Article

Stability, bifurcation and chaos control in a discrete-time prey-predator model with Holling type-II response

Muhammad Salman Khan[1], **Muhammad Asif Khan**[2], **Muhammad Sajjad Shabbir**[3], **Qamar Din**[2]
[1]Department of Mathematics, Quaid-I-Azam University Islamabad, 44230, Pakistan
[2]Department of Mathematics, University of the PoonchRawalakot, Rawalakot 12350, Pakistan
[3]Department of Mathematics, Air University Islamabad, 44230, Pakistan
E-mail: mskhan@math.qau.edu.pk, asif31182@gmail.com, sajjadmust@g mail.com, qamar.sms@gmail.com

Received 11 May 2019; *Accepted 10 June 2019*; *Published 1 September 2019*

Abstract
This paper deals with the qualitative study of a discrete-time prey-predator model with Holling type-II response. Particularly, we obtained a dynamically consistent prey-predator discrete-time model by applying a nonstandard difference scheme. We explore the novel explicate parametric conditions for the local stability of positive equilibrium point. Moreover, it is shown that there exists Nimark-sacker bifurcation for the unique positive steady-state of given system. In order to control the bifurcation, we introduced a new control strategy. Moreover, some interesting numerical simulations are provided in order to verify the theoretical discussion and to explore the effectiveness and feasibility of new design control strategy.

Keywords prey-predator model; Holling type-II response; nonstandard difference scheme; Nimark-sacker bifurcation; comparison of control techniques.

Network Biology
ISSN 2220-8879
URL: http://www.iaees.org/publications/journals/nb/online-version.asp
RSS: http://www.iaees.org/publications/journals/nb/rss.xml
E-mail: networkbiology@iaees.org
Editor-in-Chief: WenJun Zhang
Publisher: International Academy of Ecology and Environmental Sciences

1 Introduction

The modeling of prey-predator interaction is first time introduced by Lotka-Volterra (Malthus, 1798; Verhulst, 1838; Brauer and Sanchez, 1975; Lotka, 1925; Volterra, 1926). The response function used by them was the most simplest case in which they consider simple proportionality between response function and number of predators. It is well described by many researchers that prey-predator species follow different growth functions, among these growth functions, Logistic function of growth is important one (Gause, 1934; Holling, 1959; Xiao and Ruan, 2001). Verhulst (1838) use the Logistic function to explain human growth. Later Feller (1940) explained that almost each population which has asymptotically increasing behavior, satisfies Logistic law of growth of some degree. Gompertz (1825) and May (1973) suggested some other growth functions. The response function between predator and prey species plays an important role for long term survival of life in ecosystem. The response function has many types some of them are, Logistic response, ratio dependent response, Holling types of responses (Holling, 1959) and Beddington response functions (Beddington and May, 1975).

The direct consequence of stability of species in ecological system is the stability of ecological system. Moreover, this is the fundamental concern in study of ecology. Ecological models such as Angelis et al. (1975) Reeve (1988), Murdoch et al. (1992) and Holling (1965) are stability describing mathematical models for verity of systems. The dynamical relationship between predator and its prey is the topic of interest of many researchers from many years in the past. In recent years it will continue to be one of the major topics in the study of mathematical ecology (Berryman, 1992; Din et al., 2018). Except lots of good work done on Lotka-Volterra predator-prey models there are many mistakes and unreliability emerge out this study. Holling (1965) explained three types of functional response of predator specie, and they are now named as Holling type-I, type-II and type-III response respectively (Ping and Hong1986). Holling type-II response proposed by Holling (1965) is described mathematically as

$$\Psi(x) = \frac{sx}{a+x}, 0 \leq s < 1,$$

is any constant. The Holling type-II response is often illustrated by a slow ingestion rate, which tracks from the supposition that the consumer is restricted by its capability to produce food. This response if frequently modeled mathematically by using rectangular hyperbola. Albert et al. (1980) explained the use of Holling type response function in prey-predator interaction models. Later Kaung (1988) explained that the study of prey-predator interactions by using Holling functional response is much batter than study of prey-predator interactions without it. Notice that, due to rich dynamics and remarkable computing results, discrete dynamical system is more suitable then continuous one (Murray, 1989; Agarwal and Wong, 1997). Furthermore, in case of non-overlapping species models this arguments works more efficiently (Din, 2018a, 2018b, 2018c, 2018d).

Recently, many researchers explore the dynamics of discrete-time prey-pradator models (Din et al., 2018; Din, 2017). Din (2017) explore the complexity and chaos in prey-predator model in discrete form. Furthermore, Roy and Ghosh (2013) explored the stability and chaotic motion in discrete-time prey-predator model with generalized Holling response. For study of some interesting theory and analysis of functional response in discrete form we refer reader to Huang et al. (2018) and Cui et al. (2016). In this paper, we consider the continuous-time prey-predator model presented by Jha and Ghorai (2017). They considered four continuous time prey-predator models with three types of Holling type response and selective harvesting of prey and predator. Existing work on these models does not cover all dynamical properties produced by Holling type-II response. Hence, we consider the two dimensional continuous-time prey-predator model with Holling type-II proposed by Jha and Ghorai (2017) and its mathematical form is given as follows:

$$\begin{aligned} \frac{dx}{dt} &= \alpha x\left(1-\frac{x}{k}\right)-\frac{\beta xy}{a+x}, \\ \frac{dy}{dt} &= \frac{\delta xy}{a+x}-\gamma y. \end{aligned} \qquad (1)$$

According to Strogatz et al. (1994), chaos can exists in 3-dimensional phase space continuous system at least. Therefore it is clear that, in system (1) chaos can not be observed. While in case of counter discrete-time map, chaos can be observed in 1-dimension. Motivated by aforemention rich properties of discrete-time dynamical systems, it is necessary and interesting to study qualitative the discrete-time version of system (1).

Therefore, we explored the stability analysis and bifurcation of discrete-time version of system (1). The investigation of such discrete-time models can be found in Din (2017a, 2017b). Moreover, by applying a non-standard finite difference scheme to (1) we have the following discrete-time prey-predator model:

$$x_{n+1}=\frac{x_n(1+h\alpha x_n)}{1+h\left(\alpha\frac{x_n}{k}+\beta\frac{y_n}{a+x_n}\right)},$$
$$y_{n+1}=\frac{y_n+h\left(\frac{\delta x_n y_n}{a+x_n}\right)}{1+hr}. \qquad (2)$$

Here in (2), $0<h<1$ is step size, α represent intrinsic growth rate of prey, β is the rate of predation of prey, γ is the death rate of predator, a is half saturation constant. Moreover, the conversion rate of predator is represented by δ and all constants $\alpha,\beta,k,a,\gamma,\delta$ are positive. Recently, Din (2018a, b) explored the qualitative behavior of discrete-time chemical reactions models and shows that OGY methodm cannot be applied for controlling the bifurcation while keeping the step size h as bifurcation parameter, see Ott (1990). Therefore, to overcome this deficiency we introduce a new hybrid control methodology for controlling the bifurcation.

The novel assistance of this article are given as follows;

A novel discrete-time prey-predator model with Holling type-II (2) is obtained which is dynamically consistent.

Novel explicit parametric conditions are obtained for possible qualitative behavior of system (2).

A generalized Hybrid control technique is proposed for controlling the bifurcation.

In order to shows the effectiveness of newly introduced technique, a comparison is given with existing hybrid control methodology and rich numerical simulation are given.

The remaining part of this manuscript organized as; the existence of equilibrium point and local stability of model (2) is investigated in section 2, bifurcation analysis of unique positive equilibrium point of system (2) is discussed in section 3. In order to control Neimark-sacker bifurcation we proposed a modified-hybrid control strategy in section 4. Finally, a comprehensive numerical simulations are provided to support each theoretical investigation.

The next section is dedicated to existence of fixed points and their stability.

2 Existence of Positive Fixed Point and Local Stability

By performing simple algebraic manipulation, it is easy to see that system (2) has three fixed points $F_T=(0,0)$, $F_B=(k,0)$ and the unique positive equilibrium point $F_U=\left(\frac{a\gamma}{\delta-\gamma},\frac{a\alpha\delta(k\delta-(a+k)\gamma)}{k\beta(\delta-\gamma)^2}\right)$ whenever $(a+k)\gamma<k\delta$ and $\delta>\gamma$.

