Article

Reliability analysis of flow networks with an ecological perspective

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Abstract
This paper attempts to set the stage for a prospective interplay between ecology and reliability theory concerning the common issue of the concept of a capacitated or flow network. The paper treats the problem of species survivability, which pertains to the ability of a specific species to avoid local extinction by migrating from a critical habitat patch to more suitable destination habitat patches via perfect stepping stones and heterogeneous imperfect corridors. The paper proposes various types of techniques for analyzing a capacitated ecological network for the process of migration in a metapopulation landscape network that arises when paths to destination habitat patches share common corridors. These techniques include (a) Karnaugh maps, which are crucial in providing not only the visual insight necessary to write better future software but also constitute an adequate means of verifying such software and, (b) a generalization of the max-flow min-cut theorem that is applicable through the identification of minimal cut-sets and minimal paths in the ecological flow network. Care is taken to ensure that the reliability expressions obtained are as compact as possible and to check them for correctness. The ecological network capacity is a random pseudo-Boolean (-switching) function of the corridor successes; and hence, its expected value is easily obtainable from its sum-of-products formula. This network capacity has obvious benefits in the representation of nonbinary discrete random functions, which commonly arise during the analysis of flow networks. A tutorial example demonstrates these methods and illustrates their computational merits with ample details.

Keywords capacitated networks; map method; max-flow min-cut theorem; pseudo-switching function; habitat patch; ecological corridor; species migration; species survivability.

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1 Introduction

System reliability analysis is a notable field of study within reliability engineering dealing with expressing the reliability of a system in terms of the reliabilities of its constituent components. The field encompasses several important issues more than its name suggests. It pertains not only to system analysis as such, but also to system design and optimization, quantification of uncertainty, selection of most important components and optimal allocation of redundancy.

Ecosystem reliability analysis might be described as an advanced application of probability theory. It is basically based on the algebra of events (a version of set algebra), which is isomorphic to the bivalent or 2-valued Boolean algebra (switching algebra) (Crama and Hammer, 2011; Mano, 2017; Nabulsi et al., 2017). Instead of using the algebra of events, modern system reliability analysis utilizes switching algebra by employing the indicator variables for probabilistic events instead of the events themselves. Though most ecological studies merely consider ecological connectivity, this paper deals with the more advanced concept of ecological network capacity which represents the number of individual organisms of a migratory wildlife species flowing through certain ecological corridors. This network capacity is a pseudo-Boolean (-switching) function of the corridor successes (Hammer and Rudeanu, 2012; Rushdi, 1987b; 1987c, 1988, 1989, 1990); thus, its mean value is easily obtainable from its sum-of-products formula (Rushdi, 1988). This network capacity has obvious benefits in representation of nonbinary discrete random functions, which commonly arise in the analysis of flow networks.

We consider a network of habitat patches, stepping stones and corridors. Long-term survival is possible only in habitats or habitat patches, while corridors and stepping stones can only help migration by supporting short-term survival. We assume that one of the habitat patches (identified as a critical habitat patch) is strongly disturbed, where survival becomes impossible and the local population has to emigrate to other habitat patches. Other habitat patches and stepping stones are available for migration from the critical habitat patch, if they are connected to this patch by ecological corridors.

Ecosystem reliability analysis often assumes that the system under study is represented by a probabilistic graph in a two-state model, and the ecological system operates successfully if there exists at least one successful path from a critical habitat patch to more suitable destination habitat patches. According to this point of view, reliability is considered, in fact, as a matter of connectivity only, and hence it does not seem to reasonably reflect the nature or capture the essence of most real-life ecological systems. In fact, connectivity measures implicitly assume unrealistically that a perfect corridor allows the passage of an unlimited or infinite number of individual organisms of the migrating species.

Many physical systems such as ecological systems and transportation systems which play important roles in our modern society can be regarded, in fact, as capacitated-flow networks whose capacities of arcs are independent, limited, and real-valued random variables. These networks are usually modeled by a stochastic graph $G \equiv (V, E)$ (where V and E are the sets of vertices (nodes) and edges (branches) of G) on which a set $K \subseteq V$ is distinguished (Politof and Satyanarayana, 1986). A very important special case arises when the set K is an ordered set of just two nodes, namely a source node (s) and a terminal node (t). In the landscape graph, vertices (nodes) represent habitat patches and stepping stones, while edges (branches) represent corridors.

There are two main parameters that are generally used to quantify the performance of ecological network. These are:

(a) Ecological network reliability which is simply a measure of probabilistic connectivity since it equals the probability that certain connections on directed or undirected general or special graphs, with dependent or independent components (patches such as habitats and stepping stones or corridors) exist in G among the nodes (patches) in K (and from s to t in the particular st case which represent in the landscape graph from a critical habitat patch to destination habitat patches via stepping stones and imperfect heterogeneous corridors).There are two extreme situations, namely (I) the afore-mentioned st case when K contains only two nodes (patches), the source s and the destination t, or (II) when K contains all nodes (patches) of the graph for which (K =V), typically depicted as the overall reliability case (Rushdi, 1984). The capacity limitations of the various links (corridors) and the overall flow requirements of the network are generally ignored.

(b) Network s-t capacity which equals the maximum flow that can be passed from a source node (critical patch) to a terminal node (destination patch) so that no branch capacity is violated, and under the assumption that all branches (corridors) are working (Ford and Fulkerson, 2009; 2015; Tanenbaum, 2003; Tucker, 2012; Madry, 2016; Williamson, 2019; Riis and Gadouleau, 2019). The failure probabilities of the communication links (corridors) as well as the nodes (patches) are implicitly ignored.

Researchers (Aggarwal, 1985; Trstensky and Bowron, 1984; Rushdi, 1988; Ramírez and Gebre, 2007; Yeh, 2002; Patra and Misra, 1996; Fusheng, 2009; El and Yeh, 2016; Kabadurmus and Smith, 2017; Cancela et al., 2019; Rushdi and Alsalami, 2020a, 2020b) have suggested methods to define a composite performance index for a network that integrates the important measures of connectivity and capacity. Moreover, there are some papers that attempted to utilize the concept of flow networks in ecology (Phillips et al., 2008; Zhang and Wu, 2013; Haruna, 2013; Taylor et al., 2016). The main thrust of all these methods is the observation that, while the connectivity between the source and the sink nodes is a necessary condition for successful operation of a communication network, it is not a sufficient condition. The success of the s-t connection should also ensure the availability of the required s-t capacity. Success in the ecological sense does not necessarily mean the single source-to-terminal successes, but it generally means the ORing of several source-to-terminal successes (Rushdi and Hassan, 2015, 2016a, 2016b, 2020). It is quite similar to broadcasting success, which is the ANDing of such source-to-terminal successes (Rushdi and Hassan, 2016a).

To set some foundation for future interaction between the generalized reliability concepts of flow networks and ecological networks, this paper presents a tutorial exposition of the various methods for analyzing capacitated networks as applied to ecology.

The first method is a map procedure that results in simple symbolic expressions for the performance indexes. However, this technique can only be applied manually to small or moderate networks (Rushdi and Ghaleb, 2015; Rushdi and Badawi, 2017a, 2017b; Rushdi and Ba-Rukab, 2017a; Rushdi, 1988, 2018a; Rushdi and Rushdi, 2018; Rushdi and Alsalami, 2020a, 2020b).

