Article

The predator-prey models for the mechanism of autocatalysis, pair wise interactions and movements to free places

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Abstract
In this paper we aim to develop the modeled equations for different types of mechanism of the predator-prey interactions with the help of a quasi chemical approach while taking a special study case of foxes and rabbits, these mechanisms include autocatalysis mechanism, pair wise interactions and the mechanism of their movements to some free places. The chemical reactions representing the interactions obey the mass action law. The territorial animal like fox is assigned a simple cell as its territory. Under the proper relations between coefficients, this system may demonstrate globally stable dynamics.

Keywords predator prey interactions; autocatalysis mechanism; modeling of fox and rabbit interaction.

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1 Introduction

We have introduced the quasi chemical approach to represent the different types of mechanisms between the predator and its prey by taking a case study of foxes and rabbits and in the second publication we determined the modeled equations for these mechanisms including the mechanism of circulation, mechanisms of attraction and repulsion and mechanism of sharing place.

The Lotka Voltera equations are a pair of first order non linear differential equations, and these are also known as the predator prey equations .i.e. when growth rate of one population is decreased and the other increased then these populations are said to be in a predator-prey situation. The Lotka–Volterra equations are frequently used to describe the dynamics of biological system in which two species interact, one as a predator and the other as a prey (Murray, 2003).

A biologist Humberto D’Ancona was doing a statistical study in the 1920’s, on the numbers of each species sold at three main Italian ports. During his study from 1914-1923 of these fish species, he came across a surprising conclusion. He believed that these predator prey relationships between the sharks, rays, and fish were in their natural states outside of human interaction, namely before and after the war. D'Ancona asked his father in-law, Vito Volterra, a very successful mathematician, to analyze the data and draw some conclusions. Volterra spent weeks developing a series of models for interactions of two or more species, the first and simplest of these is the subject of this development (Volterra, 1926).

Alfred J.Lotka was an American mathematical biologist who discovered and development many of the same conclusions and models as Volterra and around the same time. His key study was on a species of plant and an herbivores species which relied heavily on that plant as a feeding source.

The Lotka Volterra predator prey models were originally introduced by Alfred J. Lotka in the theory of autocatalytic chemical reactions. This was effectively the logistic equation which was originally derived by Pierre Francois Verhulst. In 1920 Lotka extended the model to "organic systems" using a plant species and an herbivorous animal species as an example and in 1925 he utilized the equations to analyze predator-prey interactions in his book on biomathematics arriving at the equations that we know today (Lotka 1920).

In 1926 Vito Volterra (Volterra, 1926), made a statistical analysis of fish catches in the Adriatic independently investigated the equations. V. Voltera applied these equations to predator prey interactions; consist of a pair of first order autonomous ordinary differential equations.

Since that time the Lotka Voltera model has been applied to problems in chemical kinetics, population biology, epidemiology and neural networks. These equations also model the dynamic behavior of an arbitrary number of competitors. The Lotka–Volterra system of equations is an example of a Kolmogorov model, which is a more general framework that can model the dynamics of ecological systems with predator-prey interactions, competition, disease, and mutualism.

Lotka Volterra is the most famous predator prey model. According to Volterra if $x(t)$ is the prey population and $y(t)$ that of the predator at time t then Volterra's model is (Murray, 2003),

$$\frac{dx}{dt} = x(a-by); \tag{1}$$

$$\frac{dy}{dt} = y(cx-d). \tag{2}$$

Here *a*, *b*, *c* and *d* are positive constants.

The reaction diffusion system is a mathematical model which clarifies how the concentrations of one or more substances which are distributed in the space, change under the influence of two processes. In the local chemical reactions, the substances are transformed into each other and the diffusion occur which causes the substances to spread out over a surface in space. Such systems are naturally applied in chemistry, however; the system can also explain dynamical processes of non chemical nature. Mathematically, the reaction diffusion system takes the form of the semi-linear parabolic partial differential equation.

2 A Quasi-chemical Approach

In the existence of hundreds of different animals on the island, the modeling of the interaction among these animals is not an easy task. However, the interaction between two species is possible by taking one as a predator and the other as a prey. The same single population growth model can be used for predators (foxes), if prey (rabbits) is considered as an unlimited food supply to foxes. But Foxes are territorial animals, and in general each fox claims its own territory by marking their territories with signals that other foxes will recognize e. g., by leaving their droppings in prominent positions but they also pair up in winter season. So it will be difficult for the foxes to cover the whole island without defining their own territories.