Firstly, we describe a general results for local stability of fixed points of system (2). Let F_* be any arbitrary fixed point of (2) then we have the next discussion about the local stability of F_*. Let

$$F_J(F_*)=\begin{bmatrix} e_{11} & e_{12} \\ e_{21} & e_{22}\end{bmatrix}$$

be the variational matrix evaluated at F_*. Then the quadratic characteristic polynomial of the matrix $F_J(F_*)$ is:

$$\mathbb{H}(\varrho)=\varrho^2-T_1\varrho+D_1, \qquad (3)$$

where

$$T_1 = (e_{11} + e_{22}),$$

and

$$D_1 = e_{11}e_{22} - e_{12}e_{21}.$$

In order to discuss the stability of fixed points, we have the following Lemma (see Liu and Xiao, 2007).

Lemma 2.1 *Let* $\mathbb{H}(\varrho) = \varrho^2 - T_1\varrho + D_1$, *and* $\mathbb{H}(1) > 0$. *Moreover,* ϱ_1, ϱ_2 *are root of* $\mathbb{H}(\varrho) = 0$, *then:*

(i) $|\varrho_1| < 1$ and $|\varrho_2| < 1$ if and only if $\mathbb{H}(-1) > 0$ and $D_1 < 1$;

(ii) $|\varrho_1| > 1$ and $|\varrho_2| > 1$ if and only if $\mathbb{H}(-1) > 0$ and $D_1 > 1$;

(iii) $|\varrho_1| < 1$ and $|\varrho_2| > 1$ or $(|\varrho_1| > 1$ and $|\varrho_2 < |1)$if and only if $\mathbb{H}(-1) < 0$;

(iv) ϱ_1 and ϱ_2 represent complex conjugates with $|\varrho_1| = 1 = |\varrho_2|$ if and only if ${T_1}^2 - 4D_1 < 0$ and $D_1 = 1$.

As ϱ_1 and ϱ_2 are eigenvalues of (2) then we have the following topological type results related to the stability of F_*. F_* is known as sink if $|\varrho_1| < 1$ and $|\varrho_2| < 1$, as sink is the point of suction hence it is locally asymptotic stable. F_* is known as source if $|\varrho_1| > 1$and $|\varrho_2| > 1$, as source is repeller hence it remains unstable. F_* is known as saddle if $|\varrho_1| < 1$ and $|\varrho_2| > 1$ or $|\varrho_1| > 1$ and $|\varrho_2| < 1$. F_* is said to be non hyperbolic if condition (iv) of the Lemma 2.1 is satisfied.

Firstly, we study the behavior of model (2) at F_T, we assume that $F_J(F_T)$ be the variational matrix of model (2) at point F_T such that;

$$F_J(F_T) = \begin{pmatrix} 1 + h\alpha & 0 \\ 0 & \frac{1}{1+h\gamma} \end{pmatrix}.$$

Then the quadratic characteristic polynomial of the matrix $F_J(F_T)$ is:

$$\mathbb{H}_1(\varrho) = \varrho^2 - \left(1 + h\alpha + \frac{1}{1+h\gamma}\right)\varrho + \frac{1+h\alpha}{1+h\gamma}, \quad (4)$$

Furthermore, by performing some acceptable algebraic manipulation it follows that $\mathbb{H}_1(\varrho) = 0$ has two roots $\varrho_1 = 1 + h\alpha$ and $\varrho_2 = \frac{1}{1+h\gamma}$. Clearly, F_T is always saddle in nature.

Furthermore, to describe nature of system (2) about F_B, we assume that $F_J(F_B)$ be the variational matrix of model (2) at point F_B such that:

$$F_J(F_B) = \begin{pmatrix} \frac{1}{1+h\alpha} & -\frac{hk\beta}{(a+k)(1+h\alpha)} \\ 0 & \frac{1+\frac{hk\delta}{a+k}}{1+h\gamma} \end{pmatrix}.$$

Then the quadratic characteristic polynomial of the matrix $F_J(F_B)$ is given as:

$$\mathbb{H}_2(\varrho) = \varrho^2 - \left(\frac{1}{1+h\alpha} + \frac{1+\frac{hk\delta}{a+k}}{1+h\gamma}\right)\varrho + \frac{a+k+hk\delta}{(a+k)(1+h\alpha)(1+h\gamma)} \quad (5)$$

Moreover, $\mathbb{H}_2(\varrho) = 0$ has two values $\varrho_1 = \frac{1}{1+h\alpha}$ and $\varrho_2 = \frac{a+k+hk\delta}{(a+k)(1+h\gamma)}$. In addition, F_B is stable and saddle if and only if $k\delta < \gamma(a + k)$ and $k\delta > \gamma(a + k)$ respectively.

Finally, let $F_J(F_U)$ be variational matrix of the system (2) about F_U. Then $F_J(F_U)$ is given as:

$$F_J(F_U) = \begin{pmatrix} \frac{ah\alpha\gamma(\gamma+\delta)+k(\gamma-\delta)(\delta+h\alpha(\gamma+\delta))}{k(1+h\alpha)(\gamma-\delta)\delta} & -\frac{h\beta\gamma}{\delta+h\alpha\delta} \\ \frac{h\alpha((-a-k)\gamma+k\delta)}{k\beta(1+h\gamma)} & 1 \end{pmatrix}.$$

Then, the characteristic polynomial of $F_J(F_U)$ is given as:

$$\mathbb{H}(\varrho) = \varrho^2 - \left(1 + \frac{ah\alpha\gamma(\gamma+\delta) + k(\gamma-\delta)(\delta + h\alpha(\gamma+\delta))}{k(1+h\alpha)(\gamma-\delta)\delta}\right)\varrho$$
$$+ \frac{ah\alpha\gamma(\gamma+\delta+2h\gamma\delta)+k(\gamma-\delta)(h\alpha\gamma+\delta+h(\alpha+\gamma+2h\alpha\gamma)\delta)}{k(1+h\alpha)(1+h\gamma)(\gamma-\delta)\delta},$$

Furthermore, by performing simple algebraic manipulation and letting $k\delta > \gamma(a+k)$ and $\delta > \gamma$, we get:

$$\mathbb{H}(1) = \frac{h^2\alpha\gamma(k\delta-(a+k)\gamma)}{k(1+h\alpha)(1+h\gamma)\delta} > 0 \quad (6)$$

$$\mathbb{H}(-1) = \frac{ah\alpha\gamma(2\delta+\gamma(2+h(\gamma+3\delta)))+k(\gamma-\delta)(4\delta+h(2\alpha\gamma+4(\alpha+\gamma)\delta+h\alpha\gamma(\gamma+5\delta)))}{k(1+h\alpha)(1+h\gamma)(\gamma-\delta)\delta} \quad (7)$$

$$\mathbb{H}(0) = \frac{ah\alpha\gamma(\gamma+\delta+2h\gamma\delta)+k(\gamma-\delta)(h\alpha\gamma+\delta+h(\alpha+\gamma+2h\alpha\gamma)\delta)}{k(1+h\alpha)(1+h\gamma)(\gamma-\delta)\delta} \quad (8)$$

From (6) we see that $\mathbb{H}(1) > 0$. Therefore, the following results can be deduced by applying Lemma 2.1.

Theorem 2.1 *Assume that* $k\delta > \gamma(a+k)$ *and* $\delta > \gamma$ *such that* $F_U = \left(\frac{a\gamma}{\delta-\gamma}, \frac{a\alpha\delta(k\delta-(a+k)\gamma)}{k\beta(\delta-\gamma)^2}\right)$ *represent the unique positive steady state of (2), then the following results remains true:*

(i) F_U is locally asymptotically stable if and only if

$$ah\alpha\gamma(2\delta+\gamma(2+h(\gamma+3\delta))) + k(\gamma-\delta)(4\delta+h(2\alpha\gamma+4(\alpha+\gamma)\delta+h\alpha\gamma(\gamma+5\delta))) > 0$$

and

$$h\alpha\gamma(k(\gamma-\delta)(1+h\delta)+a(\gamma+\delta+2h\gamma\delta)) < 0$$

(ii) F_U is unstable equilibrium point if and only if

$$ah\alpha\gamma(2\delta+\gamma(2+h(\gamma+3\delta))) + k(\gamma-\delta)(4\delta+h(2\alpha\gamma+4(\alpha+\gamma)\delta+h\alpha\gamma(\gamma+5\delta))) > 0$$

and

$$h\alpha\gamma(k(\gamma-\delta)(1+h\delta)+a(\gamma+\delta+2h\gamma\delta)) > 0$$