A second method generalizes the “Max-Flow Min-Cut Theorem” (Ford and Fulkerson, 2009, 2015; Tanenbaum, 2003; Tucker, 2012; Madry, 2016; Zhang, 2018a, 2018b; Williamson, 2019; Riis and Gadouleau, 2019; Rushdi and Alsalami, 2020b) for network states X other than the ideal state (X = 1). This technique is very fast when the network minimal cutsets (Rushdi, 1983a; Zhang, 2016), and possibly its minimal paths (Rushdi, 1983a) are known.

We have a fresh look at a problem that was earlier considered by Rushdi and Hassan (2016a) concerning species survivability(called survival reliability therein) which is the probability of successful migration of a specific species from a critical habitat patch to one or more destination habitat patches via imperfect heterogeneous corridors. We add an element of capacity to the aforementioned problem. Our exposition is not only a review of existing reliability techniques in an ecology setting, but it also introduces and evaluates a new reliability measure of capacitated or flow network when the ecological network has several destination habitat patches that share some edges (corridors) in common.

The paper will be of a significant impact if it succeeds in triggering some ecologists to reformulate some of their flow network problems in a network-reliability context, and then to challenge reliability experts to apply their advanced tools to these problems. Beneficial mutual interaction would arise between ecology and reliability fields, and an interdisciplinary sub-field might emerge. This paper might be thought of as equal of a series of works along the same paradigm (Rushdi and Hassan, 2015, 2016a, 2016b, 2020).

The organization of the rest of this paper is as follows. Section 2 presents the underlying assumptions for our model, the notation used as well as some useful nomenclature in the ecology and reliability domains. Section 3 reviews the concept of the algebraic decomposition formula which serves as the most fundamental theorem for a pseudo-switching function that is used to obtain the general capacity function and its mean for the ecological network. Section 4 shows how the Karnaugh map is conveniently used to represent a pseudo-switching function for the ecological network. Section 5 presents one of the crucial and old techniques in capacitated or flow networks which is the “Max-Flow Min-Cut Theorem”. We identify the required minimal cut-sets and minimal paths in the ecological flow networks. Moreover, we verify equivalence of the results between the two afore-mentioned methods. Section 6 concludes the paper.

2 Assumptions, Notation, and Nomenclature

2.1 Assumptions

1. The ecological network considered is modeled as a linear graph consisting of (a) links(corridors) of imperfect reliabilities and limited capacities and (b) nodes (destination habitat patches and stepping stones) which are perfectly reliable (not susceptible to failure) and have unconstrained capacities.
2. The analysis concerns one particular species, henceforth called the pertinent or concerned species. The analysis does not take into account any characteristic of the species. This limitation of the current paper as species-specific is a general one in most (essentially all) ecological studies.
3. The pertinent species is in danger of local extinction in a certain habitat patch called the critical habitat patch. It escapes such extinction by migrating to a new habitat patch (one out of a few destination habitat patches) through imperfect heterogeneous corridors and perfect stepping stones.
4. Each of the corridors in the ecological network is in one of two states, either good (permeable) or failed (deleted or destroyed). Corridor successes are statistically independent. This assumption does not extend to ‘equivalent' corridor to be introduced in the network, which can be of multistate natures and statistically dependent.
5. The migration system is also in one of two states, either successful or unsuccessful.
6. Certain values are assigned to each corridor $(i\,,j)$ for its reliability p_{ij} and capacity c_{ij}, where $0 \leq\ p_{ij} \leq 1\ ,\ c_{ij} \geq 0$. The corridor capacity sets an upper bound on the flow of the pertinent species through the corridor in either direction.

2.2 Notation

n Number of ecological corridors in the logic diagram of the network, $n \geq 0$.

X_i Success of corridor i = indicator that the concerned species successfully migrates through corridor i = a switching random variable that takes only one of the two discrete values 0 and 1; ($X_i = 1$ if corridor i is permeable, while $X_i = 0$ if corridor i is failed).

$\overline{X}_i$ Failure or deletion of corridor i = indicator variable for unsuccessful migration of the pertinent species through i, where $\overline{X}_i = 0$ if corridor i is good, while $\overline{X}_i = 1$ if corridor i is deleted/destroyed. The success X_i and the failure $\overline{X}_i$are complementary variables.

$\boldsymbol{X}, \boldsymbol{p}, \boldsymbol{c}$ N-dimensional vectors of corridor i successes, reliabilities and capacities: $\boldsymbol{X} = (X_1 X_2 \ldots X_n)^T$; $\boldsymbol{p} = (p_1 p_2 \ldots p_n)^T$; $\boldsymbol{c}\ \equiv (c_1 c_2 \ldots c_n)^T$.

$S(\boldsymbol{X})$ Indicator variable for the successful operation of the system (successful migration of the pertinent species), called system success. Successful operation can be equivalent to mere connectivity, or to the satisfaction of a certain flow requirement (Aggarwal, 1985; Rushdi, 1983a; Lee, 1980).

$E[\dots]$ Expectation of the random variable [. . .].

p_i, q_i Reliability and unreliability of corridor i : $p_i \equiv \Pr\{X_i = 1\}, q_i \equiv \Pr\{\overline{X}_i = 1\} = 1 - p_i$. Both p_i and q_i are real values in the closed real interval [0.0,1.0].

R, U Network reliability and unreliability;

$$R = \Pr\{S(\mathbf{X}) = 1\} = E\{S(\mathbf{X})\}, U = \Pr\{\overline{S} = 1\} = 1.0 - R, 0.0 \leq R, U \leq 1.0 .$$

c_i Flow capacity of corridor i ; $c_i \geq 0$.

$\boldsymbol{X}_k$ State k of the ecological network, denoted by a particular value of the n-dimensional vector **X**, $k = 0,1,2,\dots,2^n - 1$.

$C_{ij}(\boldsymbol{X})$ Capacity function of (i,j) which is the maximum flow interconnection from i to j in state $\boldsymbol{X}$ that does not violate branch capacities, $C_{ij}(\boldsymbol{X}) \geq 0$. For an original (i,j) : $C_{ij} = c_{ij}X_{ij}$. Since $\boldsymbol{X}$ is a switching random vector, $C_{ij}(\boldsymbol{X})$ is a discrete random variable of a probability mass function (pmf) of no more than 2^n distinct values.

C_{ij}^T Terminal- pair capacity function from node i to node j ; $C_{ij}^T \geq 0$.

C_{ijmax} Maximum capacity function of the (i,j) corridor; in the ideal case when all corridors are functioning, $C_{ijmax} = C_{ij}(1)$.

s, t Source, terminal node

$C_{ij}(\boldsymbol{X}|1_l), C_{ij}(\boldsymbol{X}|0_l)$ The function $C_{ij}(\boldsymbol{X})$ when X_l is set to 1 or 0. Meanings of $C_{ij}(\boldsymbol{X}|1_l, 1_m), \dots$ etc., follows similarity.

P_i A Random switching variable expressing the success of minimal path i (the critical habitat patch is connected to a destination one). This is a prime implicant of the success S. It is a conjunction of the successes of elements belonging to the path (tie-set).

C_j A Random switching variable expressing the failure of cut- set j (the critical habitat patch is disconnected from all destination habitat patches). This is a prime implicant of the failure $\bar{S}$. It is a conjunction of failures of elements belonging to the cut-set.

2.3 Ecology nomenclature

Habitat patch: a place where the local population of the pertinent species may reproduce and survive for a long term.

Stepping stone: a relatively small place that helps the migration of the local population of the pertinent species, but is not suitable for its long-term survival.

Ecological corridor: a physical area which connects patches (habitat patches and stepping stones) and makes migration possible for a given species between habitat patches. However, a corridor is not expected to support long-term survival for the species.