It is a constant battle among the animals on a daily basis to survive. They have to find their food and avoid becoming a food. These species can be divide up on the basis that how they get their food, if they provide it they are producers (prey), if they need to find it they are consumers (predator) and if they breakdown dead material they are decomposers. This correlation between predator and prey intertwines into a complex food chain. In the ecosystem there are various species and every one play a significant role in maintaining populations near the carrying capacity and in keeping the system in balance. In the ecosystems Predator and prey assist them through particular adaptations to compete for food resources. Predator and prey populations are directly related and they cannot survive without each other. Here this relationship is illustrated by using foxes and rabbits.

In artificial world and in the social systems non-linear behaviors are the order of the day. Rabbits and foxes share some behaviors but vary in others and their interaction is non-linear in nature. The rabbits are proverbially very good in reproduction. In case when more rabbits are there, the more foxes will prey on them, so that the number of foxes will increase, when more foxes will predate a lot such that the number of rabbits start to decrease, swiftly it follows by falling fox numbers as there are a small amount of rabbits to sustain them therefore it is observed that the populations of rabbits and foxes are locked together in an interactive population 'dance.' This system is non-linear in its behavior. In mathematics non-linear simultaneous equations are used to represent these interactions for which there exist an infinite number of solutions.

Between the fox and rabbit populations here is a complicated and natural predator-prey relationship, since rabbits thrive in the absence of foxes and foxes thrive in the presence of rabbits. Foxes like to occupy a combination of forest and open fields. Foxes usually define their territory zones and use the transition zone or "edge" between these habitats as hunting areas. Foxes do not interfere in the defined areas of each others.

The cell-jump models may be considered as the proper diffusion models by themselves, the territorial animal like fox is given a simple cell as its territory. The sense in which the discrete equations for cells converge to the partial differential equations of diffusion is that the cell models give the semi-discrete approximation of the partial differential equations for diffusion which result in a system of ordinary differential equations in cells. There are jumps of concentrations on the boundary of cells. The system of semi discrete models for cells with “no-flux” boundary conditions has all the nice properties of the chemical kinetic equations for closed systems. Under the proper relations between coefficients, like complex balance or detailed balance, this system demonstrated globally stable dynamics. Here we specially refer this modeling by Gorban et al. (2011) who developed the idea and modeled diffusion equations for different mechanisms and proved the dissipation inequalities.

We have introduced the quasi chemical approach to represent the different types of mechanisms between the predator and its prey by taking a case study of foxes and rabbits and in the second publication we determined the modeled equations for these mechanisms including the mechanism of circulation, mechanisms of attraction and repulsion and mechanism of sharing place (Shakil et al., 2015). Here we briefly discuss the assumptions again for convenience of the readers.

Let us suppose that our space is divided on two territories, a system represented as a chain of territories each with homogeneous composition and elementary acts of transfers on the boundary (for us there are only two territories). On each territory we have some concentrations (number of foxes and rabbits) for these processes. Let F^{I} (fox/ foxes in first territory) and R^{I} (rabbit/ rabbits in first territory) is the vector concentration in the first territory *I* and F^{II} (fox/ foxes in second territory) and R^{II} (rabbit/ rabbits in second territory) is the vector of concentration in the second territory *II*. Then in our study case the following mechanisms are possible between the predator (foxes) and its prey (rabbits).

3 Pairwise Interaction

3.1 Here in pair wise interactions the movement of rabbits and foxes is being taken together. If a fox and a rabbit R_i are present in territory one and a fox and a rabbit R_j is in the territory two. As the foxes are territorial while rabbits move freely in nature and rabbits change their territories regularly while foxes do not do so. So if rabbit R_i moves from territory one to territory two and rabbit R_j moves from territory two to territory one then it is given by stoichiometric equations of the form:

$$(F^I + R_i^I) + (F^{II} + R_j^{II}) \rightarrow (F^I + R_j^I) + (F^{II} + R_i^{II}) \qquad (3)$$

This movement will not disturb the system and there will be a normal and usual interaction between the predator and its prey.