(iii) F_U is saddle point if and only if

$$ah\alpha\gamma(2\delta+\gamma(2+h(\gamma+3\delta))) + k(\gamma-\delta)(4\delta+h(2\alpha\gamma+4(\alpha+\gamma)\delta+h\alpha\gamma(\gamma+5\delta))) > 0$$

(iv) The roots of equation (5) are complex conjugates with magnitude one if and only if

$$\left(1+\frac{ah\alpha\gamma(\gamma+\delta)+k(\gamma-\delta)(\delta+h\alpha(\gamma+\delta))}{k(1+h\alpha)(\gamma-\delta)\delta}\right)^2 < \frac{4(ah\alpha\gamma(\gamma+\delta+2h\gamma\delta)+k(\gamma-\delta)(h\alpha\gamma+\delta+h(\alpha+\gamma+2h\alpha\gamma)\delta))}{k(1+h\alpha)(1+h\gamma)(\gamma-\delta)\delta} \quad (9)$$

and

$$h = \frac{k\delta - a\gamma - k\gamma - a\delta}{\delta(2a\gamma + k\gamma - k\delta)}.$$

3 Bifurcation Analysis

In this section, the parametric conditions for existence of Niemark-Scaker bifurcation of unique positive equilibrium point of the system are discussed. Considering the system (2) about the equilibrium point $F_U = \left(\frac{a\gamma}{\delta-\gamma}, \frac{a\alpha\delta(k\delta-(a+k)\gamma)}{k\beta(\delta-\gamma)^2}\right)$ then the characteristic polynomial matrix $F_J(F_U)$ of (2) can be evaluated as

$$\mathbb{H}(\varrho) = \varrho^2 - \left(1 + \frac{ah\alpha\gamma(\gamma+\delta)+k(\gamma-\delta)(\delta+h\alpha(\gamma+\delta))}{k(1+h\alpha)(\gamma-\delta)\delta}\right)\varrho + \frac{ah\alpha\gamma(\gamma+\delta+2h\gamma\delta)+k(\gamma-\delta)(h\alpha\gamma+\delta+h(\alpha+\gamma+2h\alpha\gamma)\delta)}{k(1+h\alpha)(1+h\gamma)(\gamma-\delta)\delta}. \quad (10)$$

According to Lemma 2.1, the characteristic equation (10) has two complex conjugate roots with modulus one, if condition (iv) of Theorem 2.1 is satisfied. Hence F_U undergoes Neimark-sacker bifurcation if parameters varies in the neighborhood of the following set;

$$\aleph_S = \left\{\alpha, k, \beta, a, \delta, \gamma, h \in \Re^+ : (9)\ holds, 0 < h < 1\ with\ h = \frac{k\delta - a\gamma - k\gamma - a\delta}{\delta(2a\gamma + k\gamma - k\delta)}\right\}.$$

Let $(\alpha, k, \beta, a, \delta, \gamma, \bar{h}) \in \aleph_S$ then system (2) can be written as;

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} \frac{x+\bar{h}\alpha x}{1+\bar{h}\left(\frac{\alpha x}{k}+\frac{\beta y}{a+x}\right)} \\ \frac{y+\bar{h}\left(\frac{\delta xy}{a+x}\right)}{1+\bar{h}\gamma} \end{pmatrix}. \quad (11)$$

where $\bar{h} = \frac{k\delta-a\gamma-k\gamma-a\delta}{\delta(2a\gamma+k\gamma-k\delta)}$. Let $|\tilde{h}| \ll 1$ be a perturbation parameter, then map (11) can be expressed as;

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} \frac{x+(\bar{h}+\tilde{h})\alpha x}{1+(\bar{h}+\tilde{h})\left(\frac{\alpha x}{k}+\frac{\beta y}{a+x}\right)} \\ \frac{y+(\bar{h}+\tilde{h})\left(\frac{\delta xy}{a+x}\right)}{1+(\bar{h}+\tilde{h})\gamma} \end{pmatrix}. \quad (12)$$

Next, under transformations $(H, P) = \left(x - \frac{a\gamma}{\delta-\gamma}, y - \frac{a\alpha\delta(k\delta-(a+k)\gamma)}{k\beta(\delta-\gamma)^2}\right)$, the model (12) can be described by the following system:

$$\begin{pmatrix} H \\ P \end{pmatrix} \to \begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix}\begin{pmatrix} H \\ P \end{pmatrix} + \begin{pmatrix} N_1(H,P) \\ N_2(H,P) \end{pmatrix}, \quad (13)$$

where $x = \frac{a\gamma}{\delta-\gamma}$ and $y = \frac{a\alpha\delta(k\delta-(a+k)\gamma)}{k\beta(\delta-\gamma)^2}$ implies

$$N_1(H,P) = r_{13}H^2 + r_{14}HP + r_{15}P^2 + r_{16}H^3 + r_{17}H^2P + r_{18}HP^2 + r_{19}P^3 + O((|H| + |P|)^4,$$

$N_2(H,P) = r_{23}H^2 + r_{24}HP + r_{25}H^3 + r_{26}H^2P + O(|H|+|P|)^4,$

$$r_{11} = \frac{k^2(1+h\alpha)((a+x)^2+h(a+2x)y\beta)}{((a+x)(k+hx\alpha)+hky\beta)^2}, r_{12} = -\frac{hk^2x(a+x)(1+h\alpha)\beta}{((a+x)(k+hx\alpha)+hky\beta)^2},$$

$$r_{13} = -\frac{hk^2(1+h\alpha)((a+x)^3\alpha+y(-ak+h(a^2+3ax+3x^2)\alpha)\beta-hky^2\beta^2)}{((a+x)(k+hx\alpha)+hky\beta)^3},$$

$$r_{14} = -\frac{hk^2(1+h\alpha)\beta((a+x)(ak-hx(a+2x)\alpha)+hk(a+2x)y\beta)}{((a+x)(k+hx\alpha)+hky\beta)^3},$$

$$r_{15} = \frac{h^2k^3x(a+x)(1+h\alpha)\beta^2}{((a+x)(k+hx\alpha)+hky\beta)^3},$$

$$r_{16} = \frac{\left(hk^2(1+h\alpha)(+y(-ak^2-2ahk(a+2x)\alpha+h^2(a+2x)(a^2+2ax+2x^2)\alpha^2)\beta)\right)}{/((a+x)(k+hx\alpha)+hky\beta)^4}$$

$$+\frac{(h(a+x)^4\alpha^2+hk^2(1+h\alpha)(-hky^2(k+2h(a+2x)\alpha)\beta^2)}{((a+x)(k+hx\alpha)+hky\beta)^4},$$

$$r_{17} = \frac{\left(hk^2(1+h\alpha)\beta\left((a+x)(ak^2+2ahk(a+2x)\alpha-h^2x(a^2+3ax+3x^2)\alpha^2)\right)\right)}{((a+x)(k+hx\alpha)+hky\beta)^4}$$

$$+\frac{(hk^2(1+h\alpha)\beta(2hky(kx+h(a+2x)^2\alpha)\beta-h^2k^2y^2\beta^2)}{((a+x)(k+hx\alpha)+hky\beta)^4},$$

$$r_{18} = \frac{h^2k^3(1+h\alpha)\beta^2(-(a+x)(k(-a+x)+2hx(a+2x)\alpha)+hk(a+2x)y\beta)}{((a+x)(k+hx\alpha)+hky\beta)^4},$$

$$r_{19} = -\frac{h^3\beta^3(a+x)k^4(h\alpha+1)x}{(x\alpha\, ha+x^2\alpha\, h+\beta\, hyk+ka+kx)^4}$$

$$r_{21} = \frac{ahy\delta}{(a+x)^2(1+h\gamma)}, r_{22} = \frac{(\delta\, hx+a+x)}{(a+x)(\gamma\, h+1)}, r_{23} = -\frac{y\delta\, ha}{(a+x)^3(h\gamma+1)},$$

$$r_{24} = \frac{\delta\, ha}{(a+x)^2(h\gamma+1)}, r_{25} = \frac{\delta\, ha}{(a+x)^2(h\gamma+1)}, r_{26} = -\frac{\delta\, ha}{(a+x)^3(h\gamma+1)}.$$

