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Dynamics of fractional order modified Morris-Lecar neural model

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Abstract
Most of the beautiful biological functions in neural systems are expected to happen considering the system with memory effect. Fractional differential equations are very useful to investigate long-range interacting systems or systems with memory effect. In this paper, a fractional order nonlinear three dimensional modified Morris-Lecar neural system (M-L system) has been studied. The fractional order M-L system is a generalization of the integer order M-L system. The paper presents an approximate analytical solution of the fractional order M-L system, using Homotopy Perturbation Method (HPM) and Variational Iteration Method (VIM). The fractional derivatives are described in the Caputo sense. We have used the above methods as they show very efficient result for very small time region. Solutions are obtained in the form of rapidly convergent infinite series and only a few iterations are needed to obtain the approximate solutions. Comparison of both HPM and VIM reveals that the two present methods of solution are elegant and powerful for solving the nonlinear fractional order biological as well as neural systems.

Keywords Morris-Lecar neural model; fractional order; homotopy perturbation method; variational iteration method.

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1 Introduction

Human brain is the most complicated part of the body that can display behavior which is periodic or chaotic of varying dimensionality (Destexhe, 1994; Freeman, 1994). Neurons are basic function units in nervous systems. Billions of neurons are connected in a complex network. They are the key elements in signal processing. Neurons respond to stimuli by generating sequences of brief electrical pulses, referred to as action potentials (Tateno and Pakdaman, 2004). The form of action potentials varies little, so that information concerning the stimulus cannot be readily conveyed by their shape. Conversely, the timing of these electrical discharges is stimulus dependent (Tateno and Pakdaman, 2004). The information is encoded, transmitted and decoded through firing activity of neurons, which is characterized by relaxation oscillation process of producing and transmitting action potential (Alexander et al., 2011). Bifurcation mechanisms involved in the generation of action potentials by neurons were reviewed by Izhikevich (2000).

Many neuron models have been proposed to understand a real nervous system that exhibits complicated dynamics. The following are some of the popular models: (i) Hodgkin-Huxley (H-H) model (Hodgkin and Huxley, 1952), (ii) FitzHugh-Nagumo (F-H-N) model (FitzHugh, 1955; Aqil et al., 2012), (iii) Wilson-Cowan (W-C) model (Wilson and Cowan, 1972) and (iv) Hindmarsh-Rose (H-R) model (Hindmarsh and Rose, 1984). The Morris-Lecar (M-L) model (1981) belongs to the family of conductance based membrane models of which H-H model is a well-known model. It represents an electrical circuit equivalent to a cellular membrane crossed by three different transmembraner currents referred to the voltage gated Ca^{+2} current, delayed rectifier K^{+} current and the leakage current (Morris and Lecar, 1981). Experiments on Barnacle muscle fiber with extra current show that such neuron model could produce much complex oscillation behavior (Gutkin and Ermentrout, 1998; Xie et al., 2004; Hodgkin and Rushton, 1946), but the generation mechanism of such behaviors is different from H-H model. The M-L system supports only the two types of dynamics, resting and sustained spiking activity and does not exhibit chaotic oscillations. Larter et al. (1999) developed a coupled ODE lattice model for the CA3 region of the Hippocampus for the simulation of epileptic seizures. The model consists of a hexagonal lattice of nodes, each describing a subnetwork consisting of a group of prototypical excitatory pyramidal cells with membrane potential V and a group of prototypical inhibitory interneurons with membrane potential Z interconnected via on/off excitatory and inhibitory synapses and both feed by current from the excitatory pathway (see figure 1 of Larter et al., 1999). These two types of neurons are included in the subnetwork which forms the basis of the model. The growth of mean membrane potential of excitatory principal cells is controlled by that of the inhibitory interneurons. The dynamical behavior of this subnetwork was described by a system of three differential equations based on a two variable reduction of the H-H model. To model the behavior of the entire subnetwork Larter et al. (1999) have added an equation to the original M-L model system to simulate the effect of a population of inhibitory interneurons synapsing on pyramidal cells. Rai et al. (2013) reported the existence of phase coupled oscillations in electrical activity of the neuronal cells that carry together amplitude, phase and time information for cellular signaling in the model developed by Larter et al. (1999). Nadar and Rai (2012) numerically simulated this model and commented that chaos is vital for functioning of a healthy brain and synchronization of the neural system occurs when all its regions are in transient periodicity represented by chaotic saddles in state space. This is how the intermittent pathology of epileptic seizure is created.

In this paper, we have modified the model developed by Larter et al. (1999) and studied by Nadar and Rai (2012) and Rai et al. (2013). We have considered a three dimensional fractional order nonlinear modified M-L neuron model. It is a generalization of the integer order modified M-L model with fast – slow variables, where the fractional order derivative is used for the memory effect and power law of neural cell membranes. The V-W subsystem has faster time scale than that of variable Z. Interesting behavior of the model system resulted from an interaction between the faster subsystem and a slower one with the population of inhibitory interneurons. The excitatory neurons carry information flow (long range projections) and inhibitory neurons are responsible for regulating the activation of excitatory neurons. Recently, Shi and Wang (2014) proposed a fractional order M-L neuron model and different types of bursting patterns were investigated using the bifurcation theory of fast-slow dynamical systems. It helps us in understanding the neuronal activities, efficient information processing and stimulus anticipation as well as in frequency independent phase shifts of oscillatory neuronal firing. Lundstrom et al. (2008) found that the dynamics, underlined spike rate adaptation to stimulus steps in single cortical neurons, functionally approximated fractional differentiation. This provides a general model for the firing- rate response to time-varying stimulus statistics or envelope coding. Fractional order dynamics have been observed in the vestibular-ocular systems (Paulin et al., 2004; Anastasio, 1994) and in the fly motion sensitive neuron H1 (Fairhall et al., 2001). A range of mechanisms which contribute to fractional order dynamics and power law includes synaptic mechanisms (Fusi et al., 2005), geometrical properties of cells (Thorson and Biederman-Thorson, 1974), circuits (Anastasio, 1998) and dendrites (Anastasio, 1994).

In recent years, fractional order differential equations have been used to solve physical, biological and neural systems. Since there are no methods available to obtain exact solution of fractional order differential equations, approximate analytical techniques are used to solve these equations. Several analytical and numerical methods have been proposed to solve fractional order differential equations. The most commonly used are Adomian Decomposition Method (ADM) (Adomian, 1998), Homotopy perturbation method (HPM) (He, 1999), Variational iteration method (VIM) (He, 1999a), Fractional Difference Method (FDM) (Podlubny, 1999) etc. To find an approximate analytical solution of the three dimensional nonlinear fractional order modified M-L neural system, we have applied the Homotopy perturbation method (HPM) (He, 2000; He, 2005; He, 2005a; Momani and Odibat, 2007; Abdulaziz et al., 2008; Chakraverty and Tapaswini, 2013) and Variational iteration method (VIM) (Momani and Odibat, 2007; Odibat and Momani, 2008). HPM is an approach for finding the approximate solution of linear or nonlinear differential and integral equations and for fractional order differential equations (Abdulaziz et al., 2008; Chakraverty and Tapaswini, 2013). This method was first proposed by He (1999) and was successfully applied to solve nonlinear wave equations (He, 2005; He, 2005a). The method which is a coupling of the traditional perturbation method and homotopy in topology deforms continuously to simple problems which can easily be solved. The method does not require small parameters in the equation which overcomes the limitations of the traditional perturbation method. The VIM, which was first applied by He (1999), is relatively a new approach to provide analytical approximate solutions to nonlinear problems and it provides visible symbolic terms of analytical solutions to fractional differential equations. The principles of VIM and its applicability for various kinds of differential equations are given by Momani and Odibat (2007) and Odibat and Momani (2008).

The organization of the paper is as follows: Section 2 describes the details about the modified Morris-Lecar neural model and the proposed fractional order modified M-L model based on its integer order model. Section 3 presents the analytical solutions using HPM and VIM. Numerical results are presented in Section 4. Finally, in Section 5, discussions and conclusions are presented.

2 The Modified Morris- Lecar Neuron Model

Consider a population of neurons as a dynamical system (Speelman, 1997) in which the neurons interact with each other. The differential equations that represent the dynamical system were derived by considering the behavior of prototypical single neurons (M-L neurons). The interactions between the populations of excitatory cells and inhibitory interneurons are governed by the system of nonlinear differential equations (Larter et al., 1999; Rai et al., 2013; Nadar and Rai, 2012)

$$\frac{dV}{dt} = -g_{Ca} m_\infty(V)(V-1) - g_K W(V - V^K) - g_L(V - V^L) + I - \alpha_{inh}(Z)Z, \tag{1a}$$

$$\frac{dW}{dt} = \phi(W_\infty - W)/\tau_W, \tag{1b}$$

$$\frac{dZ}{dt} = b'(c' I + \alpha_{exc}(V)V), \tag{1c}$$

where $\tau_W = 1/\cosh((V - V_3)/2V_4)$, $m_\infty(V) = 0.5(1 + \tanh((V - V_1)/V_2))$, $W_\infty(V) = 0.5(1 + \tanh((V - V_3)/V_4))$,

$$\alpha_{exc}(V) = \alpha_{exc}(1 + \tanh((V - V_5)/V_6)),\ \alpha_{inh}(Z) = \alpha_{inh}(1 + \tanh((Z - V_7)/V_6)). \tag{2}$$

The variables V and Z are mean membrane potentials for excitatory and inhibitory cells respectively. The variable W is the fraction of open potassium channels at any point of time. The detailed meanings of the parameters, their values and other details are given by Larter et al. (1999).

Fractional order Modified Morris-Lecar neural model

Resistor-capacitor circuit theory is applied to the modified M-L system (1). Since the total volume of all ionic currents Ca^{+2}, K^{+} and leaky currents varies, the mean membrane potential of excitatory neuron V changes reflecting the firing activity of the neuron. It is governed by the equation (Larter et al., 1999)

$$C\frac{dV}{dt} = I_{ion} + I_{ext} - \alpha_{inh}(Z)Z = I_{Ca} + I_K + I_L + I - \alpha_{inh}(Z)Z, \tag{3}$$

where $I_{Ca} = -g_{Ca} m_\infty(V)(V-1), I_K = -g_K W(V - V^K), I_L = -g_L(V - V^L),$ I and α_{inh} denote the applied current and dimensionless synaptic strength of inhibitory interneurons respectively. From electrical circuit theory, we have

$$I = C\, dV/dt, \tag{4}$$

where the capacitance C is a constant, $I = dQ/dt$ and $Q = CV$.

Normal capacitive behavior of real dielectrics and insulators exhibit the fractional differentiation relationship (Westerlund and Ekstam, 1994)

$$I = C\, d^\alpha V/dt^\alpha, \quad 0 < \alpha < 1. \tag{5}$$

Assume that a step voltage $V_0 u(t)$ is applied at $t = 0$. Using Laplace transforms, solutions of (4) and (5) under this initial conditions are given by (Westerlund and Ekstam, 1994; Magin, 2004)

$$I(t) = CV_0 \delta(t), \tag{6a}$$

$$I(t) = \frac{CV_0}{\Gamma(1-\alpha)} t^{-\alpha}, \quad 0 < \alpha < 1, \tag{6b}$$

where $\delta(t)$ is the Dirac delta function. From (6b), observe that the current follows a power law and decays at a rate of $t^{-\alpha}$ with time. In the limiting case as $\alpha \to 1$, (6b) is consistent with (6a) as $1/\Gamma(1-\alpha) \to 0$ as $\alpha \to 1$.