2.4 Reliability nomenclature

A Boolean (Switching) function $S(\boldsymbol{X})$: A mapping $\{0,1\}^n \to \{0,1\}$, i.e., $S(\boldsymbol{X})$ is any one particular assignment of the two functional values (0 or 1) for all possible 2^n values of $\boldsymbol{X}$ (Crama and Hammer, 2011; Mano, 2017; Nabulsi et al., 2017).

Pseudo-Boolean (Switching) function $C(\boldsymbol{X})$: A mapping $\{0,1\}^n \to R$ where R is the field of real numbers, i.e. $C(\boldsymbol{X})$ is an assignment of a real number for each of the possible 2^n values of $\boldsymbol{X}$ (Rushdi, 1987b, 1987c, 1988, 1989, 1990; Hammer and Rudeanu, 2012; Anthony et al., 2016; Roy, 2020).

Multiaffine function of n variables $\boldsymbol{R(p_1, p_2 \ldots, p_n)}$: An algebraic function which is a first-degree polynomial in each of its variables, i.e. if fixed values are given to any $(n-1)$variables, the function reduces to a first-degree polynomial in the remaining variable. Examples of multi-affine functions involve:

1. Definite algebraic functions such as
   (a) System reliability/unreliability as a function of component reliability/unreliability (Rushdi, 1983b; Rushdi, 1985).
   (b) System availability/unavailability (Rushdi, 1985; Modarres, 2006; Bamasak and Rushdi, 2015).
2. Pseudo-Boolean (switching) functions (Hammer and Rudeanu, 2012; Rushdi, 1987b, 1987c, 1988, 1989, 1990) such as source-to-terminal capacity or the squared capacity as a function of link successes.

Path (tie-set): an implicant of system success; a set of components (corridors) whose functioning ensures that the system functions, i.e., secures the required flow from the required habitat patch to some of the destination habitat patches.

Minimal path: a prime implicant of system success; a path for which all components must function for the system to function (Ebeling, 1997). By contrast to the mere connectivity situation, this minimal path cannot be visually drawn on the network graph as it does not correspond to the graph-theoretic concept of a "path" (Rushdi and Al-Khateeb, 1983). The disjoint paths can be found either by algebraic analysis or visually through k-map.

Cut (cut-set): an implicant of system failure; a set of components (corridors) whose failure ensures that the system fails, i.e., falls short of securing the required flow from the required habitat patch to some of the destination habitat patches.

Minimal cut: a prime implicant of system failure; a cut-set for which all components must fail for the system to fail (Ebeling, 1997). By contrast to the mere connectivity situation, this minimal cut-set cannot be visually drawn on the network graph as it does not correspond to the graph-theoretic concept of a "cut-set" (Rushdi and Al-Khateeb, 1983).

2.5 Reliability-Ready Expression (RRE)

An expression in the Boolean (Switching) domain, in which logically multiplied (ANDed) entities are statistically independent and logically added (ORed) and entities are disjoint. Such an expression can be directly transformed, on a one-to-one basis, to the algebraic or probability domain by replacing switching (Boolean) indicators by their statistical expectations, and also replacing logical multiplication and addition (ANDing and ORing) by their arithmetic counterparts (Rushdi, 1987c; Rushdi and Ba-Rukab, 2005a, 2005b; Rushdi and Hassan, 2016a; Rushdi and Rushdi, 2017).

3 Capacity and Its Mean

The source-to-terminal capacity as a function of corridor successes $C_{ij}(\boldsymbol{X})$ is a real-valued function of binary arguments, and hence it is a pseudo-switching function that obeys the algebraic decomposition formula:

$$C_{ij}(\boldsymbol{X}) = \overline{X}_l C_{ij}(\boldsymbol{X}|0_l) + X_l C_{ij}(\boldsymbol{X}|1_l)$$

$$= (1 - X_l) C_{ij}(X|0_l) + X_l C_{ij}(X|1_l)$$
$$= C_{ij}(X|O_l) + [C_{ij}(X|1_l) - C_{ij}(X|O_l)] X_l \,, \qquad l = 1,2,\ldots,n \tag{1}$$

Equation (1) can be easily proved by perfect induction over all possible values of $\boldsymbol{X}$, namely, $\{X|0_l\}$ and$\{X|1_l\}$. It means that $C_{ij}(\boldsymbol{X})$ is a multiaffine function. Hence $C_{ij}(\boldsymbol{X})$ can always be written in a sum-of-products (s-o-p) form, where the conventional arithmetic meanings for ‘sum’ and ‘product’ (rather than the logical ones) are implicitly understood. Furthermore, $C_{ij}(\boldsymbol{X})$is completely specified by the 2^n coefficients $C_{ij}(\boldsymbol{X}_k)$ corresponding to the 2^n values $\boldsymbol{X}_k$, that its argument$\boldsymbol{X}$ takes. Consequently, $C_{ij}(\boldsymbol{X})$can be conveniently expressed in the form of a truth table or a Karnaugh map of real entries (Rushdi and Rushdi, 2018).If the random function $C_{ij}(\boldsymbol{X})$ is written in s-o-p form, then its mean value: $E\{C_{ij}(\boldsymbol{X})\} = E\{C_{ij}\}(\boldsymbol{p})$, can be derived from it directly by replacing the arguments X_l, and $\overline{X}_l$ by their means $p_l$and q_l, respectively, viz,

$$C_{ij}(\boldsymbol{X})(\text{s-o-p}) \xleftrightarrow{\{X_l, \overline{X}_l\} \leftrightarrow \{p_l, q_l\}} E\{C_{ij}\}(\boldsymbol{p})(\text{s-o-p}) \tag{2}$$

Equation (2) results immediately from the fact that the mean of a sum is the sum of means, and the assumption that the $X_l's$ are statistically independent. Not only the capacity $C_{ij}(\boldsymbol{X})$ but also the capacity squared $C_{ij}^2(\boldsymbol{X})$ is a pseudo-switching function. Therefore, $C_{ij}^2(\boldsymbol{X})$ can also be put in s-o-p form, so that it becomes readily convertible into its mean:

$$C_{ij}^2(\boldsymbol{X})(\text{s-o-p}) \xleftrightarrow{\{X_l, \overline{X}_l\} \leftrightarrow \{p_l, q_l\}} E\{C_{ij}^2\}(\boldsymbol{p})(\text{s-o-p}) \tag{3}$$

Equations (2) and (3) show that computing the mean $E\{C_{ij}\}$ and the variance of the capacity

$$VAR\{C_{ij}\} = E\{C_{ij}^2\} - (E\{C_{ij}\})^2$$

can be achieved by ensuring that both the capacity itself and its square are expressed in s-o-p form.

4 A Map Procedure

The pseudo-switching function $C_{st}(\boldsymbol{X})$ can be specified by a modified Karnaugh map (Rushdi and Ghaleb, 2015; Rushdi and Badawi, 2017a, 2017b; Rushdi and Ba-Rukab, 2017a; Rushdi, 1988, 2018a; Rushdi and Rushdi, 2018; Rushdi and Alsalami, 2020a, 2020b) which is a very powerful manual tool that provides pictorial insight about the various functional properties and procedures. The map variables are the elements of $\boldsymbol{X}$ and the map entries are the real numbers $C_{st}(\boldsymbol{X}_k)$ which represent the s-t or corridor capacity from a critical habitat patch to destination habitat patches for states $\boldsymbol{X}_k$, and hence are not necessary 1's and 0's. These numbers can be obtained individually or collectively via any of the procedures in Section 5.