The characteristic equation of Jacobian matrix of map (13) computed at (0,0) can be described as follows:

$$\varrho^2 - T(\tilde{h})\varrho + D(\tilde{h}) = 0, \qquad (14)$$

where

$$T(\tilde{h}) = \left(1 + \frac{a(\bar{h}+\tilde{h})\alpha\gamma(\gamma+\delta)+k(\gamma-\delta)(\delta+(\bar{h}+\tilde{h})\alpha(\gamma+\delta))}{k(1+(\bar{h}+\tilde{h})\alpha)(\gamma-\delta)\delta}\right)$$

and

$$D(\tilde{h}) = \frac{a(\bar{h}+\tilde{h})\alpha\gamma(\gamma+\delta+2(\bar{h}+\tilde{h})\gamma\delta)+k(\gamma-\delta)((\bar{h}+\tilde{h})\alpha\gamma+\delta+(\bar{h}+\tilde{h})(\alpha+\gamma+2(\bar{h}+\tilde{h})\alpha\gamma)\delta)}{k(1+(\bar{h}+\tilde{h})\alpha)(1+(\bar{h}+\tilde{h})\gamma)(\gamma-\delta)\delta}.$$

Since $(\alpha,k,\beta,a,\delta,\gamma,\bar{h}) \in \aleph_S$ and equation (14) has pair of complex conjugate roots with unit modulus then we have;

$$\varrho_1,\varrho_2 = \frac{T(\tilde{h})}{2} \pm \frac{i}{2}\sqrt{4D(\tilde{h}) - T^2(\tilde{h})}.$$

Therefore, we have

$$|\varrho_1| = |\varrho_2| = \sqrt{D(\tilde{h})},$$

$$\left(\frac{d\sqrt{D(\tilde{h})}}{d\tilde{h}}\right)_{\tilde{h}=0} = \frac{\alpha\gamma\left(a\gamma(\bar{h}^2\alpha\gamma-1)-a(1+\bar{h}\gamma(4+\bar{h}(\alpha+2\gamma)))\delta-k(\gamma-\delta)(1-\bar{h}^2\alpha\gamma+\bar{h}(2+\bar{h}(\alpha+\gamma))\delta)\right)}{2k(1+\bar{h}\alpha)^2(1+\bar{h}\gamma)^2\delta(\delta-\gamma)\sqrt{\frac{a\bar{h}\alpha\gamma(\gamma+\delta+2\bar{h}\gamma\delta)+k(\gamma-\delta)(\bar{h}\alpha\gamma+\delta+\bar{h}(\alpha+\gamma+2\bar{h}\alpha\gamma)\delta)}{k(1+\bar{h}\alpha)(1+\bar{h}\gamma)(\gamma-\delta)\delta}}} \neq 0. \qquad (15)$$

Since $(\alpha, k, \beta, a, \delta, \gamma, \bar{h}) \in \aleph_S$, this implies that $T(0) \in]-2,2[$. Thus $T(0) \neq 0,1,\pm 2$ gives $\varrho_1.\varrho_1......m\ times \neq 1\ and\ \varrho_2.\varrho_2.......m\ times \neq 1$ for all $m \in \{1,2,3,4\}$ at $\tilde{h} = 0$. Hence, zeros of (14) do not lie in the intersection of the unit circle with the coordinate axes at $\tilde{h} = 0$ and if the following condition is true:

$$2 + \frac{h\alpha\gamma(k(\gamma-\delta)+a(\gamma+\delta))}{k(1+h\alpha)(\gamma-\delta)\delta} \neq 0,\ ah\alpha\gamma(\gamma+\delta) + k(\gamma-\delta)\big(\delta + h\alpha(\gamma+\delta)\big) \neq 0 \qquad (16)$$

Assume that $\xi = \frac{T(0)}{2},\ \zeta = \frac{1}{2}\sqrt{4D(0) - T^2(0)}$, then normal form of (13) at $\tilde{h} = 0$ can be expressed as:

$$\begin{pmatrix} H \\ P \end{pmatrix} = \begin{pmatrix} r_{12} & 0 \\ \xi - r_{11} & -\zeta \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}. \qquad (17)$$

hence, by using map (17) we get:

$$\begin{pmatrix} u \\ v \end{pmatrix} \to \begin{pmatrix} \xi & -\zeta \\ \zeta & \xi \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} \widetilde{g_1}(u,v) \\ \widetilde{g_2}(u,v) \end{pmatrix}, \qquad (18)$$

where

$$\widetilde{g_1}(u,v) = \frac{r_{16}H^3}{r_{12}} + \frac{r_{17}H^2P}{r_{12}} + \frac{r_{13}H^2}{r_{12}} + \frac{r_{18}HP^2}{r_{12}} + \frac{r_{14}HP}{r_{12}} + \frac{r_{19}P^3}{r_{12}} + \frac{r_{15}P^2}{r_{12}} + O((|u|+|v|)^4),$$

$$\widetilde{g_2}(u,v) = \left(\frac{\lambda r_{16}}{r_{12}T} - \frac{r_{25}}{T}\right)H^3 + \left(\frac{\lambda r_{17}}{r_{12}T} - \frac{r_{26}}{T}\right)H^2P + \left(\frac{\lambda r_{13}}{r_{12}T} - \frac{r_{23}}{T}\right)H^2 + \frac{\lambda r_{18}HP^2}{r_{12}T}$$
$$+ \left(\frac{\lambda r_{14}}{r_{12}T} - \frac{r_{24}}{T}\right)HP + \frac{\lambda r_{19}P^3}{r_{12}T} + \frac{\lambda r_{15}P^2}{r_{12}T} + O((|u|+|v|)^4),$$

where $\lambda = (\xi - r_{11})$, and $H = r_{12}u$ and $P = (\xi - r_{11})u - \zeta v$. Therefore, we define the following nonzero real number:

$$\Upsilon = \left(\left[Re(\varrho_2\tau_{21}) - Re\left(\frac{(1-2\varrho_1)\varrho_2^2}{1-\varrho_1}\tau_{20}\tau_{11}\right) - \frac{1}{2}|\tau_{11}|^2 - |\tau_{02}|^2\right]\right)_{\tilde{h}=0},$$

where

$$\tau_{20} = \frac{1}{8}\left[\widetilde{g_1}_{uu} - \widetilde{g_1}_{vv} + 2\widetilde{g_2}_{uv} + i\big(\widetilde{g_2}_{uu} - \widetilde{g_2}_{vv} - 2\widetilde{g_1}_{uv}\big)\right],$$

$$\tau_{11} = \frac{1}{4}\left[\widetilde{g_1}_{uu} + \widetilde{g_1}_{vv} + i\big(\widetilde{g_2}_{uu} + \widetilde{g_2}_{vv}\big)\right],$$

$$\tau_{02} = \frac{1}{8}\left[\widetilde{g_1}_{uu} - \widetilde{g_1}_{vv} - 2\widetilde{g_2}_{uv} + i\big(\widetilde{g_2}_{uu} - \widetilde{g_2}_{vv} + 2\widetilde{g_1}_{uv}\big)\right],$$

$$\tau_{21} = \frac{1}{16}\left[\widetilde{g_1}_{uuu} + \widetilde{g_1}_{uvv} + \widetilde{g_2}_{uuv} + \widetilde{g_2}_{vvv} + i\left(\widetilde{g_2}_{uuu} + \widetilde{g_2}_{uvv} - \widetilde{g_1}_{uuv} - \widetilde{g_1}_{vvv}\right)\right].$$

Hence, we have the following conclusions for direction and existence of Neimark-Sacker bifurcation according to aforemention calculation (Guckenheimer and Holmes, 1984; Robinson, 1998; Wiggins, 2003; Wan, 1978).

Theorem 3.1 *There exists Neimark-Sacker bifurcation at* $F_U = \left(\frac{a\gamma}{\delta-\gamma}, \frac{a\alpha\delta(k\delta-(a+k)\gamma)}{k\beta(\delta-\gamma)^2}\right)$ *whenever* h *varies in small neighborhood of* $\bar{h} = \frac{k\delta-a\gamma-k\gamma-a\delta}{\delta(2a\gamma+k\gamma-k\delta)}$. *In addition, if* $\Upsilon < 0, (\Upsilon > 0)$, *respectively, then an attracting or repelling invariant closed curve bifurcates from the equilibrium point for* $h > \bar{h}$ $(h < \bar{h})$, *respectively.*

4 Chaos Control

In last few centuries, chaos and unusual behavior of non-linear discrete dynamical system attracted the attention of scientists. Chaotic behavior examined in almost every field, such as chemistry, physics, ecology, biology, chemical engineering, telecommunications etc. During the past decades, the trend of publication in this field grow rapidly. Ott et al (1990) proposed a chaos control strategy which was the development of this field. The proposed methodology is known as OGY method. Latter on, many control schemes have proposed see also (Parthasarathy, 1992; Molgedey et al., 1992). In this section we proposed a generalized hybrid control technique for controlling Neimark-sacker bifurcation, period-doubling bifurcation and chaos under the influence of period-doubling bifurcation. The original scheme is first time proposed by Liu et al. (2003) for controlling the period-doubling bifurcation and considered as control strategy (Din, 2018a, 2018b; Elabbasy et al., 2014; He and Lai 2011; Ott, 1990). Therefore, we modified this existing technique to control period-doubling bifurcation, the Neimark-sacker bifurcation and chaos under the influence of period-doubling bifurcation. The proposed control strategy is more feasible and efficient as compared to other control schemes. Moreover, it is applicable to almost every discrete-time dynamical system.