From (4) and (6a), we get $V' = V_0 \delta(t)$. According to the definition of Caputo's fractional order derivative (Podlubny, 1999)

$$\frac{d^\alpha V}{dt^\alpha} = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} V'(\tau) d\tau,$$

$$= \frac{V_0}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} \delta(\tau) d\tau = \frac{V_0}{\Gamma(1-\alpha)} t^{-\alpha}.$$

Now, $I = C\dfrac{d^\alpha V}{dt^\alpha} = \dfrac{CV_0}{\Gamma(1-\alpha)} t^{-\alpha}.$

Hence, it is reasonable to use a fractional order capacitor theory with order $0 < \alpha < 1$ in modeling neural dynamics. For computations, we have assumed $C = 1$ and $V(0) = 0.01, W(0) = 0.1, Z(0) = 0.01$ at $t = 0$ (Larter et al., 1999; Nadar and Rai, 2012; Rai et al., 2013) for HPM and VIM solution. As a result, a fractional order modified M-L system with order $0 < \alpha \le 1$, in the sense of Caputo derivative can be proposed as

$$D_t^{\alpha} V(t) = -g_{Ca} m_{\infty}(V)(V-1) - g_K W(V - V^K) - g_L(V - V^L) + I - \alpha_{inh}(Z)Z, \tag{7a}$$

$$D_t^{\beta} W(t) = \phi(W_{\infty} - W)/\tau_W, \tag{7b}$$

$$D_t^{\gamma} Z(t) = b'(c' I + \alpha_{exc}(V)V), \tag{7c}$$

where $D_t^{\alpha}, D_t^{\beta}$ and D_t^{γ} are Caputo derivatives of orders $0 < \alpha, \beta, \gamma \le 1$. These fractional equations are obtained from the classical equations by replacing the first order time derivatives by fractional derivatives of orders $\alpha, \beta, \gamma,\ 0 < \alpha \le 1, 0 < \beta \le 1, 0 < \gamma \le 1$. The advantage of fractional order systems is that they allow greater degrees of freedom in the model.

The following definitions and properties of the fractional calculus (Podlubny, 1999; Miller and Ross, 1993) are used in the derivation.

Definition 1 A real function $f(t), t > 0$, is said to be in the space $C_{\mu}, \mu \in R$, if there exists a real number $p > \mu$, such that $f(t) = t^p f_1(t)$, where $f_1(t) \in C(0, \infty)$ and it is said to be in the space C_{μ}^n if and only if $f^n \in C_{\mu}, n \in N \cdot$

Definition 2 The Riemann-Liouville fractional integral operator J^{α} of order $\alpha \ge 0$, of a function $f \in C_{\mu}, \mu \ge -1$, is defined as

$$J^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau, \quad (\alpha > 0), \qquad J^0 f(t) = f(t).$$

Some of the properties of the operator J^{α} which are used are the following:

(i) $J^{\alpha} J^{\beta} f(t) = J^{\alpha+\beta} f(t)$, (ii) $J^{\alpha} J^{\beta} f(t) = J^{\beta} J^{\alpha} f(t)$, (iii) $J^{\alpha} t^{\gamma} = \dfrac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} t^{\alpha+\gamma}$.

Definition 3 Fractional derivative $D^{\alpha} f(t) = d^{\alpha} f(t)/dt^{\alpha}$ in the Caputo sense is defined as

$$D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} f^n(\tau) d\tau, \text{for}\, n-1 < \alpha \le n,\ \ n \in N,\ t > 0,\ f \in C_{-1}^n.$$

3 Analytical Solution of the Fractional Order Modified M-L neural System

3.1 HPM solution for fractional order modified Morris-Lecar equations

Applying HPM, we construct the following homotopy of the model system (7) as

$$D_t^{\alpha} V(t) = p\{-g_{Ca} m_{\infty}(V)(V-1) - g_K W(V - V^K) - g_L(V - V^L) + I - \alpha_{inh}(Z)Z\}, \tag{8a}$$

$$D_t^{\beta} W(t) = p\{\phi(W_{\infty} - W)/\tau_W\}, \tag{8b}$$

$$D_t^{\gamma} Z(t) = p\{b'(c' I + \alpha_{exc}(V)V)\}, \tag{8c}$$

where $p, 0 \le p \le 1$ is the homotopy parameter.

$$D_t^{\gamma} Z(t) = p\{b'(c' I + \alpha_{exc}(V)V)\}, \tag{8c}$$

where $p, 0 \le p \le 1$ is the homotopy parameter.

Substituting the values of $m_\infty(V)$, $\alpha_{inh}(Z)$, $W_\infty(V)$, τ_W and $\alpha_{exc}(V)$ in (8a) to (8c), we obtain

$$D_t^{\alpha} V(t) = p\{-0.5 g_{Ca}[1+\tanh((V-V_1)/V_2)](V-1) - g_K W(V-V^K) - g_L(V-V^L) + I \\ -\alpha_{inh}[1+\tanh((Z-V_7)/V_6)]Z\}, \tag{9a}$$

$$D_t^{\beta} W(t) = p\{\phi(0.5[1+\tanh((V-V_3)/V_4)] - W)\cosh((V-V_3)/2V_4)\}, \tag{9b}$$

$$D_t^{\gamma} Z(t) = p\{b'(c' I + \alpha_{exc}[1+\tanh((V-V_5)/V_6)]V)\}. \tag{9c}$$

Linearizing (9a)-(9c) (take Maclaurin's series of hyperbolic functions and ignore second and higher order terms), we obtain

$$D_t^{\alpha} V(t) = p\{-0.5 g_{Ca}[1+((V-V_1)/V_2)](V-1) - g_K W(V-V^K) - g_L(V-V^L) + I \\ -\alpha_{inh}[1+((Z-V_7)/V_6)]Z\}, \tag{10a}$$

$$D_t^{\beta} W(t) = p\{\phi(0.5[1+((V-V_3)/V_4)] - W)\}, \tag{10b}$$

$$D_t^{\gamma} Z(t) = p\{b'(c' I + \alpha_{exc}[1+((V-V_5)/V_6)]V)\}, \tag{10c}$$

where

$$V(t) = V_0' + pV_1' + p^2 V_2' + p^3 V_3' + \cdots, \tag{11a}$$

$$W(t) = W_0' + pW_1' + p^2 W_2' + p^3 W_3' + \cdots, \tag{11b}$$

$$Z(t) = Z_0' + pZ_1' + p^2 Z_2' + p^3 Z_3' + \cdots. \tag{11c}$$

Insert the expressions from (11a)-(11c) into (10a)-(10c) and use the set of parameters values (Nadar and Rai, 2012) and the initial conditions as

$$V_1 = -0.01, V_2 = 0.15, V_3 = 0.03, V_4 = 0.3, V_5 = 0.0, V_6 = 0.4, V_7 = 0.05, g_{Ca} = 1.1, g_K = 2.0, g_L = 1.0, V^K = -0.7,$$
$$V^L = -0.5, I = 0.3, b' = 0.15, \alpha_{exc} = 1, \alpha_{inh} = 1, \phi = 0.4, c' = 0.238 \text{ and } V(0) = 0.01, W(0) = 0.1 \text{ and } Z(0) = 0.01.$$

Taking $V_0' = a = 0.01, W_0' = b = 0.1, Z_0' = c = 0.01$, we obtain the following set of differential equations for

$V_0', W_0', Z_0', \ldots$

$$p^0 : D_t^{\alpha} V_0' = 0, \tag{12a}$$

$$D_t^{\beta} W_0' = 0, \tag{12b}$$

$$D_t^{\gamma} Z_0' = 0. \tag{12c}$$

$$p^1: D_t^{\alpha} V_1' = 2.08a - 3.6685a^2 + 0.3869 - 2ab - 1.4b - 0.875c - 2.5c^2, \tag{13a}$$

$$D_t^{\beta} W_1' = 0.18 + 0.666a - 0.4b, \tag{13b}$$

$$D_t^{\gamma} Z_1' = 0.0107 + 0.15a + 0.375a^2. \tag{13c}$$

$$p^2: D_t^{\alpha} V_2' = V_1'(2.08165 - 7.3315a - 2b) - 2W_1'(a + 0.7) - Z_1'(0.875 + 5c), \tag{14a}$$

$$D_t^{\beta} W_2' = 0.67V_1' - 0.4W_1', \tag{14b}$$

$$D_t^{\gamma} Z_2' = 0.15V_1'(1 + 5a). \tag{14c}$$

$$p^3: D_t^{\alpha} V_3' = V_2'(-7.337a + 2.08165 - 2b) - 3.6685V_1'^2 - 2(a + 0.7)W_2' - 2V_1'W_1' - Z_2'(5c + 0.875) - 2.5Z_1'^2, \tag{15a}$$

$$D_t^{\beta} W_3' = 0.67V_2' - 0.4W_2', \tag{15b}$$

$$D_t^{\gamma} Z_3' = 0.15(1 + 5a)V_2' + 0.375V_1'^2, \tag{15c}$$

and so on. The method is based on applying the operators $J_t^{\alpha}, J_t^{\beta}, J_t^{\gamma}$ respectively (the inverse operators of the Caputo derivatives $D_t^{\alpha}, D_t^{\beta}, D_t^{\gamma}$ respectively) on both sides of (12a) to (15c).

Finally, we obtain the approximate solutions for $V(t)$, $W(t)$ and $Z(t)$ as

$$V(t) = \lim_{N\to\infty} \delta_N(t),\ W(t) = \lim_{N\to\infty} \lambda_N(t),\ Z(t) = \lim_{N\to\infty} \psi_N(t),$$

where $\delta_N(t) = \sum_{n=0}^{N-1} V_n'(t),\ \lambda_N(t) = \sum_{n=0}^{N-1} W_n'(t),\ \psi_N(t) = \sum_{n=0}^{N-1} Z_n'(t),\ N \geq 1.$

The above three series converge very rapidly. The rapid convergence means that only few terms are required to get the approximate solutions. Solving (12a) to (15c), we obtain

$$V_0' = 0.01,\ W_0' = 0.1,\ Z_0' = 0.01,$$

$$V_1' = P(a,b,c)\frac{t^{\alpha}}{\Gamma(\alpha+1)},\ W_1' = P(a,b)\frac{t^{\beta}}{\Gamma(\beta+1)},\ Z_1' = P(a)\frac{t^{\gamma}}{\Gamma(\gamma+1)},$$

$$V_2' = P(a,b,c)P_1(a,\ b)\ \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - 2P(a,b)(a+0.7)\frac{t^{(\alpha+\beta)}}{\Gamma(\alpha+\beta+1)} - P(a)P(c)\frac{t^{(\alpha+\gamma)}}{\Gamma(\alpha+\gamma+1)},$$

$$W_2' = 0.67P(a,b,c)\frac{t^{(\alpha+\beta)}}{\Gamma(\alpha+\beta+1)} - 0.4P(a,b)\frac{t^{2\beta}}{\Gamma(2\beta+1)},$$

$$Z_2' = 0.15(1+5a)P(a,b,c)\frac{t^{(\alpha+\gamma)}}{\Gamma(\alpha+\gamma+1)},$$

$$V_3' = P_1(a,b)(P(a,b,c)P_1(a,b)\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} - 2(a+0.7)P(a,b)\frac{t^{(2\alpha+\beta)}}{\Gamma(2\alpha+\beta+1)})$$

$$-P_1(a,b)P(a)P(c)\frac{t^{(2\alpha+\gamma)}}{\Gamma(2\alpha+\gamma+1)}-3.6685(P(a,b,c))^2\frac{t^{3\alpha}\Gamma(2\alpha+1)}{(\Gamma(\alpha+1))^2\Gamma(3\alpha+1)}$$
$$-2(a+0.7)(0.67P(a,b,c)\frac{t^{(2\alpha+\beta)}}{\Gamma(2\alpha+\beta+1)}-0.4P(a,b)\frac{t^{(\alpha+2\beta)}}{\Gamma(\alpha+2\beta+1)})$$
$$-2P(a,b,c)P(a,b)\frac{t^{(2\alpha+\beta)}\Gamma(\alpha+\beta+1)}{\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(2\alpha+\beta+1)}$$
$$-(0.55(1+5a)P(a,b,c))P(c)\frac{t^{(2\alpha+\gamma)}}{\Gamma(2\alpha+\gamma+1)}-2.5(P(a))^2\frac{t^{(\alpha+2\gamma)}\Gamma(2\gamma+1)}{(\Gamma(\gamma+1))^2\Gamma(\alpha+2\gamma+1)},$$

$$W_3'=0.67(P(a,b,c)P_1(a,b)\frac{t^{(2\alpha+\beta)}}{\Gamma(2\alpha+\beta+1)}-2(a+0.7)P(a,b)\frac{t^{(\alpha+2\beta)}}{\Gamma(\alpha+2\beta+1)})$$
$$-0.67P(a)P(c)\frac{t^{(\alpha+\beta+\gamma)}}{\Gamma(\alpha+\beta+\gamma+1)}-0.4(0.67P(a,b,c)\frac{t^{(\alpha+2\beta)}}{\Gamma(\alpha+2\beta+1)}-0.4P(a,b)\frac{t^{3\beta}}{\Gamma(3\beta+1)}),$$

$$Z_3'=0.15(1+5a)(P(a,b,c)P_1(a,b)\frac{t^{(2\alpha+\gamma)}}{\Gamma(2\alpha+\gamma+1)}-2(a+0.7)P(a,b)\frac{t^{(\alpha+\beta+\gamma)}}{\Gamma(\alpha+\beta+\gamma+1)}$$
$$-P(a)P(c)\frac{t^{(\alpha+2\gamma)}}{\Gamma(\alpha+2\gamma+1)})+0.375(P(a,b,c))^2\frac{t^{(2\alpha+\gamma)}\Gamma(2\alpha+1)}{(\Gamma(\alpha+1))^2\Gamma(2\alpha+\gamma+1)},$$

where

$$P(a,b,c)=(2.08a-3.6685a^2+0.3869-2ab-1.4b-0.875c-2.5c^2),P(a,b)=(0.18+0.666a-0.4b),$$

$$P_1(a,b)=(2.08165-7.3315a-2b),\quad P(a)=(0.0107+0.15a+0.375a^2),\ P(c)=(0.875+5c).$$

In a similar manner, the rest of the components can be obtained, and hence the approximate solutions for $V(t)$, $W(t)$ and $Z(t)$.