To express $C_{st}(\boldsymbol{X})$ in an almost minimal s-o-p form, it is necessary to cover the nonzero entries of the map by the smallest possible number of map loops. Each of these loops should be the largest that combines $2^i\{i = 0, 1, 2, \dots, n\}$ adjacent cells of the map containing as a minimum a certain (so far uncovered) value. The contribution of such a loop to the s-o-p expression of $C_{st}(\boldsymbol{X})$ equals its covered value multiplied by the usual loop term. To allow for the choice of larger loops, a cell entry may be partitioned into several values to be covered by several loops. Such a partition is usually possible for integer-valued entries in maps describing small-size networks. Once a portion of an entry is covered, that entry is replaced by its uncovered portion. In particular, if an entry is totally covered, then it is replaced by zero. The procedure terminates when all entries in the map become 0's.

The above map procedure results in capacity expressions that are simpler than those obtained by the direct state-enumeration method in (Aggarwal, 1985). It is particularly useful when the map entries belong to a small set of integral values, which is usually the case when the branch (corridor) capacities are integer valued. Though the map procedure suffers the limitation that it is capable of handling only small networks (of seven branches or less), it can be extended to handle moderate networks through the use of variable-entered Karnaugh maps (VEKMs) (Rushdi, 1983b, 1987a, 2001, 2004, 2018a, 2018b; Rushdi and Al-Shehri, 2004; Rushdi and Amashah, 2012; Rushdi and Albarakati, 2012; Rushdi and Ba-Rukab, 2017b; Rushdi and Alsalami, 2020a, 2020b).

Example 1

This example applies the map procedure to the hypothetical ecological network which has several destination habitat patches sharing some edges in common. This network is shown in Fig. 1 and its branch (corridor) capacities are: $c = [\,2\;\;3\;\;4\;\;5\;\;6\;\;7\;\;8\;\;9\;\;10\,]^{\mathrm{T}}$

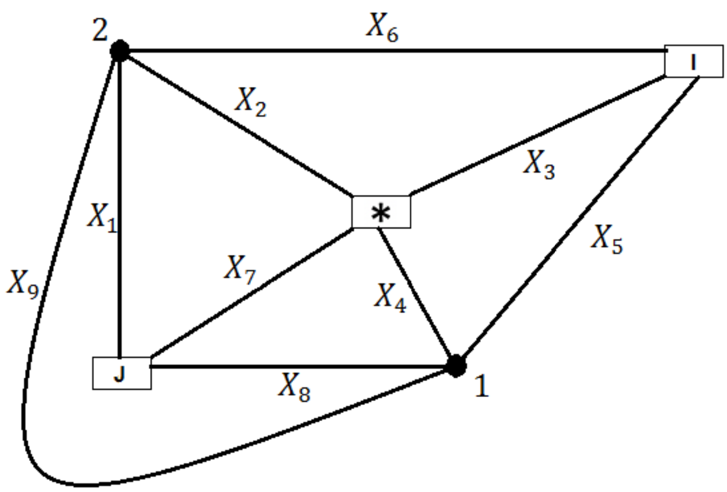


Fig. 1 A 9-branch (corridor) ecological network of a capacity vector $c = [\,2\;\;3\;\;4\;\;5\;\;6\;\;7\;\;8\;\;9\;\;10\,]^T$.

	X_1
$C_{st}(X\|0_1,0_6) = \bar{X}_2\bar{X}_7[4X_3 + 5X_4(X_5 + \bar{X}_5X_8)] + \bar{X}_2X_7[8 + 4X_3 + 5X_4(X_5 + \bar{X}_5X_8)] + X_2\bar{X}_7[X_4X_5\bar{X}_8X_9 + 3X_9(X_8 + \bar{X}_4X_5\bar{X}_8) + 4X_3 + 5X_4(X_5 + \bar{X}_5X_8)] + X_2X_7[8 + X_4X_5\bar{X}_8X_9 + 3X_9(X_8 + \bar{X}_4X_5\bar{X}_8) + 4X_3 + 5X_4(X_5 + \bar{X}_5X_8)]$	$C_{st}(X\|1_1,0_6) = \bar{X}_2\bar{X}_7[2X_4\bar{X}_5\bar{X}_8X_9 + 4X_3 + 5X_4(X_5 + \bar{X}_5X_8)] + \bar{X}_2X_7[8 + 2X_4\bar{X}_5\bar{X}_8X_9 + 4X_3 + 5X_4(X_5 + \bar{X}_5X_8)] + X_2\bar{X}_7[2 + X_9(X_8 + X_5\bar{X}_8) + 4X_3 + 5X_4(X_5 + \bar{X}_5X_8)] + X_2X_7[10 + X_9(X_8 + X_5\bar{X}_8) + 4X_3 + 5X_4(X_5 + \bar{X}_5X_8)]$
X_6: $C_{st}(X\|0_1,1_6) = \bar{X}_2\bar{X}_7[4X_3 + 5X_4(X_5 + \bar{X}_5(X_8 + \bar{X}_8X_9))] + \bar{X}_2X_7[8 + 4X_3 + 5X_4(X_8 + \bar{X}_8(X_9 + X_5\bar{X}_9))] + X_2\bar{X}_7[3 + 4(X_3 + X_4\bar{X}_5\bar{X}_8X_9) + 5X_4(X_5 + \bar{X}_5X_8)] + X_2X_7[11 + 4(X_3 + X_4\bar{X}_5\bar{X}_8X_9) + 5X_4(X_5 + \bar{X}_5X_8)]$	$C_{st}(X\|1_1,1_6) = \bar{X}_2\bar{X}_7[4X_3 + 5X_4(X_5 + \bar{X}_5(X_8 + \bar{X}_8X_9))] + \bar{X}_2X_7[8 + 4X_3 + 5X_4(X_5 + \bar{X}_5(X_9 + X_8\bar{X}_9))] + X_2\bar{X}_7[3 + 4X_3 + 5X_4(X_5 + \bar{X}_5(X_8 + \bar{X}_8X_9))] + X_2X_7[11 + 4X_3 + 5X_4(X_5 + \bar{X}_5(X_9 + X_8\bar{X}_9))]$

$C_{st}(\boldsymbol{X})$

Fig. 2 A variable-entered Karnaugh map for the pseudo-Boolean function $C_{st}(\boldsymbol{X})$ with map variables X_1 and X_6 corresponding to the two elements in the ecological network of Fig. 1