Consider the following an n-dimensional discrete dynamical system:

$$x_{n+1} = f(x_n, \omega) \quad (19)$$

where $x_n \in R^n$, $n \in Z$ and the bifurcation parameter for system (19) is $\omega \in R$. Assuming that there exists bifurcation for parameter ω.

The main objective of this technique is to restore the stability of system (19). For this purpose, we proposed the following modified hybrid control strategy by applying state feedback along with parameter perturbation;

$$x_{n+k} = \theta^{*3} f^{(k)}(x_n, \omega) + (1 - \theta^{*3})x_n \quad (20)$$

where $0 < \theta^* < 1$ is control parameter, $f^{(k)}$ is kth iteration of $f(.)$ and k is positive integer. For $\theta^* = 1$ one has the original system (19). Applying technique (20) on model (2) we have the following control model:

$$x_{n+1} = \theta^{*3}\left(\frac{x_n(1+h\alpha x_n)}{1+h\left(\alpha\frac{x_n}{k}+\beta\frac{y_n}{a+x_n}\right)}\right)+(1-\theta^{*3})x_n,$$
$$y_{n+1} = \theta^{*3}\left(\frac{y_n+h\left(\frac{\delta x_n y_n}{a+x_n}\right)}{1+h\gamma}\right)+(1-\theta^{*3})y_n, \quad (21)$$

where $0<\theta^{*}<1$ is control parameter. Furthermore, the controlled system (21) and original model (2) has the same equilibrium points. The variational matrix for (21) at positive fixed point $F_U=\left(\frac{a\gamma}{\delta-\gamma},\frac{a\alpha\delta(k\delta-(a+k)\gamma)}{k\beta(\delta-\gamma)^2}\right)$ is written below as:

$$\begin{pmatrix} 1+\frac{h\alpha\gamma(k(\gamma-\delta)+a(\gamma+\delta))\theta^{*3}}{k(1+h\alpha)(\gamma-\delta)\delta} & -\frac{h\beta\gamma\theta^{*3}}{\delta+h\alpha\delta} \\ -\frac{h\alpha((a+k)\gamma-k\delta)\theta^{*3}}{k\beta(1+h\gamma)} & 1 \end{pmatrix}. \quad (22)$$

The following result properly describes the conditions for local asymptotic stability of positive equilibrium F_U of the controlled system (21).

Theorem 4.1 *Assume that* $(a+k)\gamma<k\delta$ *and* $\delta>\gamma$, *then the equilibrium* $\left(\frac{a\gamma}{\delta-\gamma},\frac{a\alpha\delta(k\delta-(a+k)\gamma)}{k\beta(\delta-\gamma)^2}\right)$ *of controlled system (21) is locally asymptotically stable if and only if the following conditions hold.*

$$\left|2+\frac{h\alpha\gamma(k(\gamma-\delta)+a(\gamma+\delta))\theta^{*3}}{k(1+h\alpha)(\gamma-\delta)\delta}\right|<2+\frac{h\theta^{*3}(\alpha\gamma(1+h\gamma)(k(\gamma-\delta)+a(\gamma+\delta))-(\gamma-\delta)((a+k)\gamma-k\delta))}{k(1+h\alpha)(1+h\gamma)(\gamma-\delta)\delta},$$

and

$$hk\alpha(1+h\alpha)\gamma(1+h\gamma)(\gamma-\delta)\delta\theta^{*}\left(-(1+h\gamma)(k(\gamma-\delta)+a(\gamma+\delta))+h(\gamma-\delta)((a+k)\gamma-k\delta)\theta^{*3}\right)>0.$$

5 Numerical Simulation and Discussion

The present part of the manuscript is related to verification of above theoretical work. In first example, we explored the existence of bifurcation and the direction of bifurcation numerically. Second example is related to the verification of controllability of system (2).

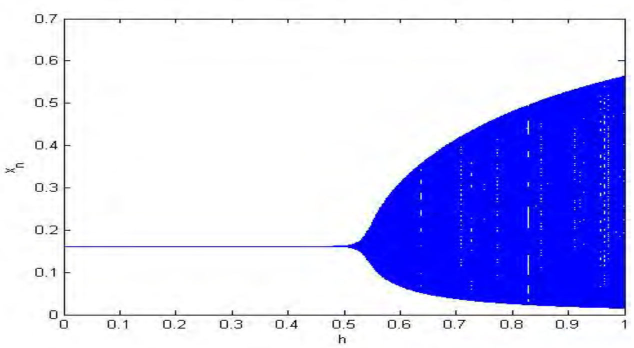

Fig. 1 Bifurcation diagram for x_n

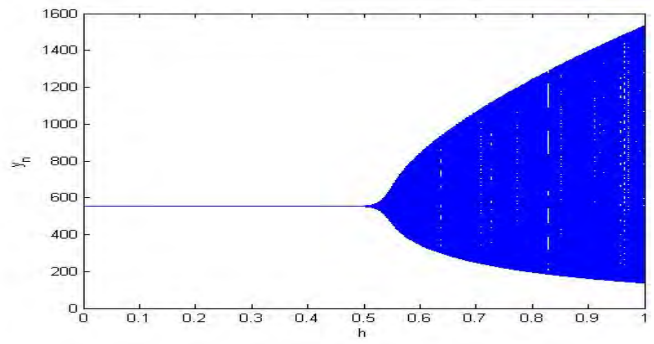

Fig. 2 Bifurcation diagram for y_n

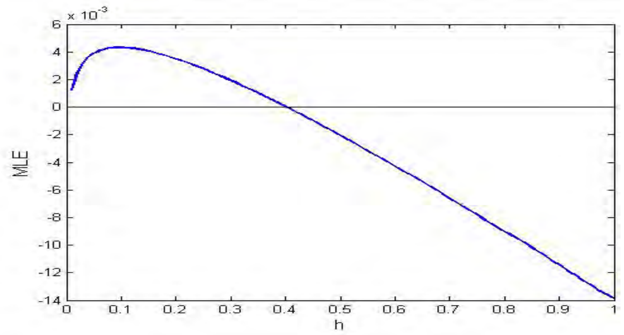

Fig. 3 Maximum Lyapunov exponents

Fig. 1-3 are bifurcation diagrams and MLE for system (2) for $h \in (0,1)$ with $(\alpha, k, \beta, a, \delta, \gamma, h) = (26.995, 0.78, 0.29, 7.35, 0.6, 27.9, 0.540711777)$ and initial conditions $x_0 = 0.140773584, y_0 = 554.41161850$

