right hand sides of the ordinary differential equations for concentrations in cells: there are flows in both directions: from cell 1 to cell II and from cell II to cell I. These flows have a simple mass action law construction and the resulting diffusion flow is the difference between them. The kinetic constants should be scaled with the cell size to keep $kl = D$ (k is the kinetic constant, l is the cell size and D is the diffusion coefficient for the particular mechanism).

This is the approximation of the right hand sides. The approximation of solutions is a more difficult problem and depends upon the properties of solutions of PDEs. It seems a good hypothesis that for obtained diffusion systems with convex Lyabunov functional and "no-flux" boundary conditions in bounded areas with smooth boundaries the cell model gives the uniform approximation to the solution of the correspondent PDE.

The system of semi-discrete models for cells with "no-flux" boundary conditions has all the nice properties of the chemical kinetic equations for closed systems. Under the proper relations between coefficients, like complex balance or detailed balance, this system demonstrated globally stable dynamics. This global stability property can help with the study of the related PDE.

Finally, the cell-jump models may be considered as the proper diffusion models by themselves, for the finite physically reasonable cell size, without limit. This size may be quite large for the coarse-grained models (it depends on the medium microstructure and on the smoothness of the concentration fields).

8 Summary

Our goals for the present study are:

- ❖ To build up a brief study of complex biological systems by taking a study case of foxes and rabbits.
- ❖ To tackle key research questions about our study case by proposing new techniques and algorithms that are inspired by those complex biological systems.
- ❖ Further in our study case we want to study and aim to extend these ideas to all other possible mechanisms complete in all aspects between foxes and rabbits. We also aim to check the stability of these interactions along with finding the solutions of the above determined modeled equations.

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