X_1 X_3 X_5 X_6 X_7 X_8 X_9 X_4 X_2

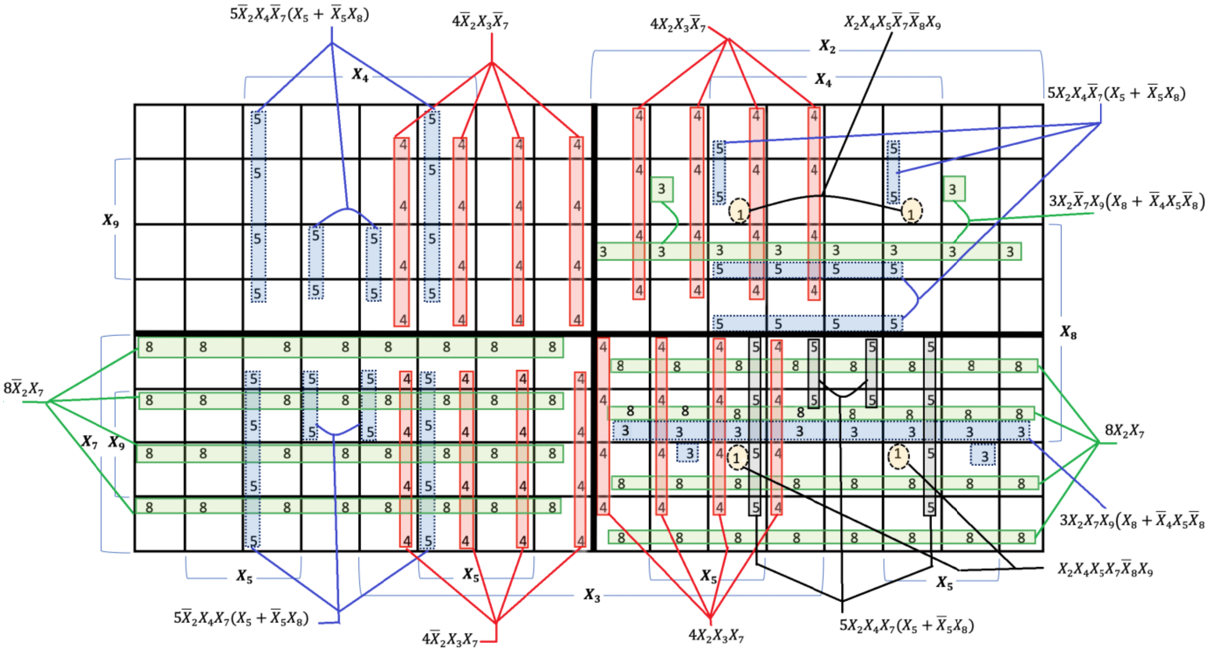
0	0	5	0	4	9	4	4	4	4	9	4	0	5	0	0	2	2	7	2	6	11	6	6	4	4	9	4	0	5	0	0
0	0	5	0	4	9	4	4	4	7	10	4	0	6	3	0	2	3	8	2	6	12	7	6	4	4	9	6	2	5	0	0
0	0	5	5	9	9	4	4	7	7	12	12	8	8	3	3	3	3	8	8	12	12	7	7	4	4	9	9	5	5	0	0
0	0	5	5	9	9	4	4	4	4	9	9	5	5	0	0	2	2	7	7	11	11	6	6	4	4	9	9	5	5	0	0
8	8	13	13	17	17	12	12	12	12	17	17	13	13	8	8	10	10	15	15	19	19	14	14	12	12	17	17	13	13	8	8
8	8	13	13	17	17	12	12	15	15	20	20	16	16	11	11	11	11	16	16	20	20	15	15	12	12	17	17	13	13	8	8
8	8	13	8	12	17	12	12	12	15	18	12	8	14	11	8	10	11	16	10	14	20	15	14	12	12	17	14	10	13	8	8
8	8	13	8	12	17	12	12	12	12	17	12	8	13	8	8	10	10	15	10	14	19	14	14	12	12	17	12	8	13	8	8
8	8	13	8	12	17	12	12	15	15	20	15	11	16	11	11	11	11	16	11	15	20	15	15	12	12	17	12	8	13	8	8
8	8	13	13	17	17	12	12	15	15	20	19	15	16	11	11	11	11	16	16	20	20	15	15	12	12	17	17	13	13	8	8
8	8	13	13	17	17	12	12	15	15	20	20	16	16	11	11	11	11	16	16	20	20	15	15	12	12	17	17	13	13	8	8
8	8	13	13	17	17	12	12	15	15	20	20	16	16	11	11	11	11	16	16	20	20	15	15	12	12	17	17	13	13	8	8
0	0	5	5	9	9	4	4	7	7	12	12	8	8	3	3	3	3	8	8	12	12	7	7	4	4	9	9	5	5	0	0
0	0	5	5	9	9	4	4	7	7	12	12	8	8	3	3	3	3	8	8	12	12	7	7	4	4	9	9	5	5	0	0
0	0	5	5	9	9	4	4	7	7	12	11	7	8	3	3	3	3	8	8	12	12	7	7	4	4	9	9	5	5	0	0
0	0	5	0	4	9	4	4	7	7	12	7	3	8	3	3	3	3	8	3	7	12	7	7	4	4	9	4	0	5	0	0

$C_{st}(\boldsymbol{X})$

Fig. 3 Karnaugh map representation of the capacity pseudo-Boolean function $C_{st}(\boldsymbol{X})$.

Fig. 2 shows A variable-entered Karnaugh map for the pseudo-Boolean function $C_{st}(\boldsymbol{X})$ with map variables X_1 *and* X_6. In addition, Fig. 3 shows the Karnaugh map representation of the capacity pseudo-Boolean function $C_{st}(\boldsymbol{X})$.

The map has $2^9 = 512$ cells such that each one of them depicts a definite state of the flow network. The Karnaugh map entries are the real numbers, which correspond to the integer values of the capacity pseudo-switching function $C_{st}(\boldsymbol{X})$ for states $\boldsymbol{X}$.



$C_{st}(\boldsymbol{X}|0_1,0_6)$

Fig. 4 A multi- Step Map Procedure to cover the s-t capacity function for subnetwork $C_{st}(\boldsymbol{X}|0_1,0_6)$for nonzero entries.

Fig. 4 demonstrates a multi-step map procedure to cover entries other than zero in the map representing thepseudo-switching function of the subnetwork $C_{st}(\boldsymbol{X}|0_1,0_6)$. This map is divided into four sections, each section representing the cells of the flow capacity with map variables X_2 *and* X_7.

In step 1, this procedure covers every cell in each section in the map that possesses an entry that is at least 8, i.e., entries that turn out to be of values 8, 11, 12, 13, 14, 15, 16, 17, 18 and 20; thus, the remaining entries in these cells will be 0,3,4,5,6,7,8,9,10 and 12, respectively. Therefore, the task in each of the next steps will be to cover the remaining nonzero map entries which turn out to be 1, 3, 4 and 5. We do the same procedures with the rest of the entries in the map.

The minimal sum-of-product equation for the pseudo-switching function $C_{st}(\boldsymbol{X})$ and the corresponding one for its mean are

$$
\begin{aligned}
C_{st}(\boldsymbol{X}) = {} & \overline{X}_1\overline{X}_6\Big[\overline{X}_2\overline{X}_7[4X_3 + 5X_4(X_5 + \overline{X}_5X_8)] + \overline{X}_2X_7[8 + 4X_3 + 5X_4(X_5 + \overline{X}_5X_8)] \\
& + X_2\overline{X}_7[X_4X_5\overline{X}_8X_9 + 3X_9(X_8 + \overline{X}_4X_5\overline{X}_8) + 4X_3 + 5X_4(X_5 + \overline{X}_5X_8)] \\
& + X_2X_7[8 + X_4X_5\overline{X}_8X_9 + 3X_9(X_8 + \overline{X}_4X_5\overline{X}_8) + 4X_3 + 5X_4(X_5 + \overline{X}_5X_8)]\Big] \\
& + \overline{X}_1X_6\Big[\overline{X}_2\overline{X}_7\left[4X_3 + 5X_4\left(X_5 + \overline{X}_5(X_8 + \overline{X}_8X_9)\right)\right] \\
& + \overline{X}_2X_7\left[8 + 4X_3 + 5X_4\left(X_8 + \overline{X}_8(X_9 + X_5\overline{X}_9)\right)\right] \\
& + X_2\overline{X}_7[3 + 4(X_3 + X_4\overline{X}_5\overline{X}_8X_9) + 5X_4(X_5 + \overline{X}_5X_8)] \\
& + X_2X_7[11 + 4(X_3 + X_4\overline{X}_5\overline{X}_8X_9) + 5\,X_4(X_5 + \overline{X}_5X_8)]\Big] \\
& + X_1\overline{X}_6\Big[\overline{X}_2\overline{X}_7[2X_4\overline{X}_5\overline{X}_8X_9 + 4X_3 + 5X_4(X_5 + \overline{X}_5X_8)] \\
& + \overline{X}_2X_7[8 + 2X_4\overline{X}_5\overline{X}_8X_9 + 4X_3 + 5X_4(X_5 + \overline{X}_5X_8)] \\
& + X_2\overline{X}_7[2 + X_9(X_8 + X_5\overline{X}_8) + 4X_3 + 5X_4(X_5 + \overline{X}_5X_8)] \\
& + X_2X_7[10 + X_9(X_8 + X_5\overline{X}_8) + 4X_3 + 5X_4(X_5 + \overline{X}_5X_8)]\Big] \\
& + X_1X_6\Big[\overline{X}_2\overline{X}_7\left[4X_3 + 5X_4\left(X_5 + \overline{X}_5(X_8 + \overline{X}_8X_9)\right)\right] \\
& + \overline{X}_2X_7\left[8 + 4X_3 + 5X_4\left(X_5 + \overline{X}_5(X_9 + X_8\overline{X}_9)\right)\right] \\
& + X_2\overline{X}_7\left[3 + 4X_3 + 5X_4\left(X_5 + \overline{X}_5(X_8 + \overline{X}_8X_9)\right)\right] \\
& + X_2X_7\left[11 + 4X_3 + 5X_4\left(X_5 + \overline{X}_5(X_9 + X_8\overline{X}_9)\right)\right]\Big]
\end{aligned}
$$