Let us suppose that c^I and c^{II} be the concentrations of foxes in the first and the second cell respectively, γ_F^I and γ_F^{II} be the stoichiometric vectors in the respective cells, w_r^I and w_r^{II} be the reactions rates in the respective cells and J_F be the total flux.

The mass action law is applicable here for the above mechanisms and we will calculate the stoichiometric vectors, reaction rates, vector of total flux and their modeled equations as,

$$\gamma_F^I = 0,\ \gamma_{R_i}^I = -1,\ \gamma_{R_j}^I = 1$$

$$w_r^I = kc^I c^{II} d_i^I d_j^{II},\ w_r^{II} = kc^I c^{II} d_j^I d_i^{II}$$

Total flux will be calculated as, $J = w_r^{II} - w_r^I$,

$$J = kc^I c^{II} d_j^I d_i^{II} - kc^I c^{II} d_i^I d_j^{II},$$

$$J = kc^I c^{II} \left[d_j^I d_i^{II} - d_i^I d_j^{II} \right],$$

$$J = kc^I \left(c^I + l\nabla c \right)\left[d_j^I \left(d_i^I + l\nabla d_i \right) - d_i^I \left(d_j^I + l\nabla d_j \right) \right],$$

First order approximation will be,

$$J = klc^2 \left[d_j \nabla d_i - d_i \nabla d_j \right],$$

$$J = Dc^2 \left[d_j \nabla d_i - d_i \nabla d_j \right], \tag{4}$$

In this approximation, $kl = D$, where l is the cell size.

Its modeled equation will be, $\dfrac{\partial c}{\partial t} = -div\left(J\right)$

$$\frac{\partial c}{\partial t} = -Dc^2 \left[d_j \Delta d_i - d_i \Delta d_j \right]. \tag{5}$$

3.2 Now if rabbit from territory two moves to territory one then in this case both the rabbits will be present in territory one along with the fox of that territory, then for this situation the possible stoichiometric equation will be:

$$(F^I + R^I) + (F^{II} + R^{II}) \rightarrow (F^I + 2R^I) + F^{II} \tag{6}$$

For the above mechanism,

$$\gamma_F^I = 0, \ \gamma_R^I = 1.$$

$$w_r^I = kc^I c^{II} d^I d^{II}, \ w_r^{II} = kc^I c^{II} \left(d^I\right)^2.$$

Total flux will be, $J = kc^I c^{II} \left(d^I\right)^2 - kc^I c^{II} d^I d^{II}$,

$$J = -kc^I c^{II} \left[d^I \left(d^{II} - d^I \right) \right],$$

$$J = -klc^I d^I \left(c^I + l\nabla c \right) \left[\frac{d^{II} - d^I}{l} \right],$$

$$J = -klc^I d^I \left(c^I + l\nabla c \right) \nabla d,$$

First order approximation will be,

$$J = -Dc^2 d\nabla d. \tag{7}$$

And its modeled equation will be,

$$\frac{\partial c}{\partial t} = Dc^2 d\Delta d. \tag{8}$$

3.3 Now if rabbit from territory one moves to territory two, then in this case both the rabbits will be present in territory two along with the fox of that territory, then for this situation the possible stoichiometric equation will be:

$$(F^I + R^I) + (F^{II} + R^{II}) \to F^I + (F^{II} + 2R^{II}). \tag{9}$$

For the above mechanism,

For the above mechanism,

$$\gamma_F^I = 0,\ \gamma_R^I = -1,\ w_r^I = kc^I c^{II} d^I d^{II},\ w_r^{II} = kc^I c^{II} \left(d^{II}\right)^2.$$

Total flux will be, $J = kc^I c^{II} \left(d^{II}\right)^2 - kc^I c^{II} d^I d^{II}$,

$$J = klc^I c^{II} d^{II} \left[\frac{d^{II} - d^I}{l}\right],$$

$$J = klc^I \left(c^I + l\nabla c\right)\left(d^I + l\nabla d\right)\nabla d,$$

Its first order approximation will be,

$$J = Dc^2 d\nabla d. \tag{10}$$

And its modeled equation will be,

$$\frac{\partial c}{\partial t} = -Dc^2 d\Delta d. \tag{11}$$

4 Movements to Free Places

4.1 Case study of foxes

If fox/foxes is/are in territory one/ two and due to some reasons its neighbouring territory *(Z)* is laying vacant, if this fox/foxes moves to that vacant territory (as it is stated that foxes are territorial animal and they stay in their own territory zones so it may or may not be possible in real) then for this situation it is given by stoichiometric equations of the form:

$$F^{I} + Z^{II} \rightarrow F^{II} + Z^{II}$$
$$F^{II} + Z^{I} \rightarrow F^{I} + Z^{I} \quad (12)$$