3.2 VIM solution for fractional order modified Morris-Lecar equations

The variational iteration method is also valid for solving fractional differential equations. In this paper, we extend the application of the variational iteration method to solve the fractional order modified M-L neural model as follows: The standard VIM formula for (7a) - (7c) leads to the following iteration formula (after inserting the values of $g_{Ca},g_K,g_L,I,\alpha_{exc},\alpha_{inh},\phi,b',c'$)

$$V'_{n+1}=V_n'-J_t^{\alpha}[D_t^{\alpha}V_n'+0.55\{1+(V_n'-V_1)/V_2\}(V_n'-1)+2W_n'(V_n'-V^K)+(V_n'-V^L)-0.3+\{1+(Z_n'-V_7)/V_6\}Z_n'],\tag{16a}$$

$$W'_{n+1}=W_n'-J_t^{\beta}[D_t^{\beta}W_n'-0.2\{1+(V_n'-V_3)/V_4\}+0.4W_n'],\tag{16b}$$

$$Z'_{n+1}=Z_n'-J_t^{\gamma}[D_t^{\gamma}Z_n'-0.01071-0.15\{1+(V_n'-V_5)/V_6\}V_n'].\tag{16c}$$

Assume $V_0'=a=0.01,W_0'=b=0.1,Z_0'=c=0.01.$ The values of the parameters are taken as in Nadar and Rai (2012). For $n=0,1,2,3$, we obtain

$$V_1'=a-A\frac{t^{\alpha}}{\Gamma(\alpha+1)},\ W_1'=b-B\frac{t^{\beta}}{\Gamma(\beta+1)},\ Z_1'=c-G\frac{t^{\gamma}}{\Gamma(\gamma+1)},$$

where

$$A=(3.6685a^2-2.08165a-0.38685+2ab+1.4b+2.5c^2+0.875c),$$
$$B=(-0.18-0.666a+0.4b),\ G=(-0.01071-0.15a-0.375a^2),$$

$$V_2' = 2.0817(a\frac{t^{\alpha}}{\Gamma(\alpha+1)} - A\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}) - 3.6685(a^2\frac{t^{\alpha}}{\Gamma(\alpha+1)} + A^2\frac{t^{3\alpha}\Gamma(2\alpha+1)}{(\Gamma(\alpha+1))^2\Gamma(3\alpha+1)})$$
$$+7.337aA\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - 2(ab\frac{t^{\alpha}}{\Gamma(\alpha+1)} - Ab\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - aB\frac{t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)})$$
$$+2AB\frac{t^{2\alpha+\beta}\Gamma(\alpha+\beta+1)}{\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(2\alpha+\beta+1)} - 1.4(b\frac{t^{\alpha}}{\Gamma(\alpha+1)} - B\frac{t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}) + 0.3869\frac{t^{\alpha}}{\Gamma(\alpha+1)}$$
$$-0.875(c\frac{t^{\alpha}}{\Gamma(\alpha+1)} - G\frac{t^{\alpha+\gamma}}{\Gamma(\alpha+\gamma+1)}) - 2.5(ac\frac{t^{\alpha}}{\Gamma(\alpha+1)} - cB\frac{t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} - bG\frac{t^{\alpha+\gamma}}{\Gamma(\alpha+\gamma+1)})$$
$$-2.5BG\frac{t^{2\alpha+\gamma}\Gamma(\alpha+\gamma+1)}{\Gamma(\alpha+1)\Gamma(\gamma+1)\Gamma(2\alpha+\gamma+1)}.$$

Simplifying V_2', we have

$$V_2' = H\frac{t^{\alpha}}{\Gamma(\alpha+1)} + K\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - 3.6685A^2\frac{t^{3\alpha}\Gamma(2\alpha+1)}{(\Gamma(\alpha+1))^2\Gamma(3\alpha+1)} + J\frac{t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}$$
$$-2AB\frac{t^{2\alpha+\beta}\Gamma(\alpha+\beta+1)}{\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(2\alpha+\beta+1)} + P\frac{t^{\alpha+\gamma}}{\Gamma(\alpha+\gamma+1)} - 2.5BG\frac{t^{2\alpha+\gamma}\Gamma(\alpha+\gamma+1)}{\Gamma(\alpha+1)\Gamma(\gamma+1)\Gamma(2\alpha+\gamma+1)},$$
$$W_2' = L\frac{t^{\beta}}{\Gamma(\beta+1)} - 0.066A\frac{t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + 0.4B\frac{t^{2\beta}}{\Gamma(2\beta+1)},$$
$$Z_2' = M\frac{t^{\gamma}}{\Gamma(\gamma+1)} + N\frac{t^{\alpha+\gamma}}{\Gamma(\alpha+\gamma+1)} + 0.375A^2\frac{t^{2\alpha+\gamma}\Gamma(2\alpha+1)}{(\Gamma(\alpha+1))^2\Gamma(2\alpha+\gamma+1)},$$

where

$H = (2.0817a - 3.6685a^2 - 2ab - 1.4b + 0.3869 - 0.875c - 2.5ac),\ K = (-2.0817A + 7.337aA + 2Ab),$
$J = (2aB + 1.4B + 2.5cB),\ P = (0.875G + 2.5bG),\ L = (0.18 + 0.066a - 0.4b),$
$M = (0.01071 + 0.15a + 0.375a^2),\ N = (-0.15A - 0.75aA).$

For $n = 2$, we obtain the next iterated values V_3', W_3' and Z_3' as

$$V_3' = 2.0817J_t^{\alpha}V_2' - 3.6685J_t^{\alpha}V_2'^2 + 0.38685\frac{t^{\alpha}}{\Gamma(\alpha+1)} - 2J_t^{\alpha}V_2'W_2' - 1.4J_t^{\alpha}W_2' - 0.875J_t^{\alpha}Z_2' - 2.5J_t^{\alpha}Z_2'^2,$$
$$W_3' = 0.18\frac{t^{\beta}}{\Gamma(\beta+1)} + 0.666J_t^{\beta}V_2' - 0.4J_t^{\beta}W_2',$$
$$Z_3' = 0.01071\frac{t^{\gamma}}{\Gamma(\gamma+1)} + 0.15J_t^{\gamma}V_2' + 3.75J_t^{\gamma}V_2'^2.$$

Inserting the values of V_2', W_2' and Z_2', we can simplify the above expressions. In a similar manner, the rest of the components can be obtained. Finally, we get the approximate solutions for $V(t), W(t)$ and $Z(t)$.

Detailed expressions for V_3', W_3' and Z_3' are given in the Appendix A1.

4 Numerical Simulation Results

In this section, numerical results of the modified Morris-Lecar equations given by (1a) to (1c), for different fractional order derivatives $\alpha = \beta = \gamma = 1/3, 1/2, 2/3$ and for the standard derivative $\alpha = \beta = \gamma = 1$ are calculated using the methods HPM and VIM for $t \in [0,1]$. Numerical results are depicted graphically by varying $t \in [0,1]$ and keeping γ constant and by taking $\gamma = 0.5$ and 1 for different values of fractional orders $\alpha = \beta = 1/3, \alpha = \beta = 1/2, \alpha = \beta = 2/3$ and for integer order $\alpha = \beta = 1$. Next, fixing $t = 0.1$ and varying

$\gamma \in [0,1]$, the obtained results are shown graphically for various order fractional derivatives $\alpha = \beta = 1/3$, $\alpha = \beta = 1/2$, $\alpha = \beta = 2/3$, and $\alpha = \beta = 1$. Varying both $t, \gamma \in [0,1]$ for $\alpha = \beta = 1/3$, $\alpha = \beta = 1/2$, $\alpha = \beta = 2/3$, and $\alpha = \beta = 1$, the results are depicted graphically using the above two methods.

In the two methods, it was found that only four terms are sufficient to obtain the approximate solutions. The accuracy of the results can be improved by introducing more terms.

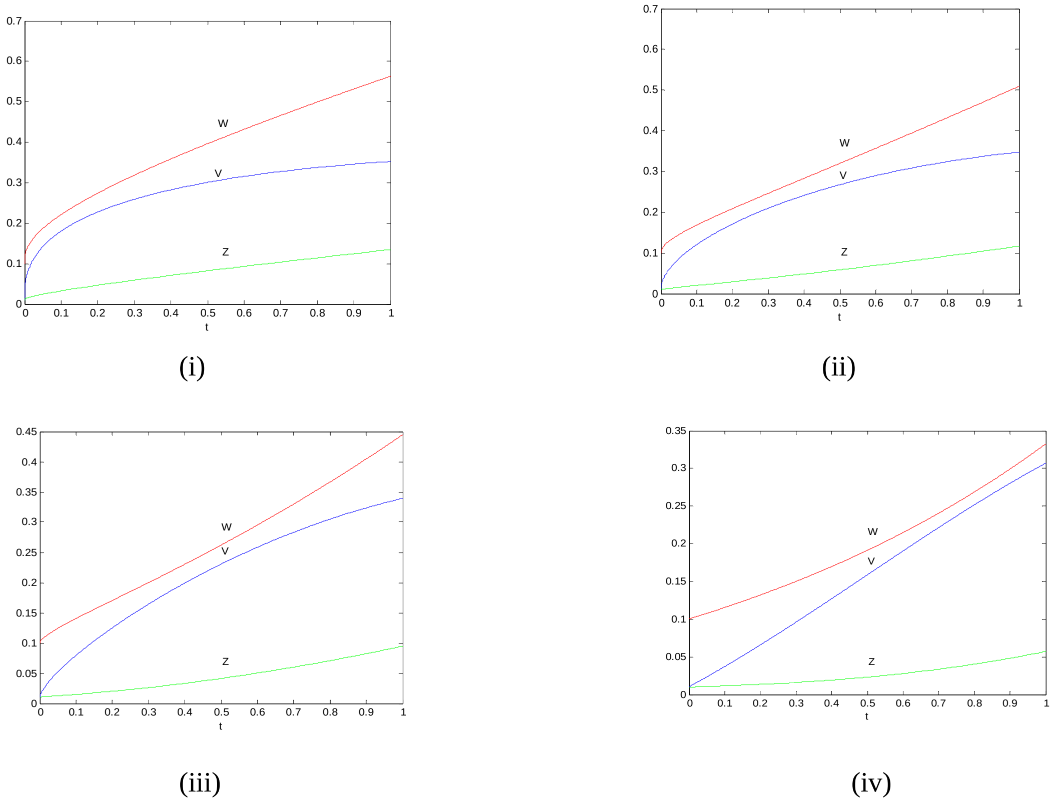
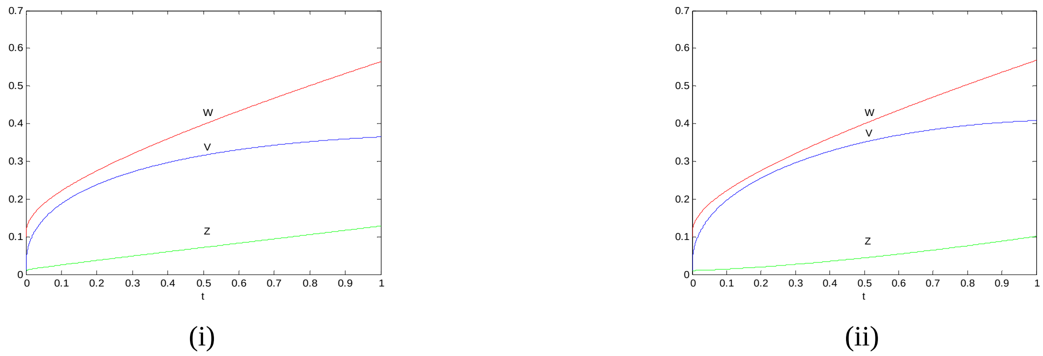
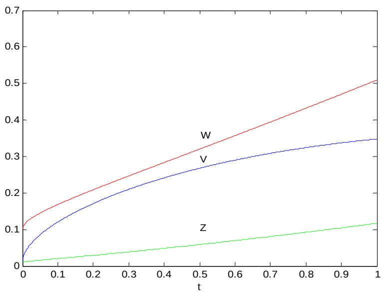
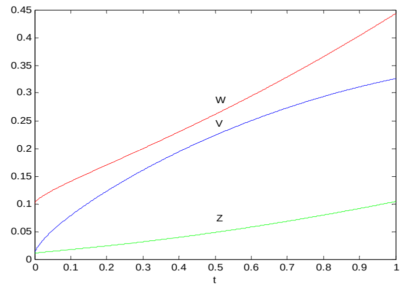


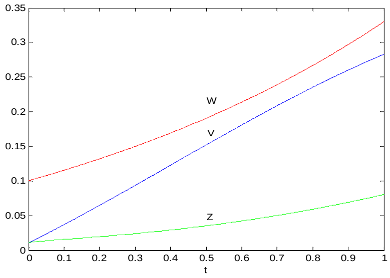
Fig. 1 Fractional order HPM solutions for $V(t), W(t), Z(t)$ for $t \in [0,1]$ and (i) $\alpha = \beta = \gamma = 1/3$, (ii) $\alpha = \beta = \gamma = 1/2$, (iii) $\alpha = \beta = \gamma = 2/3$, (iv) $\alpha = \beta = \gamma = 1$.