$$
\begin{aligned}
C_{st}(\boldsymbol{X}) = {} & 11\,X_2X_6X_7 + 10\,X_1X_2\overline{X}_6X_7 + 8(\overline{X}_2X_7 + \overline{X}_1X_2\overline{X}_6X_7) \\
& + 5X_4\left(X_5 + \overline{X}_5\left(X_8 + X_6\overline{X}_8X_9(X_1 + \overline{X}_1\overline{X}_2)\right)\right) + 4(X_3 + \overline{X}_1X_2X_4\overline{X}_5X_6\overline{X}_8X_9) \\
& + 3X_2\left(X_6\overline{X}_7 + \overline{X}_1\overline{X}_6X_9(X_8 + \overline{X}_4X_5\overline{X}_8)\right) + 2X_1\overline{X}_6(X_2\overline{X}_7 + \overline{X}_2X_4\overline{X}_5\overline{X}_8X_9) \\
& + X_2\overline{X}_6X_9\left(X_1X_8 + X_5\overline{X}_8(X_1 + \overline{X}_1X_4)\right)
\end{aligned} \quad (4)
$$

$$E\{C_{st}\}(\boldsymbol{p}) = 11\, p_2p_6p_7 + 10\, p_1p_2q_6p_7 + 8(q_2p_7 + q_1p_2q_6p_7) + 5p_4\left(p_5 + q_5\big(p_8 + p_6q_8p_9(p_1 + q_1q_2)\big)\right) + 4(p_3 + q_1p_2p_4q_5p_6q_8p_9) + 3p_2\big(p_6q_7 + q_1q_6p_9(p_8 + q_4p_5q_8)\big) + 2p_1q_6(p_2q_7 + q_2p_4q_5q_8p_9) + p_2q_6p_9\big(p_1p_8 + p_5q_8(p_1 + q_1p_4)\big) \quad (5)$$

The pseudo-switching function $C_{st}(\boldsymbol{X})$ can be converted into the switching function of success $S_{st}(\boldsymbol{X})$ by suppressing all non-unity numerals and replacing the arithmetic operators $\{+,\bullet\}$ by their logic counterparts, viz,

$$\begin{aligned} S_{st}(\boldsymbol{X}) = {} & X_2X_6X_7 \vee X_1X_2\overline{X}_6X_7 \vee \overline{X}_2X_7 \vee \overline{X}_1X_2\overline{X}_6X_7 \vee X_4\left(X_5 \vee \overline{X}_5\left(X_8 \vee X_6\overline{X}_8X_9\big(X_1 \vee \overline{X}_1\overline{X}_2\big)\right)\right) \vee X_3 \\ & \vee \overline{X}_1X_2X_4\overline{X}_5X_6\overline{X}_8X_9 \vee X_2\left(X_6\overline{X}_7 \vee \overline{X}_1\overline{X}_6X_9\big(X_8 \vee \overline{X}_4X_5\overline{X}_8\big)\right) \vee X_1\overline{X}_6\big(X_2\overline{X}_7 \vee \overline{X}_2X_4\overline{X}_5\overline{X}_8X_9\big) \\ & \vee X_2\overline{X}_6X_9\left(X_1X_8 \vee X_5\overline{X}_8\big(X_1 \vee \overline{X}_1X_4\big)\right) \end{aligned} \quad (6)$$

The final sum-of-product equation for the capacity squared $C_{st}^2(\boldsymbol{X})$ can be successfully obtained either by squaring expression (4) or by using the map technique in which all the map cell entries are squared for those of the Karnaugh map in Fig. 3. The equation for the pseudo-switching function $C_{st}^2(\boldsymbol{X})$ and its mean are

$$\begin{aligned} C_{st}^2(\boldsymbol{X}) = {} & 121\, X_2X_6X_7 + 100\, X_1X_2\overline{X}_6X_7 + 64\big(\overline{X}_2X_7 + \overline{X}_1X_2\overline{X}_6X_7\big) \\ & + 25X_4\left(X_5 + \overline{X}_5\left(X_8 + X_6\overline{X}_8X_9\big(X_1 + \overline{X}_1\overline{X}_2\big)\right)\right) + 16\big(X_3 + \overline{X}_1X_2X_4\overline{X}_5X_6\overline{X}_8X_9\big) \\ & + 9X_2\left(X_6\overline{X}_7 + \overline{X}_1\overline{X}_6X_9\big(X_8 + \overline{X}_4X_5\overline{X}_8\big)\right) + 4X_1\overline{X}_6\big(X_2\overline{X}_7 + \overline{X}_2X_4\overline{X}_5\overline{X}_8X_9\big) \\ & + X_2\overline{X}_6X_9\left(X_1X_8 + X_5\overline{X}_8\big(X_1 + \overline{X}_1X_4\big)\right) \end{aligned} \quad (7)$$

$$\{C_{st}^2\}(\boldsymbol{p}) = 121\, p_2p_6p_7 + 100\, p_1p_2q_6p_7 + 64(q_2p_7 + q_1p_2q_6p_7) + 25p_4\left(p_5 + q_5\big(p_8 + p_6q_8p_9(p_1 + q_1q_2)\big)\right) + 16(p_3 + q_1p_2p_4q_5p_6q_8p_9) + 9p_2\big(p_6q_7 + q_1q_6p_9(p_8 + q_4p_5q_8)\big) + 4p_1q_6(p_2q_7 + q_2p_4q_5q_8p_9) + p_2q_6p_9\big(p_1p_8 + p_5q_8(p_1 + q_1p_4)\big)$$

(8)

5 A Generalized Cutset Procedure

One of the important problems in flow or capacitated networks is the problem of maximum flow, which is simply the problem of finding the maximum number of total flow units from the source node (the critical habitat patch) to the terminal nodes (the destination habitat patches) taken collectively, so that no branch (corridor) capacity is violated. Two implicit assumptions in that problem is that (a) flow terminates at one, and only one, of the destination nodes, and (b) all network branches (corridors) are good, i.e., the network state is $\boldsymbol{X} = 1$ and the maximum flow is actually equal to $C_{st}(\mathbf{1})$. An elegant approach for solving the maximum flow problem is the maximum flow algorithm of Ford and Fulkerson (Rushdi, 1988; Ford and Fulkerson, 2009; 2015; Tanenbaum, 2003; Tucker, 2012; Madry, 2016; Zhang, 2018a, 2018b; Williamson, 2019; Riis and Gadouleau, 2019; Rushdi and Alsalami, 2020a, 2020b). A corollary of this approach is the "Max-Flow Min-Cut Theorem", which can be generalized for all network states as follows

$$C_{st}(\boldsymbol{X}) = \min\left\{\sum_{l \in M_i} c_l X_l\right\}, \tag{9}$$

where M_i is the set of branches (corridors) constituting the minimal s-t cutset number$i$ for the network (Rushdi, 1983a). Equation (9) includes certain series and parallel reduction rules as special cases (Rushdi, 1988).