For the mechanism $F^{I} + Z^{II} \rightarrow F^{II} + Z^{II}$

$$\gamma_F^I = -1, \ \gamma_z^I = 0, \ w_r^I = kc^I z^{II}.$$

Where z is supposed to be the concentration of free zones.

For the mechanism $F^{II} + Z^{I} \rightarrow F^{I} + Z^{I}$

$$\gamma_F^I = 1, \ \gamma_z^I = 0, \ w_r^{II} = kc^{II} z^{I}.$$

Total flux will be, $J_F = kc^{II} z^I - kc^I z^{II}$.

$$J_F = k\left[(c^I + l\nabla c)z^I - c^I\left(z^I + l\nabla z\right)\right] = kl\left[z\nabla c - c\nabla z\right].$$

$$J_F = D\left[z\nabla c - c\nabla z\right]. \quad (13)$$

Its modeled equation will be

$$\frac{\partial c}{\partial t} = D\left[c\Delta z - z\Delta c\right]. \quad (14)$$

4.2 Case study of rabbits

If rabbit/rabbits is/are in territory one/ two and due to some reasons its neighbouring territory is laying vacant, if this rabbit/rabbits moves to that vacant territory, then for this situation it is given by stoichiometric equations of the form:

$$R^{I} + Z^{II} \rightarrow R^{II} + Z^{II}$$
$$R^{II} + Z^{I} \rightarrow R^{I} + Z^{I} \quad (15)$$

For the mechanism $R^I + Z^{II} \rightarrow R^{II} + Z^{II}$

$$\gamma_R^I = 1,\ \gamma_z^I = 0,\ w_r^{II} = kd^{II} z^I.$$

Total flux will be, $J_R = D\left[z\nabla d - d\nabla z\right].$ (16)

Its modeled equation will be, $\dfrac{\partial d}{\partial t} = D\left[d\Delta z - z\Delta d\right].$ (17)

4.3 Mixed study case for foxes and rabbits

If a fox/foxes and rabbit/rabbits are in territory one/ two and due to some reasons their neighbouring territory is laying vacant, if all the foxes/ fox and rabbit/rabbits moves to that vacant territory, then for this situation it is given by stoichiometric equations of the form:

$$\begin{aligned} &F^I + R^I + Z^{II} \rightarrow F^{II} + R^{II} + Z^{II} \\ &F^{II} + R^{II} + Z^I \rightarrow F^I + R^I + Z^I \end{aligned} \tag{18}$$

For the mechanism $F^I + R^I + Z^{II} \rightarrow F^{II} + R^{II} + Z^{II}$.

$$\gamma_F^I = -1,\ \gamma_R^I = -1,\ \gamma_z^I = 0,\ w_r^I = kc^I d^I z^{II}.$$

For the mechanism $F^{II} + R^{II} + Z^I \rightarrow F^I + R^I + Z^I$,

$$\gamma_F^I = 1,\ \gamma_R^I = 1,\ \gamma_z^I = 0,\ w_r^I = kc^{II} d^{II} z^I.$$

Total flux will be, $J = k\left[c^{II} d^{II} z^I - c^I d^I z^{II}\right]$,

$$J = k\left[\left(c^I + l\nabla c\right)\left(d^I + l\nabla d\right)z^I - c^I d^I\left(z^I + l\nabla z\right)\right],$$

$$J = kl\left[zc\nabla d + zd\nabla c - cd\nabla z + zl\nabla c\nabla d\right],$$

Its first order approximation will be,

$$J = D\left[c\left(z\nabla d - d\nabla z\right) + zd\nabla c\right]. \tag{19}$$

Its modeled equation will be,

$$\frac{\partial c}{\partial t} = -D\left[c\left(z\Delta d - d\Delta z\right) + zd\Delta c\right]. \quad (20)$$

5 Autocatalysis Mechanisms

Autocatalysis is the procedure whereby a chemical is involved in its own production. Here in our study case we will see that how two foxes or two rabbits interact together to increase their populations.