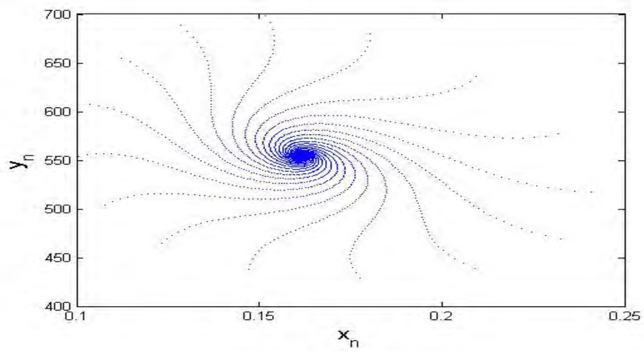

Fig. 4

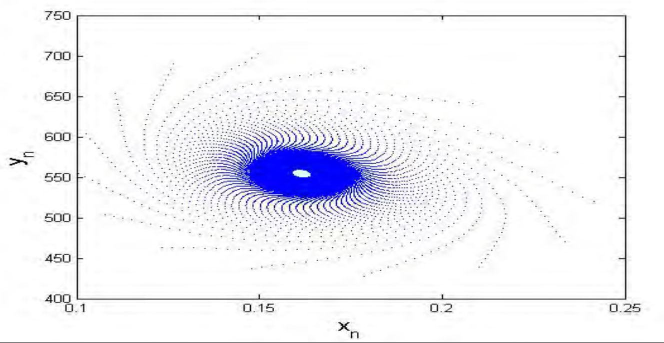

Fig. 5

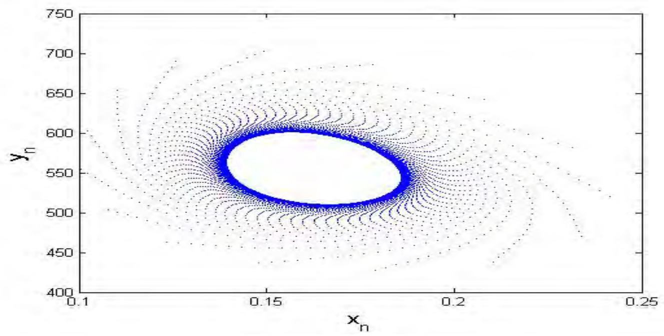

Fig. 6

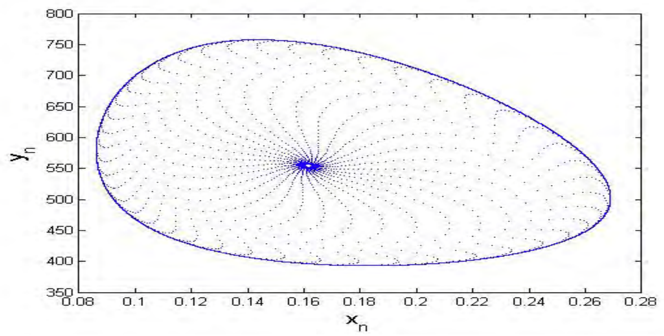

Fig. 7

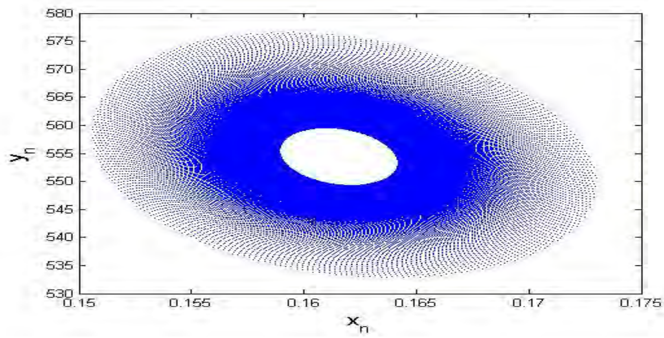

Fig. 8

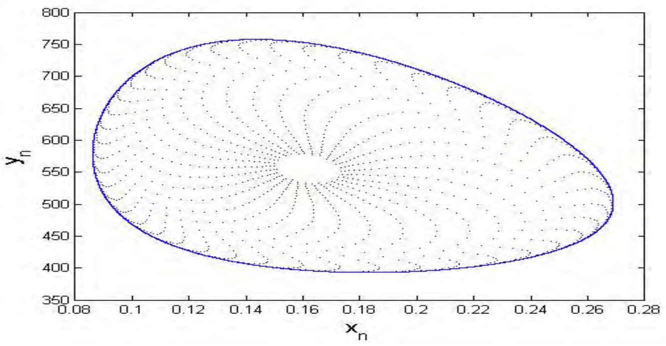

Fig. 9

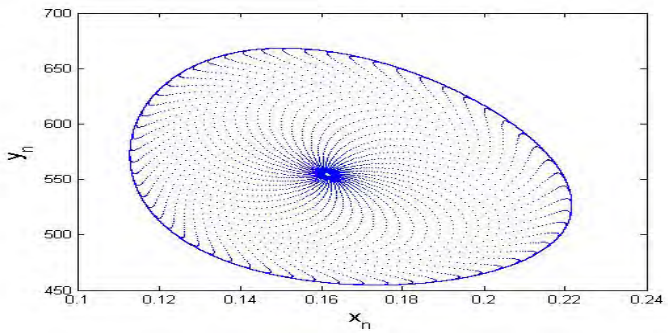

Fig. 10

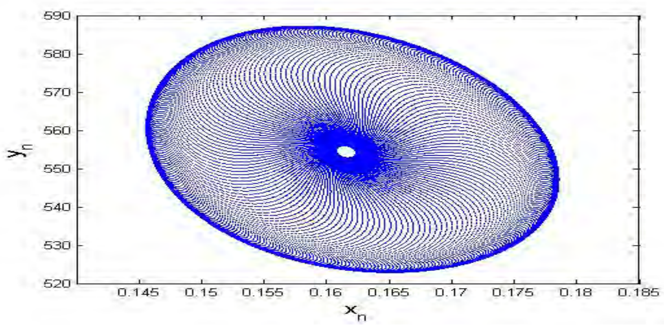

Fig. 11

Fig. 4-11 are phase portrait for system (2). when $h \in (0,1)$ with $(\alpha, k, \beta, a, \delta, \gamma, h) = (26.995, 0.78, 0.29, 7.35, 0.6, 27.9, 0.5407117772)$ and initial conditions $x_0 = 0.140773584, y_0 = 554.41161850$.

Example 5.1 First we assume the special case of system *(2)* by taking $x_0 = 0.1407735849056605$, and $y_0 = 554.4116185079969$) and parametric values $(\alpha, k, \beta, a, \delta, \gamma) = (26.995, 0.78, 0.29, 7.35, 0.6, 27.9)$,where$0 < h < 1$ is taken as bifurcation parameter. Particularly, if $h = 0.5407117772709169$ then system (2) undergoes Neimark-sacker bifurcation. Fig.1-3 shows that both populations undergo Neimark-sacker bifurcation

about $F_U = (0.1407735849056605,\ 554.4116185079969)$(see Fig. 1 and Fig. 2) and corresponding maximum Lyapunov exponents are shown in Fig.3. Furthermore, for $h = 0.5407117772709169$, the unique positive fixed point $F_U = (0.1407735849056605,\ 554.4116185079969)$, undergoes Neimrk-sacker bifurcation (see Fig.1 and Fig. 2). Finally, some phase portrait are given in Fig. 4-11 for$0 < h < 1$.

The characteristic polynomial of system (2) computed at fixed point $F_U = (0.1407735, 554.4116185)$ is given by

$$\mathbb{H}(\varrho) = \varrho^2 - 1.82213635874\varrho + 1.$$

Then roots of $\mathbb{H}(\varrho) = 0$ are $\varrho_{1,2} = 0.9110681162 \pm 0.4122576585i$ with $|\varrho_1| = 1 = |\varrho_2|$ so that $(\alpha, k, \beta, a, \delta, \gamma, h) = (26.995, 0.78, 0.29, 7.35, 0.6, 27.9, 0.5407117772709169) \in \aleph_S$. Next, we observe that $T(0) = 1.8221363587481079 \neq 0,1$ thus condition (16) is satisfied. Moreover, some careful calculation gives

$$\mathbb{H}(1) = 0.1778636412518920 > 0$$

with

$$N_1(H,P) = 0.14042388074 + 0.8607478357\,H - 0.01980727908\,P - 0.9711662538H^2 - 0.09325300047\,HP + 0.02719991324P^2 + 1.082124448H^3 + 0.02540617346H^2P + 0.009379928897\,HP^2 - 0.003735168660P^3 + O((|H| + |P|)^4),$$

$$N_2(H,P) = 554.4477758 + 827.1585276\,H + 0.9694020796\,P - 110.4206974H^2 + 1.491957419\,HP + 14.74050017H^3 - 0.1991673582H^2P + O(|H| + |P|)^4,$$

$$\widetilde{g_1}(u,v) = 0.0001974767127u^3 + 0.0001626365117u^2v + 0.0001419592457u^2 + 0.0002078002466\,uv^2 + 0.0004414176552\,uv - 0.0001321270057v^3 - 0.02333888815v^2 + O((|u| + |v|)^4),$$

$$\widetilde{g_2}(u,v) = -0.0002396156133u^3 + 0.0003990084800u^2v + 0.0006390664085u^2 - 0.001623804978\,uv^2 - 0.0002416366123\,uv - 0.0001612745780v^3 - 0.002848750955v^2 + O((|u|+|v|)^4),$$

$$\tau_{20} = \frac{1}{8}\left[\widetilde{g_1}_{uu} - \widetilde{g_1}_{vv} + 2\widetilde{g_2}_{uv} + i\left(\widetilde{g_2}_{uu} - \widetilde{g_2}_{vv} - 2\widetilde{g_1}_{uv}\right)\right] = 0.00334279 - 0.00872559i,$$