To facilitate the computation of $C_{st}(\boldsymbol{X})$ via (9), it is noted that $C_{st}(\boldsymbol{X})$= 0 if state **X** is an s-t cutset (i.e., if there is no s-t connection) and $C_{st}(\boldsymbol{X}) \neq 0$ if state **X** is an s-t path (i.e., if there is some s-t connection). Therefore, if $\{P_j\}$ is a (preferably minimal) set of exhaustive and disjoint s-t paths (Rushdi, 1988), i.e., if

$$S_{st} = \bigvee_{j=1}^{n_p} P_j , \tag{10}$$

$$P_j \bigwedge P_k = 0 , for\ all\ j \neq k , \tag{11}$$

where $S_{st}$is the indicator variable for successful operation of the flow network which can be equivalent to connectivity (Lee, 1980; Rushdi, 1983a, 1983b; Rushdi, 1985; Hammer and Rudeanu, 2012), then $C_{st}(\boldsymbol{X})$ is:

$$C_{st}(\boldsymbol{X}) = \sum_{J=1}^{n_p} P_j C_{st}(\boldsymbol{X}|P_j = 1). \tag{12}$$

Equation (12) can be proved by the repeated application of the decomposition rule (1) (Rushdi, 1988). The subfunction $C_{st}(\boldsymbol{X}|P_j = 1)$ in (12) are to be obtained by substituting $\{\boldsymbol{X} \mid P_j = 1\}$ for **X** in (9).

Example 2

The problem of Examples1 is now revisited by applying the "Max-Flow Min-Cut Theorem". The ecological capacitated network of Fig. 1 has 4 minimal cut-sets (Rushdi and Hassan, 2016a), whose capacities are given by

$C_1 = \{2X_1 + 4X_3 + 6X_5 + 7X_6 + 8X_7 + 9X_8\}$

$C_2 = \{3X_2 + 4X_3 + 5X_4 + 8X_7\}$

$C_3 = \{3X_2 + 4X_3 + 6X_5 + 8X_7 + 9X_8 + 10X_9\}$

$C_4 = \{2X_1 + 4X_3 + 5X_4 + 7X_6 + 8X_7 + 10X_9\}$

Thus, expression (9) takes the form:

$$C_{st}(\boldsymbol{X}) = min(c_1X_1 + c_3X_3 + c_5X_5 + c_6X_6 + c_7X_7 + c_8X_8\,, c_2X_2 + c_3X_3 + c_4X_4 + c_7X_7\,, c_2X_2 + c_3X_3 + c_5X_5 + c_7X_7 + c_8X_8 + c_9X_9\,, c_1X_1 + c_3X_3 + c_4X_4 + c_6X_6 + c_7X_7 + c_9X_9)$$

$$C_{st}(\boldsymbol{X}) = min(2X_1 + 4X_3 + 6X_5 + 7X_6 + 8X_7 + 9X_8\,, 3X_2 + 4X_3 + 5X_4 + 8X_7\,, 3X_2 + 4X_3 + 6X_5 + 8X_7 + 9X_8 + 10X_9\,, 2X_1 + 4X_3 + 5X_4 + 7X_6 + 8X_7 + 10X_9) \quad (13)$$

which can be successively simplified by decomposition about various expansion variables in accordance with (1). By decomposing the capacity function $C_{st}(\boldsymbol{X})$ with respect to the indicator variables (X_1, X_6) that represent two elements (corridors) in the ecological network of Fig. 1, the following special case of (1) is obtained:

$$C_{st}(\boldsymbol{X}) = \overline{X}_1\overline{X}_6C_{st}(\boldsymbol{X}|0_1, 0_6) + \overline{X}_1X_6C_{st}(\boldsymbol{X}|0_1, 1_6) + X_1\overline{X}_6C_{st}(\boldsymbol{X}|1_1, 0_6) + X_1X_6C_{st}(\boldsymbol{X}|1 \quad (14)$$

Therefore, the sub-functions in (14) are obtained via (13) as:

$$C_{st}(\boldsymbol{X}|0_1, 0_6) = min(4X_3 + 6X_5 + 8X_7 + 9X_8\,, 3X_2 + 4X_3 + 5X_4 + 8X_7\,, 3X_2 + 4X_3 + 6X_5 + 8X_7 + 9X_8 + 10X_9\,, 4X_3 + 5X_4 + 8X_7 + 10X_9) \quad (15)$$

The sub-function in expression (15) can be decomposed further with respect to the indicator variables (X_2, X_7) that represent two elements (corridors) in the ecological sub-network $C_{st}(\boldsymbol{X}|0_1, 0_6)$. The following special case of (1) is obtained:

$$C_{st}(\boldsymbol{X}|0_1,0_6) = \overline{X}_2\overline{X}_7C_{st}(\boldsymbol{X}|0_2,0_7) + \overline{X}_2X_7C_{st}(\boldsymbol{X}|0_2,1_7) + X_2\overline{X}_7C_{st}(\boldsymbol{X}|1_2,0_7) + X_2X_7C_{st}(\boldsymbol{X}|1_2,1_7) \quad (16)$$

The new or lower sub-functions in (16) are obtained via the sub-function in (15). The first of them is

$$\begin{aligned}
C_{st}(\boldsymbol{X}|0_1,0_2,0_6,0_7) &= min(4X_3 + 6X_5 + 9X_8\,, 4X_3 + 5X_4\,, 4X_3 + 6X_5 + 9X_8 + 10X_9\,, 4X_3 + 5X_4 + 10X_9) \\
&= X_3 \min(4 + 6X_5 + 9X_8\,, 4 + 5X_4, 4 + 6X_5 + 9X_8 + 10X_9, 4 + 5X_4 + 10X_9) \\
&\quad + \overline{X}_3 \min(\,6X_5 + 9X_8\,,\ 5X_4\,, 6X_5 + 9X_8 + 10X_9, 5X_4 + 10X_9) \\
&= X_3\big[X_4\, min\,(\,4 + 6X_5 + 9X_8, 9, 4 + 6X_5 + 9X_8 + 10X_9, 9 + 10X_9) \\
&\quad + \overline{X}_4\, min\,(\,4 + 6X_5 + 9X_8, 4, 4 + 6X_5 + 9X_8 + 10X_9, 4 + 10X_9)\big] \\
&\quad + \overline{X}_3[X_4\, min\,(6X_5 + 9X_8, 5, 6X_5 + 9X_8 + 10X_9, 5 + 10X_9)] \\
&= X_3\Big[X_4\big[X_5 \min(10 + 9X_8, 9, 10 + 9X_8 + 10X_9, 9 + 10X_9) \\
&\quad + \overline{X}_5 \min(4 + 9X_8, 9, 4 + 9X_8 + 10X_9, 9 + 10X_9)\big] \\
&\quad + \overline{X}_4\big[X_5 \min(10 + 9X_8, 4,10 + 9X_8 + 10X_9, 4 + 10X_9) \\
&\quad + \overline{X}_5 \min(4 + 9X_8, 4, 4 + 9X_8 + 10X_9, 4 + 10X_9)\big]\Big] \\
&\quad + \overline{X}_3\Big[X_4\big[\,X_5 \min(6 + 9X_8, 5, 6 + 9X_8 + 10X_9, 5 + 10X_9) \\
&\quad + \overline{X}_5 \min(9X_8, 5,9X_8 + 10X_9, 5 + 10X_9)\big]\Big]
\end{aligned}$$