5.1 Case study of foxes

As it is stated in the early lines that the foxes are the territorial animals and they define their own territories to prey, but they also pair up in winter seasons. when in winter seasons two foxes pair up in territory one/two together, let a male fox $F_{(m)}$ and a female fox $F_{(f)}$ interact, let n be the number of foxes which born as a result of that interaction, they all can be male foxes, females or a combination of both the sexes, this circle repeats itself. In this case the possible mechanism will be of the form,

$$\begin{aligned} &2F^{I} \to n_1 F^{I}, n_1 > 2 \\ &2F^{II} \to n_2 F^{II}, n_2 > 2 \end{aligned} \quad (21)$$

For the mechanism $2F^{I} \to n_1 F^{I}$

$$\gamma_F^I = n_1 - 2,\ \ w_r^I = k\left(c^I\right)^2.$$

For the mechanism $2F^{II} \to n_2 F^{II}$

$$\gamma_F^I = 0,\ \ w_r^{II} = k\left(c^{II}\right)^2.$$

Total flux will be, $J_F = k\left(c^{II}\right)^2 - k\left(c^I\right)^2 = k\left[\left(c^{II} - c^I\right)\left(c^{II} + c^I\right)\right].$

$$J_F = kl\left[\left(\frac{c^{II} - c^I}{l}\right)\left(c^I + l\nabla c + c^I\right)\right] = D\nabla c\left(2c + l\nabla c\right).$$

Its first order approximation will be, $J_F = 2cD\nabla c.$ (22)

Its modeled equation for this case will be, $\frac{\partial c}{\partial t} = -2cD\Delta c.$ (23)

5.2 Case study of rabbits

Now in the study case of rabbits when two rabbits are in territory one /two, let a male rabbit $R_{(m)}$ and a female rabbit $R_{(f)}$ interact, let n be the number of rabbits which born as a result of that interaction, they all can be male rabbits, females or a combination of both the sexes, this circle repeats itself. In this case the possible mechanism will be of the form,

$$\begin{aligned} &2R^{I} \to n_3 R^{I}, n_3 > 2 \\ &2R^{II} \to n_4 R^{II}, n_4 > 2 \end{aligned} \tag{24}$$

For the mechanism $2R^{I} \to n_3 R^{I}$

$$\gamma_R^I = n_3 - 2,\ \ w_r^I = k\left(d^{\,I}\right)^2.$$

For the mechanism $2R^{I} \to n_4 R^{I}$

$$\gamma_R^I = n_4 - 2,\ \ w_r^{II} = k\left(d^{\,II}\right)^2.$$

Total flux will be, $J_R = 2dD\nabla c.$ (25)

Modeled equation for this case will be, $\dfrac{\partial d}{\partial t} = -2dD\Delta d.$ (26)

6 The Observations

Thus the discrete equations for cells converge result in a system of ODEs in cells. The jumps of concentrations on the boundary of cells are considered the discrete system of ODEs for cells produces the second order in the cell size approximation to the continuous diffusion equation These jumps are flows in both directions: from cell 1 to cell II and from cell II to cell I. Different types of mechanism were considered here and obtained the modeled equations for the predator prey interactions. These jumps or the integrations are supposed to obey the mass action law. The kinetic constants may be scaled with the cell size to keep $kl = d$ (k is the kinetic constant, l is the cell size, d is the diffusion coefficient for the particular mechanism). It seems a good hypothesis that for obtained modeled systems with convex Lyabunov functional and “no-flux” boundary conditions in bounded areas with smooth boundaries the cell model gives the uniform approximation to the solution of the correspondent PDE. The system of semi-discrete models for cells with “no-flux” boundary conditions has all the nice properties of the chemical kinetic equations for closed systems. Under the proper relations between coefficients, like complex balance or detailed balance, this system may demonstrate a globally stable dynamics. This global stability property can help with the study of the related PDE.

7 Future Work

In the future, we aim:

- To build up a concise study of complex biological systems and to tackle some key research questions concerning our study case by proposing some new techniques and algorithms that are inspired by those complex biological systems.
- To check the stability of all the mechanisms discussed in our study case.
- To find the solutions of the modeled equations determined for all these interactions.
- To check the existence and uniqueness of those solutions.

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