$$\tau_{11} = \frac{1}{4}\left[\widetilde{g_1}_{uu} + \widetilde{g_1}_{vv} + i\left(\widetilde{g_2}_{uu} + \widetilde{g_2}_{vv}\right)\right] = -0.00457148179001 + 0.0001770956565i,$$

$$\tau_{02} = \frac{1}{8}\left[\widetilde{g_1}_{uu} - \widetilde{g_1}_{vv} - 2\widetilde{g_2}_{uv} + i\left(\widetilde{g_2}_{uu} - \widetilde{g_2}_{vv} + 2\widetilde{g_1}_{uv}\right)\right] = 0.000154246 + 0.000133453i,$$

$$\xi_{21} = \frac{1}{16}\left[\widetilde{g_1}_{uuu} + \widetilde{g_1}_{uvv} + \widetilde{g_2}_{uuv} + \widetilde{g_2}_{vvv} + i\left(\widetilde{g_2}_{uuu} + \widetilde{g_2}_{uvv} - \widetilde{g_1}_{uuv} - \widetilde{g_1}_{vvv}\right)\right] = 0.000052609321087625 - 0.000020903940143125i,$$

and

$$\Upsilon = \left(\left[-Re\left(\frac{(1-2\omega_1)\omega_2^2}{1-\omega_1}\tau_{20}\tau_{11}\right) - \frac{1}{2}|\tau_{11}|^2 - |\tau_{02}|^2 + Re(\omega_2\tau_{21})\right]\right)_{\tilde{c}=0} = -0.00004374348897384201 < 0.$$

which proves the correctness of Theorem 3.1.

Example 5.2 In this example we implements the modified hybrid control technique to control the Neimark-sacker bifurcation. Thus by applying the modified hybrid control technique for controlling the Neimark-sacker bifurcation, we have the following scheme;

$$\begin{aligned} x_{n+1} &= \theta^{*3}\left(\frac{x_n(1+(0.760711777)26.995x_n)}{1+0.760711777\left(26.995\frac{x_n}{0.78}+0.29\frac{y_n}{7.35+x_n}\right)}\right) + (1-\theta^{*3})x_n, \\ y_{n+1} &= \theta^{*3}\left(\frac{y_n+0.760711777\left(\frac{0.6x_ny_n}{7.35+x_n}\right)}{1+(0.760711777)27.9}\right) + (1-\theta^{*3})y_n, \end{aligned} \quad (23)$$

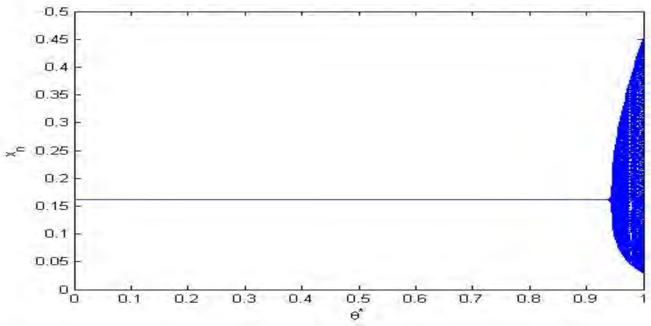

Fig. 12 Controlled diagram for x_n

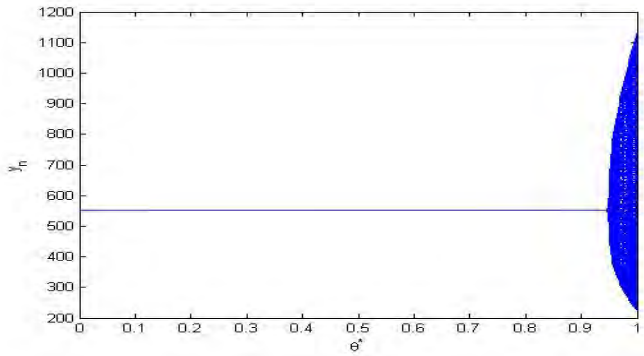

Fig. 13 Controlled diagram for y_n

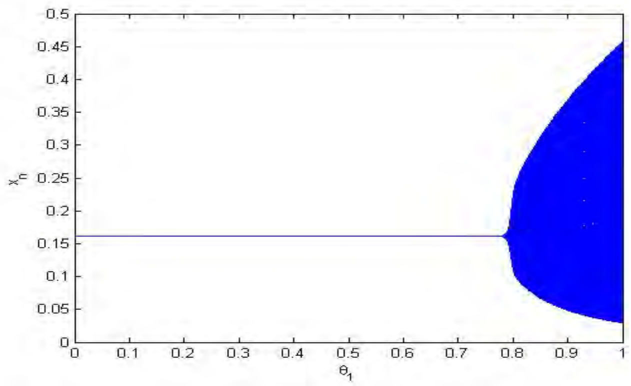

Fig. 14 Controlled diagram for x_n

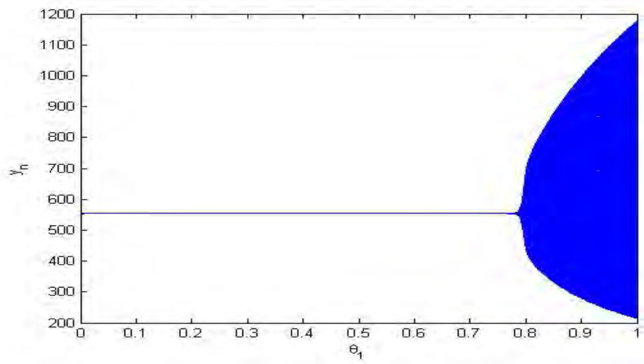

Fig. 15 Controlled diagram for y_n

Fig. 12-15 are **c**ontrolled diagrams for system (2)when $h \in (0,1)$ with $(\alpha, k, \beta, a, \delta, \gamma, h) = (26.995, 0.78, 0.29, 7.35, 0.6, 27.9, 0.760711777)$ and initial conditions $x_0 = 0.1607735849, y_0 = 554.41161850$.

Here $0 < \theta^* < 1$ is the control parameter. In this case unique positive equilibrium point $F_U = (0.1607735849, 554.41161850)$ is positive fixed point of original map (2). Hence by using Lemma 2.1 the controlled map (23) is locally asymptotically stable if and only if $0 < \theta^* < 0.920606005$. Moreover, it can be easily seen that the stability of original system (2) is successfully restored for maximum range of control parameter θ^* (see Fig. 12-13).

6 Conclusion

In this article we discussed the prey-predator discrete-time biological model proposed by Jha and Ghorai (2017). We discretized the system of differential equations and obtained dynamically consistent discrete-time model for prey-predator interaction model. Particularly, we discussed the local asymptotic stability of equilibrium point map (2). It is shown that there exists the Neimark-sacker bifurcation for equilibrium point of constructed discrete-time model by assuming the step size h as bifurcation parameter. We computed the parametric conditions for the existence and direction of the Neimark-sacker bifurcation. The novel contribution of this work is to develop a modified hybrid control strategy for controlling the Neimark-sacker bifurcation. Therefore, to show the effectiveness of newly modified technique, a comparison with existing hybrid control methodology (Luo et al., 2003) is given here. For this, we take the parametric values used in example 5.2. In this case the corresponding scheme is given as;

$$\begin{aligned} x_{n+1} &= \theta_1\left(\frac{x_n(1+(0.5407117772709169)26.995x_n)}{1+h\left(26.995\frac{x_n}{0.78}+0.29\frac{y_n}{7.35+x_n}\right)}\right)+(1-\theta_1)x_n, \\ y_{n+1} &= \theta_1\left(\frac{y_n+0.5407117772709169\left(\frac{27.9x_ny_n}{7.35+x_n}\right)}{1+(0.5407117772709169)0.6}\right)+(1-\theta_1)y_n, \end{aligned} \qquad (24)$$

where $\theta_1 \in]0,1[$ is control parameter. Moreover, the Jacobian matrix of (24) about equilibrium point $F_U = (0.1407735, 554.4116185)$ is given by

$$\begin{pmatrix} 1-0.177864\theta_1 & -0.000216214\theta_1 \\ 822.629\theta_1 & 1 \end{pmatrix}.$$

According to Lemma 2.1 system (24) is stable if and only if $0 < \theta_1 < 0.780227783$ (see Fig. 14 and Fig.