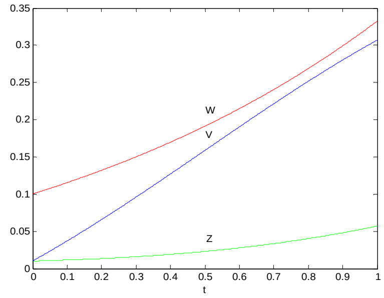






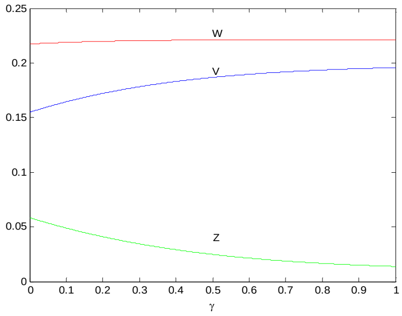
(iii) (iv)

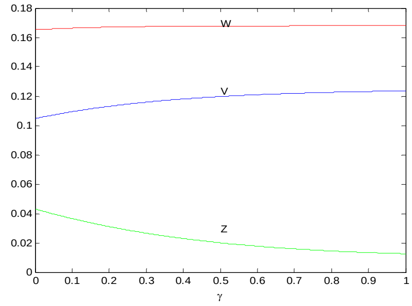




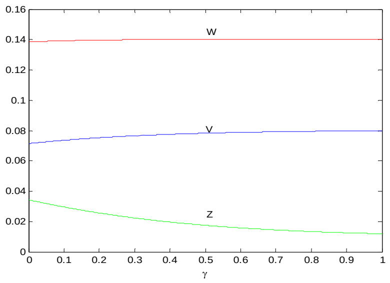
(v) (vi)

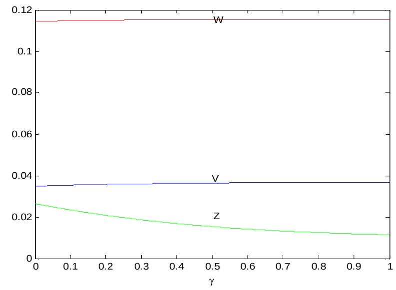
Fig. 2 Fractional order HPM solutions for $V(t),W(t),Z(t)$ for $t\in[0,1]$ and (i) $\alpha=\beta=1/3,\gamma=0.5$, (ii) $\alpha=\beta=1/3,\gamma=1$, (iii) $\alpha=\beta=1/2,\gamma=0.5$, (iv) $\alpha=\beta=2/3,\gamma=0.5$, (v) $\alpha=\beta=1,\gamma=0.5$, (vi) $\alpha=\beta=1,\gamma=1$.





(i) (ii)

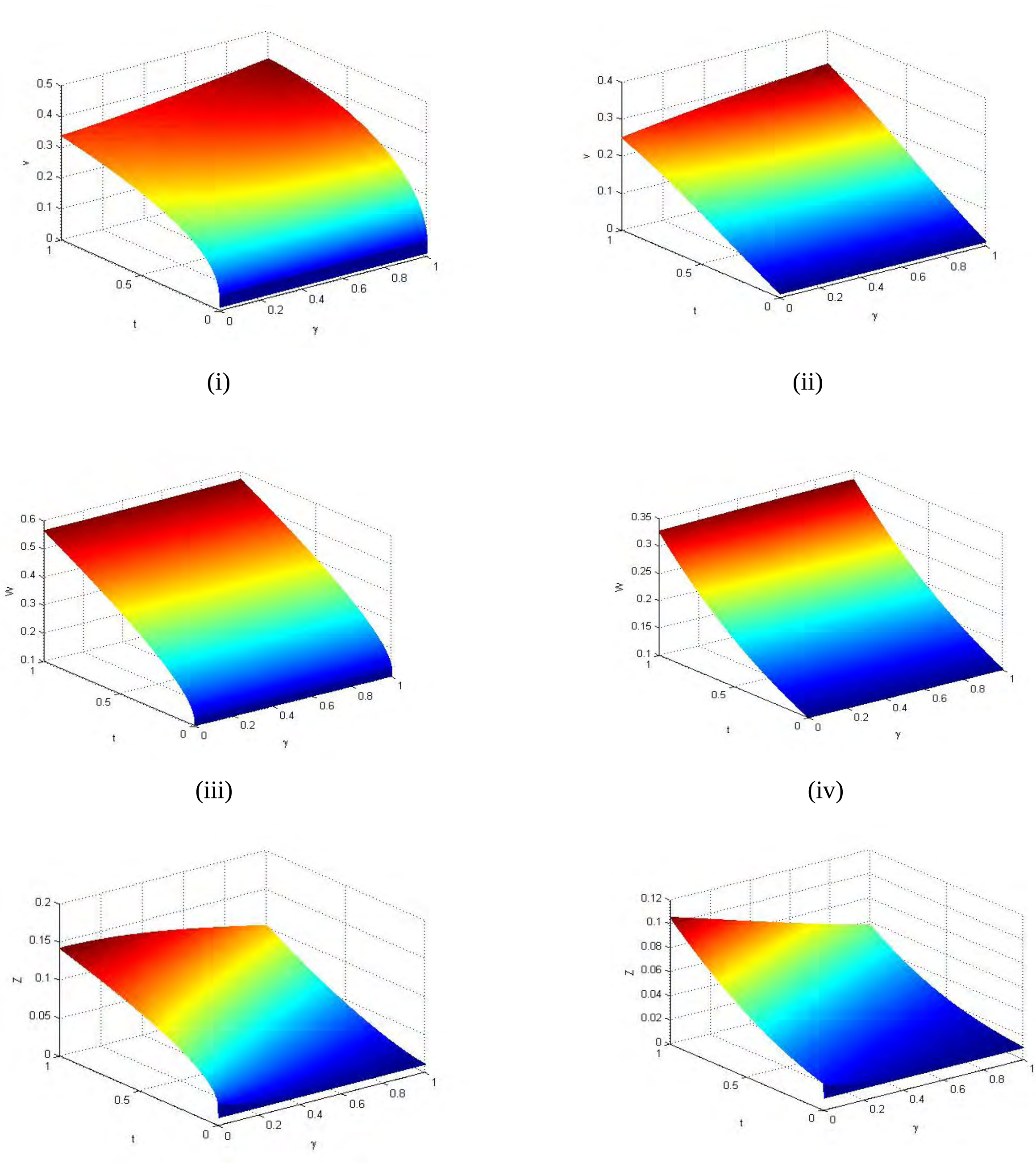




(iii) (iv)

Fig. 3 Fractional order HPM solutions for $V(t),W(t),Z(t)$ at $t=0.1$ and $\gamma\in[0,1]$ (i) $\alpha=\beta=1/3$, (ii) $\alpha=\beta=1/2$,

(iii) $\alpha = \beta = 2/3$, (iv) $\alpha = \beta = 1$.



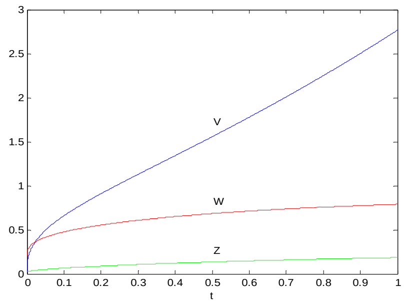
(i) (ii)

(iii) (iv)

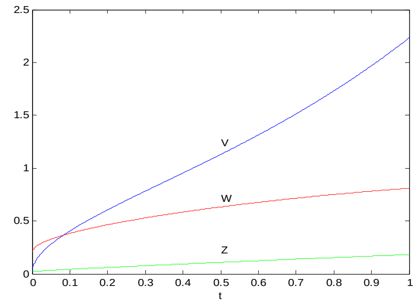
(v) (vi)

Fig. 4 Fractional order HPM solutions. $V(t)$ for $\gamma \in [0,1]$, $t \in [0,1]$ and (i) $\alpha = \beta = 1/3$, (ii) $\alpha = \beta = 1$. $W(t)$ for $\gamma \in [0,1]$, $t \in [0,1]$ and (iii) $\alpha = \beta = 1/3$, (iv) $\alpha = \beta = 1$. $Z(t)$ for $\gamma \in [0,1]$, $t \in [0,1]$ and (v) $\alpha = \beta = 1/3$, (vi) $\alpha = \beta = 1$.

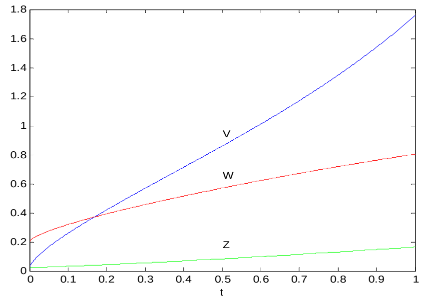
It is observed from Figs. 1 and 2, that the inhibitory and excitatory neurons are classified on the nature of chemical substances that are released in the synaptic cleft. The mean membrane potential of excitatory and inhibitory neurons increase but the rate of change is different. In Fig. 3, the variations of $V(t), W(t), Z(t)$, with respect to γ (fractional order of (1c)) are plotted. It is observed that $W(t)$ does not change with γ, whereas $V(t)$ increases and $Z(t)$ decreases for $\alpha = \beta = 1/3, 1/2$. However, for $\alpha = \beta = 2/3$ and 1, $V(t)$ and $W(t)$ behave similarly, which remain fixed in the whole range of γ whereas in all the cases, decrease in $Z(t)$ is reported. In Fig. 4, rectangular patterns are observed showing temporal variations of $V(t), W(t), Z(t)$, with respect to γ. In all cases, we observe that $V(t)$ dominates over $W(t)$ and $Z(t)$ and the density of $Z(t)$ is clubbed at left top corner points and it is well below 0.15 mV. However, $W(t)$ is uniformly distributed horizontally and inclined towards the left side in the domain and its value is lying between 0.31 and 0.58. $V(t)$ is concentrated on the upper top corner points and it varies from 0.25mV to 0.4 mV.



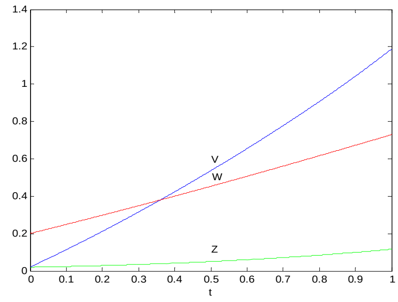
(i)



(ii)

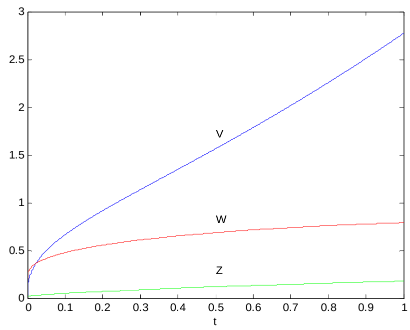


(iii)

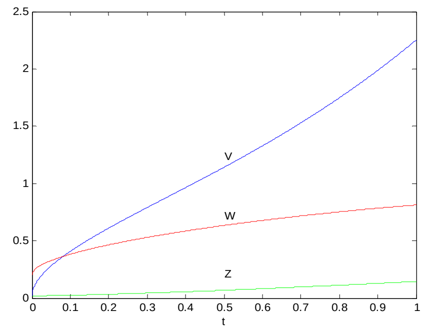


(iv)

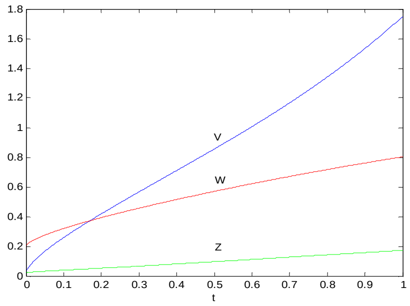
Fig. 5 Fractional order VIM solutions for $V(t), W(t), Z(t)$ for $t \in [0,1]$ and (i) $\alpha = \beta = \gamma = 1/3$, (ii) $\alpha = \beta = \gamma = 1/2$, (iii) $\alpha = \beta = \gamma = 2/3$, (iv) $\alpha = \beta = \gamma = 1$.



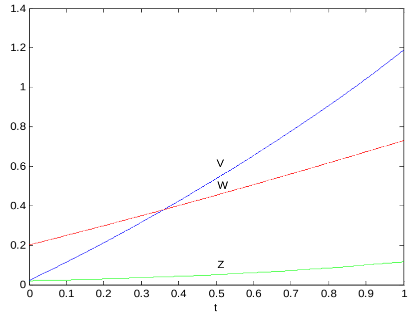
(i)



(ii)

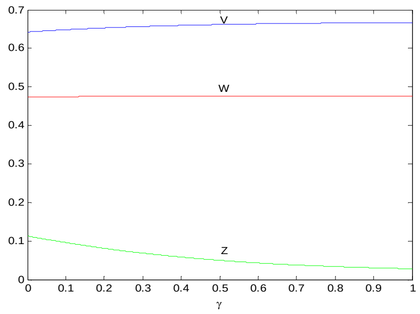


(iii)

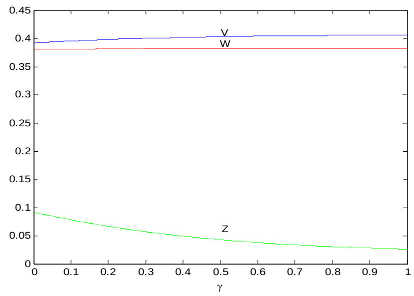


(iv)

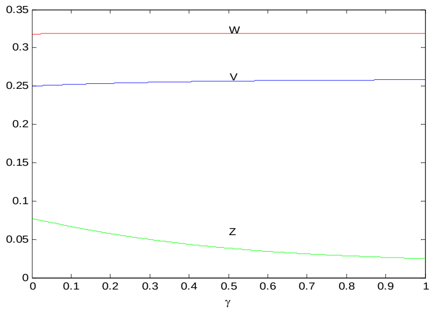
Fig. 6 Fractional order VIM solutions for $V(t), W(t), Z(t)$ for $t \in [0,1]$ and (i) $\alpha = \beta = 1/3, \gamma = 0.5$, (ii) $\alpha = \beta = 1/2, \gamma = 1$, (iii) $\alpha = \beta = 2/3, \gamma = 0.5$, (iv) $\alpha = \beta = 1, \gamma = 1$.



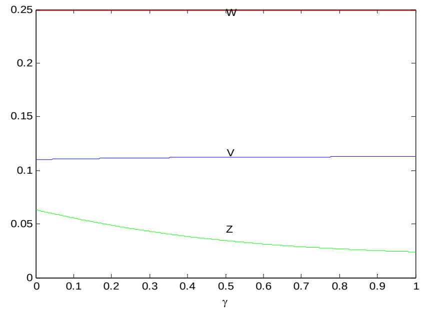
(i)



(ii)

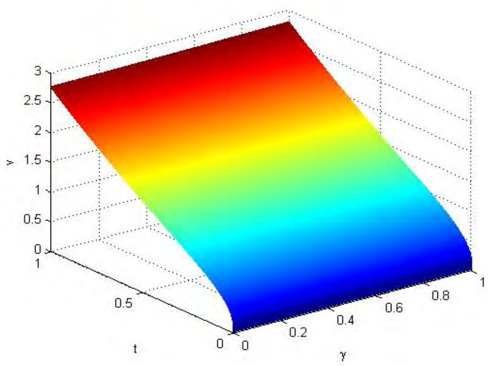


(iii)

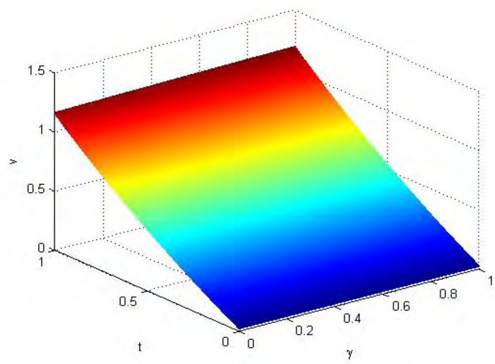


(iv)

Fig. 7 Fractional order VIM solutions for $V(t), W(t), Z(t)$ at $t = 0.1$ and $\gamma \in [0,1]$ (i) $\alpha = \beta = 1/3$, (ii) $\alpha = \beta = 1/2$, (iii) $\alpha = \beta = 2/3$, (iv) $\alpha = \beta = 1$.



(i)



(ii)

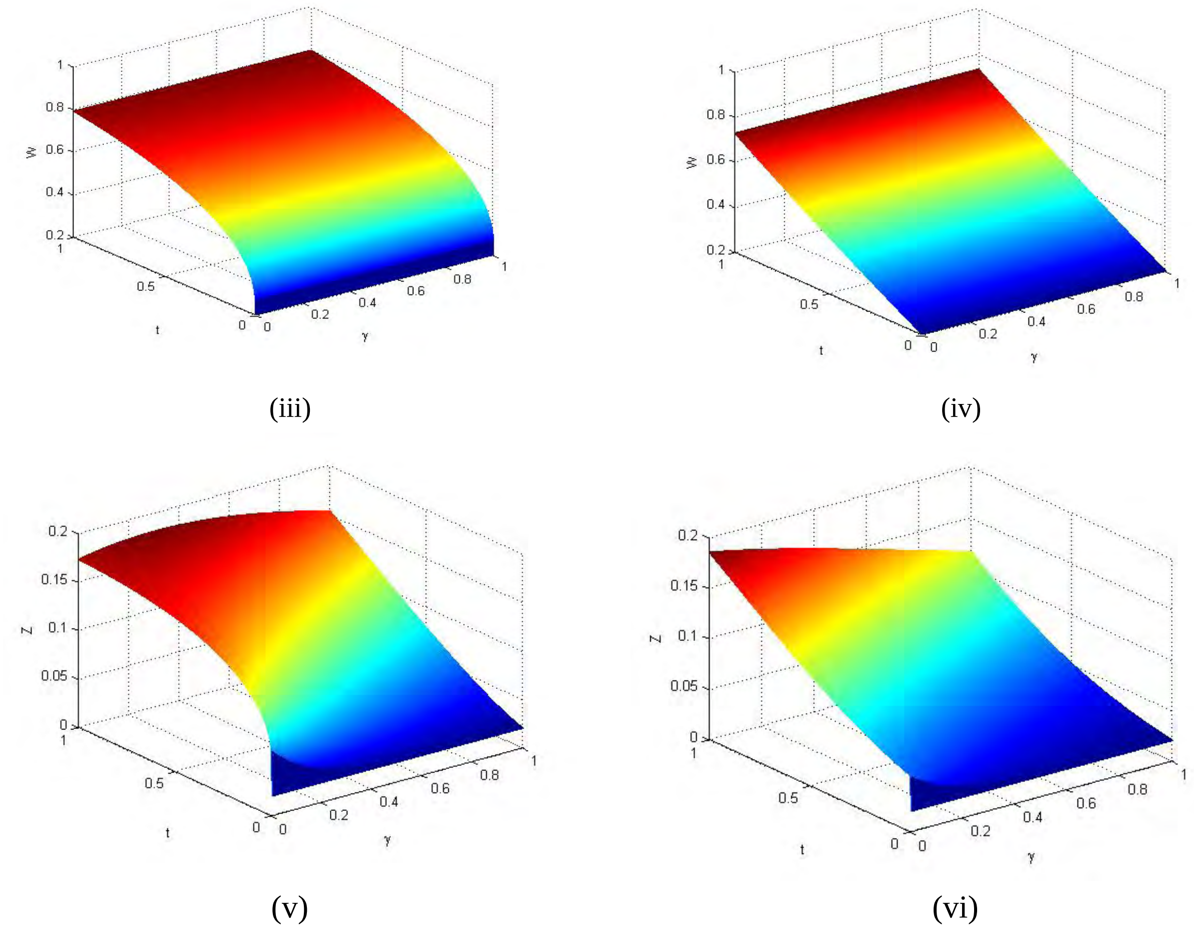


Fig. 8 Fractional order VIM solutions. $V(t)$ for $\gamma \in [0,1]$, $t \in [0,1]$ and (i) $\alpha = \beta = 1/3$, (ii) $\alpha = \beta = 1$. $W(t)$ for $\gamma \in [0,1]$, $t \in [0,1]$ and (iii) $\alpha = \beta = 1/3$, (iv) $\alpha = \beta = 1$. $Z(t)$ for $\gamma \in [0,1]$, $t \in [0,1]$ and (v) $\alpha = \beta = 1/3$, (vi) $\alpha = \beta = 1$.

In VIM solution, we have taken the initial condition as $V(t) = 0.01, W(t)=0.1$ and $Z(t) = 0.01$ at $t = 0$. It is observed from Figs. 5 and 6 that the mean membrane potential of excitatory and inhibitory neurons, i.e., $V(t)$ and $Z(t)$ increases but the rate of change is different. With the change in mean membrane potential of excitatory cell there is a corresponding change in the opening of potassium channels. The rate of increase of $Z(t)$ and $V(t)$ are different whereas one is very slow and other one is very fast. In Fig. 7, we have plotted the variation of $V(t), W(t)$ and $Z(t)$ with respect to γ. We observe that $W(t)$ and $V(t)$ do not change with γ, whereas $Z(t)$ decreases for all values of α and β. In Fig. 8, rectangular patterns are observed showing the temporal variation of $V(t), W(t)$ and $Z(t)$ with respect to γ. From these figures, we observe that $V(t)$ is uniformly distributed horizontally, $W(t)$ is also uniformly distributed and inclined towards left side and its value is lying between 0.6 to 0.8. The density of $Z(t)$ is clubbed at left top corner points which is well below 0.2*mV*.

Now, we present an example for a single neuron as a representative of a subnetwork model for epileptic seizures and apply the homotopy perturbation method.