15). On the other hand, the newly modified control methodology (20) for similar parametric values applied to map (2) in example 5.2 and stable region is shown in Fig. 12 and Fig. 13. Moreover, a comparison of modified Hybrid technique with original Hybrid technique for different values of h is presented in Table 1. It can be seen from Table 1 that newly given technique is very better than old Hybrid technique. In Table.1, I_1 represents the controlled interval for Hybrid method and controlled interval for modified Hybrid method is represented by I_2. Hence, the modified control methodology is more efficient then existing hybrid control scheme and applicable for every class of discrete-time maps.

Table 1 Comparison between Hybrid method and modified Hybrid method for system (2) for $h \in (0,1)$ and $(\alpha, k, \beta, a, \delta, \gamma) = (26.995, 0.78, 0.29, 7.35, 0.6, 27.9)$ with initial conditions $x_0 = 0.16077358, y_0 = 553.411618507$

Values of h	I_1	I_2	Length of I_1	Length of I_2
0.66271777	$0 < \theta_1 < 0.860996893$	$0 < \theta^* < 0.951335846$	0.860996893	0.951335846
0.76271777	$0 < \theta_1 < 0.780227783$	$0 < \theta^* < 0.920606005$	0.780227783	0.920606005
0.86271777	$0 < \theta_1 < 0.718183014$	$0 < \theta^* < 0.895526365$	0.718183014	0.895526365
0.96271777	$0 < \theta_1 < 0.669027748$	$0 < \theta^* < 0.874610547$	0.669027748	0.874610547

References

Abbasi MA, Din Q. 2019. Under the influence of crowding effects: stability, bifurcation and chaos control for a discrete-time predator-prey model. International Journal of Biomathematics (DOI: 10.1142/S179352451950044X)

Agarwal RP, Wong PJY. 1997. Advance Topics in Difference Equations. Kluwer, Dordrecht, Netherlands

Albert A, Freedman M, Perelson AS. 1980. Tumors and the immune system: the effects of a tumor growth modulator. Mathematical Biosciences, 50(1-2): 25-58

Beddington JR, May RM. 1975. Time delays are not necessarily destabilizing. Mathematical Biosciences, 27: 109-118

Berryman AA. 1992. The orgins and evolution of predator-prey theory. Ecology, 73(5): 1530-1535

Brauer F, Sanchez DA. 1975. Constant rate population harvesting: equilibrium and stability. Theoretical population biology, 8(1): 12-30

Cui Q, Zhang Q, Qiu Z, Hu Z. 2016. Complex dynamics of a discrete-time predator-prey system with Holling IV functional response. Chaos, Solitons and Fractals, 87: 158-171

De Angelis DL, Goldstein RA, Neil RV. 1975. A model for tritrophic interactions. Ecology, 56: 881-892

Din Q, Donchev T, Kolev D. 2018. Stability, bifurcation analysis and chaos control in chlorine dioxide–iodine–malonic acid reaction. MATCH Commun. Math. Comput. Chem, 79(3): 577-606

Din Q, Elsadany, AA, Ibrahim S. 2018. Bifurcation analysis and chaos control in a second-order rational difference equation. International Journal of Nonlinear Sciences and Numerical Simulation, 19(1): 53-68

Din Q. 2017. Complexity and chaos control in a discrete-time prey-predator model. Communications in

Nonlinear Science and Numerical Simulation, 49: 113-134

Din Q. 2017. Neimark-Sacker bifurcation and chaos control in Hassell-Varley model. Journal of Difference Equations and Applications, 23(4): 741-762

Din Q, Khan MA. 2017. Period-doubling bifurcation and chaos control in a discrete-time mosquito model. Computational Ecology and Software, 7(4): 153

Din Q. 2018. Bifurcation analysis and chaos control in discrete-time glycolysis models. Journal of Mathematical Chemistry, 56(3): 904-931

Din Q. 2018. Controlling chaos in a discrete-time prey-predator model with Allee effects. International Journal of Dynamics and Control, 6(2): 858-872

Din Q. 2018. Qualitative analysis and chaos control in a density-dependent host–parasitoid system. International Journal of Dynamics and Control, 6(2): 778-798

Elabbasy EM, Elsadany AA, Zhang Y. 2014. Bifurcation analysis and chaos in a discrete reduced Lorenz system. Applied Mathematics and Computation, 228: 184-194

Feller W. 1940. On the logistic law of growth and its empirical verification biology. Acta Biotheoretica, 5: 51-66

Gause GF. 1934. The Struggle for Existence. Dover Phoenix Editions. Hafner, New York, USA

Gompertz B. 1825. On the nature of the function expressive of the law of human mortality and on a new mode of determining the value of life contingencies. Philosophical Transactions of the Royal Society, 115: 513-585

Guckenheimer J, Holmes P. 1984. Nonlinear oscillations, dynamical systems and bifurcations of vector fields. J. Appl. Mech, 51(4): 947

He Z, Lai X. 2011. Bifurcation and chaotic behavior of a discrete-time predator–prey system. Nonlinear Analysis: Real World Applications, 12(1): 403-417

Holling CS. 1959. Some characteristics of simple types of predation and parasitism. The Canadian Entomologist, 91(7): 385-398

Holling CS. 1965. The functional response of predators to prey density and its role in mimicry and population regulation. The Memoirs of the Entomological Society of Canada, 97(S45): 5-60

Huang J, Liu S, Ruan S,Xiao D. 2018. Bifurcations in a discrete predator–prey model with nonmonotonic functional response. Journal of Mathematical Analysis and Applications, 464(1): 201-230

Jha Pk, Ghorai S. 2017. Stability of prey-predator model with Holling type response function; and selective harvesting. Journal of Applied and Computational Mathematics, 6(3): 1-7

Jun-Ping C, Hong-De Z. 1986. The qualitative analysis of two species predator-prey model with Holling's type III functional response. Applied Mathematics and Mechanics, 7(1): 77-86

Liu X, Xiao D. 2007. Complex dynamic behaviors of a discrete-time predator–prey system. Chaos, Solitons and Fractals, 32(1): 80-94

Lotka AJ. 1925. Elements of physical biology. Williams and Wilkins. Baltimore, Md, USA

Lu XS, Chen G, Wang BH, Fang JQ. 2003. Hybrid control of period-doubling bifurcation and chaos in discrete nonlinear dynamical systems. Chaos, Solitons and Fractals, 18(4): 775-783

Malthus TR. 1798. An Essay on The Principles of Populations. St. Paul's, London, UK

May RM. 1973. Stability and Complexity in Model Ecosystem. Monograph in Population Biology VI, Princeton University Press, USA

Molgedey L, Schuchhardt J, Schuster HG. 1992. Suppressing chaos in neural networks by noise. Physical review letters, 69(26): 3717

MurdochWW, Briggs CJ, Nisbet RM, Gurney WS, Stewart-Oaten A. 1992. Aggregation and stability in metapopulation models. The American Naturalist, 140(1): 41-58

Murray JD. 1989. Mathematical Biology. Springer, New York, USA

Ott E, Grebogi C, Yorke JA. 1990. Erratum: "Controlling chaos". Physical Review Letters, 64: 2837

Parthasarathy S. 1992. Homoclinic bifurcation sets of the parametrically driven Duffing oscillator. Physical Review A, 46(4): 2147

Reeve JD. 1988. Environmental variability, migration, and persistence in host-parasitoid systems. The American Naturalist, 132(6): 810-836

Robinson C. 1998. Dynamical Systems: Stability, Symbolic Dynamics, and Chaos. CRC Press, USA

Roy P, Ghosh N. 2013. Discrete-time Prey-Predator model with generalized Holling type interaction. International Journal of Information Technology, Modeling and Computing (IJITMC), 1(4)

Strogatz S, Friedman M, Mallinckrodt AJ, McKay S. 1994. Nonlinear dynamics and chaos: with applications to physics, biology, chemistry, and engineering. Computers in Physics, 8(5): 532-532

Verhulst PF. 1838. Notice sur la loi que la population suit dans son accroissement. Corresp. Math. Phys, 10: 113-126

Volterra V. 1926. Variazioni e fluttazionidelnumero di individui in species animaliconviventi. Memorie Accademia Nazionaledei Lincei, 2: 31-313

Wan YH. 1978. Computation of the Stability Condition for the Hopf Bifurcation of Diffeomorphisms on R^2. SIAM Journal on Applied Mathematics, 34(1): 167-175

Wiggins S. 2003. Introduction to Applied Nonlinear Dynamical Systems and Chaos (Vol. 2). Springer Science and Business Media, Netherlands

Xiao D, Ruan S. 2001. Global dynamics of a ratio-dependent predator-prey system. Journal of Mathematical Biology, 43(3): 268-290