Example Consider the following homotopy of the model system (7) for a single neuron as a representative of a subnetwork model for epileptic seizures (Larter et al., 1999). We solve the problem using HPM.

$$D_t^\alpha V(t) = -g_{Ca} m_\infty(V)(V-1) - g_K W(V - V^K) - 0.5(V - V^L) + I - \alpha_{inh}(Z)Z, \tag{17a}$$

$$D_t^\beta W(t) = 0.7\,(W_\infty - W)/\tau_W\,, \tag{17b}$$

$$D_t^\beta Z(t) = 0.1(0.165I + \alpha_{exc}(V)V). \tag{17c}$$

Applying HPM to the problem, we obtain the system as:

$$D_t^\alpha V(t) = p\{-0.5 g_{Ca}[1 + ((V - V_1)/V_2)](V-1) - g_K W(V - V^K) - 0.5(V - V^L) + I - \alpha_{inh}[1 + ((Z - V_7)/V_6)]Z\}, \tag{18a}$$

$$D_t^\beta W(t) = p\{0.7(0.5[1 + ((V - V_3)/V_4)] - W)\}, \tag{18b}$$

$$D_t^\gamma Z(t) = p\{0.1(0.165I + \alpha_{exc}[1 + ((V - V_5)/V_6)]V)\}. \tag{18c}$$

Now, using the parameter values (Larter et al., 1999)

$V_1 = -0.01, V_2 = 0.15, V_3 = 0.0, V_4 = 0.3, V_5 = 0.0, V_6 = 0.6, V_7 = 0.0, g_{Ca} = 1.1, g_K = 2.0,\ V^K = -0.7, V^L = -0.5,$
$I = 0.3,\ \alpha_{exc} = 1$ and $\alpha_{inh} = 1,$

the above system reduces to

$$D_t^\alpha V(t) = p\{-(3.6667V + 0.5867)(V-1) - 2W(V + 0.7) - 0.5V + 0.05 - (1 + 1.6667Z)Z\}, \tag{19a}$$

$$D_t^\beta W(t) = p\{0.35 + 1.1655V - 0.7W\}, \tag{19b}$$

$$D_t^\gamma Z(t) = p\{0.005 + 0.1V + 0.1667V^2\}. \tag{19c}$$

Assume the initial conditions as $V(0) = 0.01, W(0) = 0.1, Z(0) = 0.01$ (Larter et al., 1999; Nadar and Rai, 2012; Rai et al., 2013) and take $V_0' = a = 0.01, W_0' = b = 0.1, Z_0' = c = 0.01$. Solving the system (19a)-(19c) and equating the coefficients of powers of p, we have

$$p^0 : D_t^\alpha V_0' = 0,\ D_t^\beta W_0' = 0,\ D_t^\gamma Z_0' = 0.$$

$$p^1 : D_t^\alpha V_1' = X(a,b,c),\ D_t^\beta W_1' = Y(a,b),\ D_t^\gamma Z_1' = U(a).$$

$$p^2 : D_t^\alpha V_2' = V_1' D(a,b) - 2W_1'(a + 0.7) - (3.3334c + 1)Z_1',\ D_t^\beta W_2' = 1.1655V_1' - 0.7W_1',$$

$$D_t^\gamma Z_2' = (0.1 + 0.3334a)V_1'.$$

$$p^3 :\ D_t^\alpha V_3' = D(a,b)V_2' - 3.66675V_1'^2 - 2(a + 0.7)W_2' - 2V_1'W_1' - Z_2'(3.3334c + 1) - 1.6667Z_1'^2,$$

$$D_t^{\beta}W_3' = 1.1655V_2' - 0.7W_2',$$

$$D_t^{\gamma}Z_3' = (0.1+0.3334a)V_2' + 0.1667V_1'^2.$$

Solving the above equations, we obtain

$$V_1' = X(a,b,c)\frac{t^{\alpha}}{\Gamma(\alpha+1)},\ W_1' = Y(a,b)\frac{t^{\beta}}{\Gamma(\beta+1)},\ Z_1' = U(a)\frac{t^{\gamma}}{\Gamma(\gamma+1)},$$

$$V_2' = D(a,b)X(a,b,c)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - 2Y(a,b)(a+0.7)\frac{t^{(\alpha+\beta)}}{\Gamma(\alpha+\beta+1)} - (1+3.3334c)U(a)\frac{t^{(\alpha+\gamma)}}{\Gamma(\alpha+\gamma+1)},$$

$$W_2' = 1.1655X(a,b,c)\frac{t^{(\alpha+\beta)}}{\Gamma(\alpha+\beta+1)} - 0.7Y(a,b)\frac{t^{2\beta}}{\Gamma(2\beta+1)},$$

$$Z_2' = (0.1+0.3334a)X(a,b,c)\frac{t^{(\alpha+\gamma)}}{\Gamma(\alpha+\gamma+1)},$$

$$\begin{aligned}V_3' = {}& D(a,b)(D(a,b)X(a,b,c)\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} - 2Y(a,b)(a+0.7)\frac{t^{(2\alpha+\beta)}}{\Gamma(2\alpha+\beta+1)})\\ & - D(a,b)(1+3.3334c)U(a)\frac{t^{(2\alpha+\gamma)}}{\Gamma(2\alpha+\gamma+1)} - 3.6667(X(a,b,c))^2\frac{t^{3\alpha}\Gamma(2\alpha+1)}{(\Gamma(\alpha+1))^2\Gamma(3\alpha+1)}\\ & -1.1655(2a+1.4)X(a,b,c)\frac{t^{(2\alpha+\beta)}}{\Gamma(2\alpha+\beta+1)} - 0.7(2a+1.4)Y(a,b)\frac{t^{\alpha+2\beta}}{\Gamma(\alpha+2\beta+1)}\\ & -2X(a,b,c)Y(a,b)\frac{t^{(2\alpha+\beta)}\Gamma(\alpha+\beta+1)}{\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(2\alpha+\beta+1)} - (1+3.3334c)(0.1+0.3334a)X(a,b,c)\\ & \frac{t^{2\alpha+\gamma}}{\Gamma(2\alpha+\gamma+1)} - 1.6667(U(a))^2\frac{t^{\alpha+2\gamma}\Gamma(2\gamma+1)}{\left(\Gamma(\gamma+1)\right)^2\Gamma(\alpha+2\gamma+1)},\end{aligned}$$

$$\begin{aligned}W_3' = {}& 1.1655(D(a,b)X(a,b,c)\frac{t^{2\alpha+\beta}}{\Gamma(2\alpha+\beta+1)} - 2Y(a,b)(a+0.7)\frac{t^{(\alpha+2\beta)}}{\Gamma(\alpha+2\beta+1)})\\ & -1.1655(1+3.3334c)U(a)\frac{t^{(\alpha+\beta+\gamma)}}{\Gamma(\alpha+\beta+\gamma+1)} - 0.7(1.1655X(a,b,c)\frac{t^{(\alpha+2\beta)}}{\Gamma(\alpha+2\beta+1)}) + 0.49Y(a,b)\frac{t^{3\beta}}{\Gamma(3\beta+1)},\end{aligned}$$

$$\begin{aligned}Z_3' = {}& (0.1+0.3334a)(D(a,b)X(a,b,c)\frac{t^{2\alpha+\gamma}}{\Gamma(2\alpha+\gamma+1)} - 2Y(a,b)(a+0.7)\frac{t^{(\alpha+\beta+\gamma)}}{\Gamma(\alpha+\beta+\gamma+1)})\\ & -(0.1+0.3334a)(1+3.3334c)U(a)\frac{t^{(\alpha+2\gamma)}}{\Gamma(\alpha+2\gamma+1)} + 0.1667(X(a,b,c))^2\frac{t^{2\alpha+\gamma}\Gamma(2\alpha+1)}{(\Gamma(\alpha+1))^2\Gamma(2\alpha+\gamma+1)},\end{aligned}$$

where

$$X(a,b,c) = (-3.6667a^2 + 2.58a - 2ab - 1.4b + 0.6367 - c - 1.6667c^2),\ Y(a,b) = (0.35 + 1.1655a - 0.7b),$$
$$U(a) = (0.005 + 0.1a + 0.1667a^2),\ D(a,b) = (-7.3334a + 2.58 - 2b).$$

Similarly, rest of the components can be obtained. The solutions for $V(t), W(t)$ and $Z(t)$ are then obtained.

We have presented the numerical solution of the Example using HPM in Table 1, for different values of fractional orders α,β,γ for $t\in[0.2,1]$ with step size $t = 0.2$.

Similarly, we can apply the VIM method and obtain the solutions.

Table 1

$V(t)$				
	$\alpha=\beta=\gamma=1/3$	$\alpha=\beta=\gamma=2/3$	$\alpha=\beta=\gamma=1/2$	$\alpha=\beta=\gamma=1$
$t=0.2$	0.57412	0.263602	0.387733	0.12529
$t=0.4$	0.772912	0.448387	0.589202	0.259881
$t=0.6$	0.918664	0.608484	0.749258	0.402839
$t=0.8$	1.03684	0.746685	0.883042	0.5432225
$t=1$	1.13729	0.863479	0.997026	0.670105
$W(t)$				
	$\alpha=\beta=\gamma=1/3$	$\alpha=\beta=\gamma=2/3$	$\alpha=\beta=\gamma=1/2$	$\alpha=\beta=\gamma=1$
$t=0.2$	0.560412	0.26098	0.366121	0.166946
$t=0.4$	0.818475	0.420615	0.579914	0.254363
$t=0.6$	1.04772	0.605105	0.801581	0.367116
$t=0.8$	1.26203	0.816342	1.03374	0.510066
$t=1$	1.46678	1.05477	1.27662	0.688079
$Z(t)$				
	$\alpha=\beta=\gamma=1/3$	$\alpha=\beta=\gamma=2/3$	$\alpha=\beta=\gamma=1/2$	$\alpha=\beta=\gamma=1$
$t=0.2$	0.0810816	0.0297947	0.0486777	0.014746
$t=0.4$	0.12201	0.0583432	0.0862323	0.0268083
$t=0.6$	0.15638	0.0923855	0.123921	0.0465328
$t=0.8$	0.18711	0.1308	0.161803	0.0742658
$t=1$	0.215403	0.172966	0.199887	0.110353

5 Discussions and Conclusions

In this paper, we have studied a fractional order modified M-L neural model modeling the dynamics of a single neuron that interacts with other neurons through on-off excitatory and inhibitory synapses in a neural system. We have presented the solutions of a modified M-L neural model of fractional-order time derivatives with the help of two different analytical methods HPM, VIM. We observe that the methods are efficient in finding the exact as well as approximate solutions. They provide series solutions that converge very rapidly in real physical problems (Momani and Odibat, 2007). We have tested the analytical results by solving it numerically. It is observed that the inhibitory and excitatory neurons are classified on the nature of chemical substances that are released in the synaptic cleft. With the change in mean membrane potential of excitatory cell there is a corresponding opening of potassium channels. The mean membrane potential of excitatory and inhibitory neurons increases but the rate of change is different (see Figs. 1 and 2) for both HPM and VIM. Fractional differentiation is used to investigate the behavior of neural systems (Magin, 2004). Even processing of external applied stimuli by individual neurons can be expressed by fractional differentiation (Lundstrom et al., 2008). The fractional-order modified M-L neural model may help in understanding the synchronous neural activities occurring during a seizure, stimulus anticipation and in frequency independent phase shifts of oscillatory neuronal firing.

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Appendix A1

Detailed expressions for V_3', W_3' and Z_3' obtained by the VIM are the following:

$$
\begin{aligned}
V_3' = {} & 2.0817H\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + 2.0817K\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} - 7.6367A^2\frac{t^{4\alpha}\Gamma(2\alpha+1)}{(\Gamma(\alpha+1))^2\Gamma(4\alpha+1)} \\
& + 2.0817J\frac{t^{2\alpha+\beta}}{\Gamma(2\alpha+\beta+1)} - 4.1634AB\frac{t^{3\alpha+\beta}\Gamma(\alpha+\beta+1)}{\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(3\alpha+\beta+1)} + 2.0817P\frac{t^{2\alpha+\gamma}}{\Gamma(2\alpha+\gamma+1)} \\
& -5.2043BG\frac{t^{3\alpha+\gamma}\Gamma(\alpha+\gamma+1)}{\Gamma(\alpha+1)\Gamma(\gamma+1)\Gamma(3\alpha+\gamma+1)} + 0.38685\frac{t^{\alpha}}{\Gamma(\alpha+1)} - 3.6685H^2\frac{t^{3\alpha}\Gamma(2\alpha+1)}{(\Gamma(\alpha+1))^2\Gamma(3\alpha+1)} \\
& -3.6685K^2\frac{t^{5\alpha}\Gamma(4\alpha+1)}{(\Gamma(2\alpha+1))^2\Gamma(5\alpha+1)} - 3.6685^3A^4\frac{t^{7\alpha}(\Gamma(2\alpha+1))^2\Gamma(6\alpha+1)}{(\Gamma(\alpha+1))^4(\Gamma(3\alpha+1))^2\Gamma(7\alpha+1)} \\
& -7.337HK\frac{t^{4\alpha}\Gamma(3\alpha+1)}{\Gamma(\alpha+1)\Gamma(2\alpha+1)\Gamma(4\alpha+1)} + 26.9158A^2K\frac{t^{6\alpha}\Gamma(5\alpha+1)}{(\Gamma(\alpha+1))^2\Gamma(3\alpha+1)\Gamma(6\alpha+1)} \\
& +26.9158A^2H\frac{t^{5\alpha}\Gamma(2\alpha+1)\Gamma(4\alpha+1)}{(\Gamma(\alpha+1))^3\Gamma(3\alpha+1)\Gamma(5\alpha+1)} - 3.66855J^2\frac{t^{3\alpha+2\beta}\Gamma(2\alpha+2\beta+1)}{(\Gamma(\alpha+\beta+1))^2\Gamma(3\alpha+2\beta+1)} \\
& -14.674A^2B^2\frac{t^{5\alpha+2\beta}(\Gamma(\alpha+\beta+1))^2\Gamma(4\alpha+2\beta+1)}{(\Gamma(\alpha+1))^2(\Gamma(\beta+1))^2(\Gamma(2\alpha+\beta+1))^2\Gamma(5\alpha+2\beta+1)} \\
& +14.674ABJ\frac{t^{4\alpha+2\beta}\Gamma(3\alpha+2\beta+1)}{\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(2\alpha+\beta+1)\Gamma(4\alpha+2\beta+1)} - 3.6685P^2\frac{t^{3\alpha+2\gamma}\Gamma(2\alpha+2\gamma+1)}{(\Gamma(\alpha+\gamma+1))^2\Gamma(3\alpha+2\gamma+1)} \\
& -22.9281B^2G^2\frac{t^{5\alpha+2\gamma}\Gamma(4\alpha+2\gamma+1)(\Gamma(\alpha+\gamma+1))^2}{(\Gamma(\alpha+1))^2(\Gamma(\gamma+1))^2(\Gamma(2\alpha+\gamma+1))^2\Gamma(5\alpha+2\gamma+1)} \\
& +18.3425PBG\frac{t^{4\alpha+2\gamma}\Gamma(3\alpha+2\gamma+1)}{\Gamma(\alpha+1)\Gamma(\gamma+1)\Gamma(2\alpha+\gamma+1)\Gamma(4\alpha+2\gamma+1)} \\
& +36.685JBG\frac{t^{4\alpha+\beta+\gamma}\Gamma(\alpha+\gamma+1)\Gamma(3\alpha+\beta+\gamma+1)}{\Gamma(\alpha+\beta+1)\Gamma(\alpha+1)\Gamma(\gamma+1)\Gamma(2\alpha+\gamma+1)\Gamma(4\alpha+\beta+\gamma+1)} \\
& +14.674ABP\frac{t^{4\alpha+\beta+\gamma}\Gamma(3\alpha+\beta+\gamma+1)\Gamma(\alpha+\beta+1)}{\Gamma(\alpha+\gamma+1)\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(2\alpha+\beta+1)\Gamma(4\alpha+\beta+\gamma+1)} \\
& -36.685AB^2G\frac{t^{5\alpha+\beta+\gamma}\Gamma(\alpha+\gamma+1)\Gamma(\alpha+\beta+1)\Gamma(4\alpha+\beta+\gamma+1)}{(\Gamma(\alpha+1))^2\Gamma(2\alpha+\beta+1)\Gamma(\beta+1)\Gamma(\gamma+1)\Gamma(2\alpha+\gamma+1)\Gamma(5\alpha+\beta+\gamma+1)} \\
& -7.337HJ\frac{t^{3\alpha+\beta}\Gamma(2\alpha+\beta+1)}{\Gamma(\alpha+1)\Gamma(\alpha+\beta+1)\Gamma(3\alpha+\beta+1)} - 7.337KJ\frac{t^{4\alpha+\beta}\Gamma(3\alpha+\beta+1)}{\Gamma(2\alpha+1)\Gamma(\alpha+\beta+1)\Gamma(4\alpha+\beta+1)} \\
& +26.9158A^2J\frac{t^{5\alpha+\beta}\Gamma(4\alpha+\beta+1)\Gamma(2\alpha+1)}{(\Gamma(\alpha+1))^2\Gamma(\alpha+\beta+1)\Gamma(3\alpha+1)\Gamma(5\alpha+\beta+1)} \\
& +14.674ABH\frac{t^{4\alpha+\beta}\Gamma(\alpha+\beta+1)\Gamma(3\alpha+\beta+1)}{(\Gamma(\alpha+1))^2\Gamma(2\alpha+\beta+1)\Gamma(\beta+1)\Gamma(4\alpha+\beta+1)} \\
& +14.674ABK\frac{t^{5\alpha+\beta}\Gamma(4\alpha+\beta+1)\Gamma(\alpha+\beta+1)}{\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(2\alpha+1)\Gamma(2\alpha+\beta+1)\Gamma(5\alpha+\beta+1)} \\
& -53.8316A^3B\frac{t^{6\alpha+\beta}\Gamma(5\alpha+\beta+1)\Gamma(2\alpha+1)\Gamma(\alpha+\beta+1)}{(\Gamma(\alpha+1))^3\Gamma(3\alpha+1)\Gamma(\beta+1)\Gamma(2\alpha+\beta+1)\Gamma(6\alpha+\beta+1)}
\end{aligned}
$$

$$-7.337HP\frac{t^{3\alpha+\gamma}\Gamma(2\alpha+\gamma+1)}{\Gamma(\alpha+1)\Gamma(\alpha+\gamma+1)\Gamma(3\alpha+\gamma+1)}-7.337KP\frac{t^{4\alpha+\gamma}\Gamma(3\alpha+\gamma+1)}{\Gamma(\alpha+\gamma+1)\Gamma(2\alpha+1)\Gamma(4\alpha+\gamma+1)}$$

$$+26.9158A^2P\frac{t^{5\alpha+\gamma}\Gamma(4\alpha+\gamma+1)\Gamma(2\alpha+1)}{(\Gamma(\alpha+1))^2\Gamma(3\alpha+1)\Gamma(\alpha+\gamma+1)\Gamma(5\alpha+\gamma+1)}$$

$$+18.3425HBG\frac{t^{4\alpha+\gamma}\Gamma(3\alpha+\gamma+1)\Gamma(\alpha+\gamma+1)}{(\Gamma(\alpha+1))^2\Gamma(\gamma+1)\Gamma(2\alpha+\gamma+1)\Gamma(4\alpha+\gamma+1)}$$

$$+18.3425KBG\frac{t^{5\alpha+\gamma}\Gamma(4\alpha+\gamma+1)\Gamma(\alpha+\gamma+1)}{\Gamma(2\alpha+1)\Gamma(\alpha+1)\Gamma(\gamma+1)\Gamma(2\alpha+\gamma+1)\Gamma(5\alpha+\gamma+1)}$$

$$-67.2895A^2BG\frac{t^{6\alpha+\gamma}\Gamma(2\alpha+1)\Gamma(\alpha+\gamma+1)\Gamma(5\alpha+\gamma+1)}{(\Gamma(\alpha+1))^3\Gamma(3\alpha+1)\Gamma(\gamma+1)\Gamma(2\alpha+\gamma+1)\Gamma(6\alpha+\gamma+1)}$$

$$-2LH\frac{t^{2\alpha+\beta}\Gamma(\alpha+\beta+1)}{\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(2\alpha+\beta+1)}-2KL\frac{t^{3\alpha+\beta}\Gamma(2\alpha+\beta+1)}{\Gamma(2\alpha+1)\Gamma(\beta+1)\Gamma(3\alpha+\beta+1)}$$

$$+7.337A^2L\frac{t^{4\alpha+\beta}\Gamma(3\alpha+\beta+1)\Gamma(2\alpha+1)}{(\Gamma(\alpha+1))^2\Gamma(3\alpha+1)\Gamma(\beta+1)\Gamma(4\alpha+\beta+1)}-2J\frac{t^{2\alpha+2\beta}\Gamma(\alpha+2\beta+1)}{\Gamma(\beta+1)\Gamma(\alpha+\beta+1)\Gamma(2\alpha+2\beta+1)}$$

$$+4ABL\frac{t^{3\alpha+2\beta}\Gamma(2\alpha+2\beta+1)\Gamma(\alpha+\beta+1)}{\Gamma(\alpha+1)(\Gamma(\beta+1))^2\Gamma(2\alpha+\beta+1)\Gamma(3\alpha+2\beta+1)}-2P\frac{t^{2\alpha+\beta+\gamma}\Gamma(\alpha+\beta+\gamma+1)}{\Gamma(\beta+1)\Gamma(\alpha+\gamma+1)\Gamma(2\alpha+\beta+\gamma+1)}$$

$$+0.132AH\frac{t^{3\alpha+\beta}\Gamma(2\alpha+\beta+1)}{\Gamma(\alpha+1)\Gamma(\alpha+\beta+1)\Gamma(3\alpha+\beta+1)}+0.132AK\frac{t^{4\alpha+\beta}\Gamma(3\alpha+\beta+1)}{\Gamma(2\alpha+1)\Gamma(\alpha+\beta+1)\Gamma(4\alpha+\beta+1)}$$

$$-0.4842A^3\frac{t^{5\alpha+\beta}\Gamma(4\alpha+\beta+1)\Gamma(2\alpha+1)}{(\Gamma(\alpha+1))^2\Gamma(3\alpha+1)\Gamma(\alpha+\beta+1)\Gamma(5\alpha+\beta+1)}+0.132AJ\frac{t^{3\alpha+2\beta}\Gamma(2\alpha+2\beta+1)}{(\Gamma(\alpha+\beta+1))^2\Gamma(3\alpha+2\beta+1)}$$

$$-0.264A^2B\frac{t^{4\alpha+2\beta}\Gamma(3\alpha+2\beta+1)}{\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(2\alpha+\beta+1)\Gamma(4\alpha+2\beta+1)}$$

$$+0.132AP\frac{t^{3\alpha+\beta+\gamma}\Gamma(2\alpha+\beta+\gamma+1)}{\Gamma(\alpha+\beta+1)\Gamma(\alpha+\gamma+1)\Gamma(3\alpha+\beta+\gamma+1)}$$

$$-0.33ABG\frac{t^{4\alpha+\beta+\gamma}\Gamma(3\alpha+\beta+\gamma+1)\Gamma(\alpha+\gamma+1)}{\Gamma(\alpha+1)\Gamma(\gamma+1)\Gamma(2\alpha+\gamma+1)\Gamma(\alpha+\beta+1)\Gamma(4\alpha+\beta+\gamma+1)}$$

$$-0.8BH\frac{t^{2\alpha+2\beta}\Gamma(\alpha+2\beta+1)}{\Gamma(\alpha+1)\Gamma(2\beta+1)\Gamma(2\alpha+2\beta+1)}-0.8BK\frac{t^{3\alpha+2\beta}\Gamma(2\alpha+2\beta+1)}{\Gamma(2\alpha+1)\Gamma(2\beta+1)\Gamma(3\alpha+2\beta+1)}$$

$$+2.9348A^2B\frac{t^{4\alpha+2\beta}\Gamma(3\alpha+2\beta+1)\Gamma(2\alpha+1)}{(\Gamma(\alpha+1))^2\Gamma(3\alpha+1)\Gamma(2\beta+1)\Gamma(4\alpha+2\beta+1)}$$

$$-0.8BJ\frac{t^{2\alpha+3\beta}\Gamma(\alpha+3\beta+1)}{\Gamma(\alpha+\beta+1)\Gamma(2\beta+1)\Gamma(2\alpha+3\beta+1)}$$

$$+1.6AB^2\frac{t^{3\alpha+3\beta}\Gamma(2\alpha+3\beta+1)\Gamma(\alpha+\beta+1)}{\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(2\beta+1)\Gamma(2\alpha+\beta+1)\Gamma(3\alpha+3\beta+1)}$$

$$-0.8BP\frac{t^{2\alpha+2\beta+\gamma}\Gamma(\alpha+2\beta+\gamma+1)}{\Gamma(\alpha+\gamma+1)\Gamma(2\beta+1)\Gamma(2\alpha+2\beta+\gamma+1)}$$

$$+2B^2G\frac{t^{3\alpha+2\beta+\gamma}\Gamma(\alpha+\gamma+1)\Gamma(2\alpha+2\beta+\gamma+1)}{\Gamma(\alpha+1)\Gamma(\gamma+1)\Gamma(2\beta+1)\Gamma(2\alpha+\gamma+1)\Gamma(3\alpha+2\beta+\gamma+1)}$$

$$-1.4(L\frac{t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}-0.066A\frac{t^{2\alpha+\beta}}{\Gamma(2\alpha+\beta+1)}+0.4B\frac{t^{\alpha+2\beta}}{\Gamma(\alpha+2\beta+1)})$$

$$-0.875(M\frac{t^{\alpha+\gamma}}{\Gamma(\alpha+\gamma+1)}+N\frac{t^{2\alpha+\gamma}}{\Gamma(2\alpha+\gamma+1)}+0.375A^2\frac{t^{3\alpha+\gamma}\Gamma(2\alpha+1)}{(\Gamma(\alpha+1))^2\Gamma(3\alpha+\gamma+1)})$$

$$-2.5M^2\frac{t^{\alpha+2\gamma}\Gamma(2\gamma+1)}{(\Gamma(\gamma+1))^2\Gamma(\alpha+2\gamma+1)}-2.5N^2\frac{t^{3\alpha+2\gamma}\Gamma(2\alpha+2\gamma+1)}{(\Gamma(\alpha+\gamma+1))^2\Gamma(3\alpha+2\gamma+1)}$$
$$-0.3516A^4\frac{t^{5\alpha+2\gamma}\Gamma(4\alpha+2\gamma+1)(\Gamma(2\alpha+1))^2}{(\Gamma(\alpha+1))^4(\Gamma(2\alpha+\gamma+1))^2\Gamma(5\alpha+2\gamma+1)}-5MN\frac{t^{2\alpha+2\gamma}\Gamma(\alpha+2\gamma+1)}{\Gamma(\gamma+1)\Gamma(\alpha+\gamma+1)\Gamma(2\alpha+2\gamma+1)}$$
$$-1.875A^2N\frac{t^{4\alpha+2\gamma}\Gamma(3\alpha+2\gamma+1)\Gamma(2\alpha+1)}{\Gamma(\alpha+\gamma+1)(\Gamma(\alpha+1))^2\Gamma(2\alpha+\gamma+1)\Gamma(4\alpha+2\gamma+1)}$$
$$-1.875A^2M\frac{t^{3\alpha+2\gamma}\Gamma(2\alpha+1)\Gamma(2\alpha+2\gamma+1)}{\Gamma(\gamma+1)(\Gamma(\alpha+1))^2\Gamma(2\alpha+\gamma+1)\Gamma(3\alpha+2\gamma+1)},$$

$$W_3'=0.189\frac{t^{\beta}}{\Gamma(\beta+1)}+0.666H\frac{t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}+0.666K\frac{t^{2\alpha+\beta}}{\Gamma(2\alpha+\beta+1)}$$
$$-2.4432A^2\frac{t^{3\alpha+\beta}\Gamma(2\alpha+1)}{(\Gamma(\alpha+1))^2\Gamma(3\alpha+\beta+1)}+0.666J\frac{t^{\alpha+2\beta}}{\Gamma(\alpha+2\beta+1)}-1.332AB\frac{t^{2\alpha+2\beta}\Gamma(\alpha+\beta+1)}{\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(2\alpha+2\beta+1)}$$
$$+0.666P\frac{t^{\alpha+\beta+\gamma}}{\Gamma(\alpha+\beta+\gamma+1)}-1.665BG\frac{t^{2\alpha+\beta+\gamma}\Gamma(\alpha+\gamma+1)}{\Gamma(\alpha+1)\Gamma(\gamma+1)\Gamma(2\alpha+\beta+\gamma+1)}$$
$$-0.4(L\frac{t^{2\beta}}{\Gamma(2\beta+1)}-0.066A\frac{t^{\alpha+2\beta}}{\Gamma(\alpha+2\beta+1)}+0.4B\frac{t^{3\beta}}{\Gamma(3\beta+1)}),$$

$$Z_3'=0.01071\frac{t^{\gamma}}{\Gamma(\gamma+1)}+0.15H\frac{t^{\alpha+\gamma}}{\Gamma(\alpha+\gamma+1)}+0.15K\frac{t^{2\alpha+\gamma}}{\Gamma(2\alpha+\gamma+1)}$$
$$-0.5503A^2\frac{t^{3\alpha+\gamma}\Gamma(2\alpha+1)}{(\Gamma(\alpha+1))^2\Gamma(3\alpha+\gamma+1)}+0.15J\frac{t^{\alpha+\beta+\gamma}}{\Gamma(\alpha+\beta+\gamma+1)}-0.3AB\frac{t^{2\alpha+\beta+\gamma}\Gamma(\alpha+\beta+1)}{\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(2\alpha+\beta+\gamma+1)}$$
$$+0.15P\frac{t^{\alpha+2\gamma}}{\Gamma(\alpha+2\gamma+1)}-0.375BG\frac{t^{2\alpha+2\gamma}\Gamma(\alpha+\gamma+1)}{\Gamma(\alpha+1)\Gamma(\gamma+1)\Gamma(2\alpha+2\gamma+1)}$$
$$+0.375H^2\frac{t^{2\alpha+\gamma}\Gamma(2\alpha+1)}{(\Gamma(\alpha+1))^2\Gamma(2\alpha+\gamma+1)}+0.375K^2\frac{t^{4\alpha+\gamma}\Gamma(4\alpha+1)}{(\Gamma(2\alpha+1))^2\Gamma(4\alpha+\gamma+1)}$$
$$+5.0467A^4\frac{t^{6\alpha+\gamma}\Gamma(6\alpha+1)(\Gamma(2\alpha+1))^2}{(\Gamma(\alpha+1))^4(\Gamma(3\alpha+1))^2\Gamma(6\alpha+\gamma+1)}+0.75HK\frac{t^{3\alpha+\gamma}\Gamma(3\alpha+1)}{\Gamma(\alpha+1)\Gamma(2\alpha+1)\Gamma(3\alpha+\gamma+1)}$$
$$-2.7514A^2K\frac{t^{5\alpha+\gamma}\Gamma(5\alpha+1)}{(\Gamma(\alpha+1))^2\Gamma(3\alpha+1)\Gamma(5\alpha+\gamma+1)}$$
$$-2.7514A^2H\frac{t^{4\alpha+\gamma}\Gamma(2\alpha+1)\Gamma(4\alpha+1)}{(\Gamma(\alpha+1))^3\Gamma(3\alpha+1)\Gamma(4\alpha+\gamma+1)}+0.375J^2\frac{t^{2\alpha+2\beta+\gamma}\Gamma(2\alpha+2\beta+1)}{(\Gamma(\alpha+\beta+1))^2\Gamma(2\alpha+2\beta+\gamma+1)}$$
$$+1.5A^2B^2\frac{t^{4\alpha+2\beta+\gamma}\Gamma(4\alpha+2\beta+1)(\Gamma(\alpha+\beta+1))^2}{(\Gamma(\alpha+1))^2(\Gamma(\beta+1))^2(\Gamma(2\alpha+\beta+1))^2\Gamma(4\alpha+2\beta+\gamma+1)}$$
$$-1.5ABJ\frac{t^{3\alpha+2\beta+\gamma}\Gamma(3\alpha+2\beta+1)}{\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(2\alpha+\beta+1)\Gamma(3\alpha+2\beta+\gamma+1)}+0.375P^2\frac{t^{2\alpha+3\gamma}\Gamma(2\alpha+2\gamma+1)}{(\Gamma(\alpha+\gamma+1))^2\Gamma(2\alpha+3\gamma+1)}$$
$$+2.3438B^2G^2\frac{t^{4\alpha+3\gamma}\Gamma(4\alpha+2\gamma+1)(\Gamma(\alpha+\gamma+1))^2}{(\Gamma(\alpha+1))^2(\Gamma(\gamma+1))^2(\Gamma(2\alpha+\gamma+1))^2\Gamma(4\alpha+3\gamma+1)}$$
$$-1.875BGP\frac{t^{3\alpha+3\gamma}\Gamma(3\alpha+2\gamma+1)}{\Gamma(\alpha+1)\Gamma(\gamma+1)\Gamma(2\alpha+\gamma+1)}+0.75JP\frac{t^{2\alpha+\beta+2\gamma}\Gamma(2\alpha+\beta+\gamma+1)}{\Gamma(\alpha+\beta+1)\Gamma(\alpha+\gamma+1)\Gamma(2\alpha+\beta+2\gamma+1)}$$
$$-1.5ABP\frac{t^{3\alpha+\beta+2\gamma}\Gamma(3\alpha+\beta+\gamma+1)\Gamma(\alpha+\beta+1)}{\Gamma(\alpha+\gamma+1)\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(2\alpha+\beta+1)\Gamma(3\alpha+\beta+2\gamma+1)}$$
$$-1.875BGJ\frac{t^{3\alpha+\beta+2\gamma}\Gamma(3\alpha+\beta+\gamma+1)\Gamma(\alpha+\gamma+1)}{\Gamma(\alpha+\beta+1)\Gamma(\alpha+1)\Gamma(\gamma+1)\Gamma(2\alpha+\gamma+1)\Gamma(3\alpha+\beta+2\gamma+1)}$$

$$+1.875AB^2G\frac{t^{4\alpha+\beta+2\gamma}\Gamma(4\alpha+\beta+\gamma+1)\Gamma(\alpha+\gamma+1)\Gamma(\alpha+\beta+1)}{(\Gamma(\alpha+1))^2\Gamma(\gamma+1)\Gamma(\beta+1)\Gamma(2\alpha+\beta+1)\Gamma(2\alpha+\gamma+1)\Gamma(4\alpha+\beta+2\gamma+1)}$$

$$+0.75HJ\frac{t^{2\alpha+\beta+\gamma}\Gamma(2\alpha+\beta+1)}{\Gamma(\alpha+1)\Gamma(\alpha+\beta+1)\Gamma(2\alpha+\beta+\gamma+1)}+0.75KJ\frac{t^{3\alpha+\beta+\gamma}\Gamma(3\alpha+\beta+1)}{\Gamma(2\alpha+1)\Gamma(\alpha+\beta+1)\Gamma(3\alpha+\beta+\gamma+1)}$$

$$-2.7514A^2J\frac{t^{4\alpha+\beta+\gamma}\Gamma(2\alpha+1)\Gamma(4\alpha+\beta+1)}{(\Gamma(\alpha+1))^2\Gamma(3\alpha+1)\Gamma(\alpha+\beta+1)\Gamma(4\alpha+\beta+\gamma+1)}$$

$$-1.5ABH\frac{t^{3\alpha+\beta+\gamma}\Gamma(3\alpha+\beta+1)\Gamma(\alpha+\beta+1)}{(\Gamma(\alpha+1))^2\Gamma(\beta+1)\Gamma(2\alpha+\beta+1)\Gamma(3\alpha+\beta+\gamma+1)}$$

$$-1.5ABK\frac{t^{4\alpha+\beta+\gamma}\Gamma(4\alpha+\beta+1)\Gamma(\alpha+\beta+1)}{\Gamma(2\alpha+1)\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(2\alpha+\beta+1)\Gamma(4\alpha+\beta+\gamma+1)}$$

$$+5.5028A^3B\frac{t^{5\alpha+\beta+\gamma}\Gamma(5\alpha+\beta+1)\Gamma(2\alpha+1)\Gamma(\alpha+\beta+1)}{(\Gamma(\alpha+1))^3\Gamma(3\alpha+1)\Gamma(\beta+1)\Gamma(2\alpha+\beta+1)\Gamma(5\alpha+\beta+\gamma+1)}$$

$$+0.75HP\frac{t^{2\alpha+2\gamma}\Gamma(2\alpha+\gamma+1)}{\Gamma(\alpha+1)\Gamma(\alpha+\gamma+1)\Gamma(2\alpha+\gamma+1)}+0.75KP\frac{t^{3\alpha+2\gamma}\Gamma(3\alpha+\gamma+1)}{\Gamma(2\alpha+1)\Gamma(\alpha+\gamma+1)\Gamma(3\alpha+2\gamma+1)}$$

$$-2.7514A^2P\frac{t^{4\alpha+2\gamma}\Gamma(4\alpha+\gamma+1)\Gamma(2\alpha+1)}{(\Gamma(\alpha+1))^2\Gamma(3\alpha+1)\Gamma(\alpha+\gamma+1)\Gamma(4\alpha+2\gamma+1)}$$

$$-1.875BGH\frac{t^{3\alpha+2\gamma}\Gamma(3\alpha+\gamma+1)\Gamma(\alpha+\gamma+1)}{(\Gamma(\alpha+1))^2\Gamma(\gamma+1)\Gamma(2\alpha+\gamma+1)\Gamma(3\alpha+2\gamma+1)}$$

$$-6.8784A^2BG\frac{t^{5\alpha+2\gamma}\Gamma(5\alpha+\gamma+1)\Gamma(2\alpha+1)\Gamma(\alpha+\gamma+1)}{(\Gamma(\alpha+1))^3\Gamma(3\alpha+1)\Gamma(\gamma+1)\Gamma(2\alpha+\gamma+1)\Gamma(5\alpha+2\gamma+1)}$$

$$-1.875BGK\frac{t^{4\alpha+2\gamma}\Gamma(4\alpha+\gamma+1)\Gamma(\alpha+\gamma+1)}{\Gamma(\alpha+1)\Gamma(\gamma+1)\Gamma(2\alpha+1)\Gamma(2\alpha+\gamma+1)\Gamma(4\alpha+2\gamma+1)}.$$