$$\begin{aligned}
&= X_3\Big[X_4\Big[X_5\big[X_8(9) + \overline{X}_8(9)\big] + \overline{X}_5\big[X_8(9) + \overline{X}_8(4)\big]\Big] \\
&\quad + \overline{X}_4\Big[X_5\big[X_8(4) + \overline{X}_8(4)\big] + \overline{X}_5\big[X_8(4) + \overline{X}_8(4)\big]\Big]\Big] \\
&\quad + \overline{X}_3\Big[X_4\Big[X_5\big[X_8(5) + \overline{X}_8(5)\big] + \overline{X}_5[X_8(5)]\Big]\Big]
\end{aligned}$$

$$= 4X_3 + 5X_4\big(X_5 + \overline{X}_5X_8\big) \quad (16a)$$

The other lower sub-functions in (16) are obtained similarly as

$$C_{st}(\boldsymbol{X}|0_1,0_2,0_6,1_7) = 8 + 4X_3 + 5X_4(X_5 + \overline{X}_5X_8) \tag{16b}$$

$$C_{st}(\boldsymbol{X}|0_1,1_2,0_6,0_7) = X_4X_5\overline{X}_8X_9 + 3X_9\big(X_8 + \overline{X}_4X_5\overline{X}_8\big) + 4X_3 + 5X_4(X_5 + \overline{X}_5X_8) \tag{16c}$$

$$C_{st}(\boldsymbol{X}|0_1,1_2,0_6,1_7) = 8 + X_4X_5\overline{X}_8X_9 + 3X_9\big(X_8 + \overline{X}_4X_5\overline{X}_8\big) + 4X_3 + 5X_4(X_5 + \overline{X}_5X_8) \tag{16d}$$

These lower sub-functions can be used to fill in the map entries in Fig. 3 and Fig. 4. They can also be substituted into (16) to yield the following expression

$$\begin{aligned} C_{st}(\boldsymbol{X}|0_1,0_6) = {} & \overline{X}_2\overline{X}_7\big[4X_3 + 5X_4\big(X_5 + \overline{X}_5X_8\big)\big] + \overline{X}_2X_7\big[8 + 4X_3 + 5X_4\big(X_5 + \overline{X}_5X_8\big)\big] \\ & + X_2\overline{X}_7\big[X_4X_5\overline{X}_8X_9 + 3X_9\big(X_8 + \overline{X}_4X_5\overline{X}_8\big) + 4X_3 + 5X_4\big(X_5 + \overline{X}_5X_8\big)\big] \\ & + X_2X_7\big[8 + X_4X_5\overline{X}_8X_9 + 3X_9\big(X_8 + \overline{X}_4X_5\overline{X}_8\big) + 4X_3 + 5X_4\big(X_5 + \overline{X}_5X_8\big)\big] \end{aligned}$$

$$= 8X_7 + 5X_4\big(X_5 + \overline{X}_5X_8\big) + 4X_3 + 3X_2X_9\big(X_8 + \overline{X}_4X_5\overline{X}_8\big) + X_2X_4X_5\overline{X}_8X_9 \tag{17a}$$

$$\begin{aligned} C_{st}(\boldsymbol{X}|0_1,1_6) = {} & \overline{X}_2\overline{X}_7\Big[4X_3 + 5X_4\Big(X_5 + \overline{X}_5\big(X_8 + \overline{X}_8X_9\big)\Big)\Big] \\ & + \overline{X}_2X_7\Big[\,8 + 4X_3 + 5X_4\Big(X_8 + \overline{X}_8\big(X_9 + X_5\overline{X}_9\big)\Big)\Big] \\ & + X_2\overline{X}_7\big[\,3 + 4\big(X_3 + X_4\overline{X}_5\overline{X}_8X_9\big) + 5X_4\big(X_5 + \overline{X}_5X_8\big)\big] \\ & + X_2X_7\big[\,11 + 4\big(X_3 + X_4\overline{X}_5\overline{X}_8X_9\big) + 5\,X_4\big(X_5 + \overline{X}_5X_8\big)\big] \end{aligned}$$

$$= 11\,X_2X_7 + 8\overline{X}_2X_7 + 5X_4\Big(X_5 + \overline{X}_5\big(X_8 + \overline{X}_8X_9\overline{X}_2\big)\Big) + 4\big(X_3 + X_2X_4\overline{X}_5\overline{X}_8X_9\big) + 3X_2\overline{X}_7 \tag{17b}$$

$$\begin{aligned} C_{st}(\boldsymbol{X}|1_1,0_6) = {} & \overline{X}_2\overline{X}_7\big[2X_4\overline{X}_5\overline{X}_8X_9 + 4X_3 + 5X_4\big(X_5 + \overline{X}_5X_8\big)\big] \\ & + \overline{X}_2X_7\big[8 + 2X_4\overline{X}_5\overline{X}_8X_9 + 4X_3 + 5X_4\big(X_5 + \overline{X}_5X_8\big)\big] \\ & + X_2\overline{X}_7\big[2 + X_9\big(X_8 + X_5\overline{X}_8\big) + 4X_3 + 5X_4\big(X_5 + \overline{X}_5X_8\big)\big] \\ & + X_2X_7\big[10 + X_9\big(X_8 + X_5\overline{X}_8\big) + 4X_3 + 5X_4\big(X_5 + \overline{X}_5X_8\big)\big] \end{aligned}$$

$$= 10\, X_2X_7 + 8\overline{X}_2X_7 + 5X_4\left(X_5 + \overline{X}_5X_8\right) + 4X_3 + 2\left(X_2\overline{X}_7 + \overline{X}_2X_4\overline{X}_5\overline{X}_8X_9\right) + X_2X_9\left(X_8 + X_5\overline{X}_8\right) \quad (17c)$$

$$\begin{aligned} C_{st}(\boldsymbol{X}|1_1, 1_6) = {} & \overline{X}_2\overline{X}_7\left[4X_3 + 5X_4\left(X_5 + \overline{X}_5\left(X_8 + \overline{X}_8X_9\right)\right)\right] \\ & + \overline{X}_2X_7\left[8 + 4X_3 + 5X_4\left(X_5 + \overline{X}_5\left(X_9 + X_8\overline{X}_9\right)\right)\right] \\ & + X_2\overline{X}_7\left[3 + 4X_3 + 5X_4\left(X_5 + \overline{X}_5\left(X_8 + \overline{X}_8X_9\right)\right)\right] \\ & + X_2X_7\left[11 + 4X_3 + 5X_4\left(X_5 + \overline{X}_5\left(X_9 + X_8\overline{X}_9\right)\right)\right] \end{aligned}$$

$$= 11\, X_2X_7 + 8\overline{X}_2X_7 + 5X_4\left(X_5 + \overline{X}_5\left(X_8 + \overline{X}_8X_9\right)\right) + 4X_3 + 3X_2\overline{X}_7 \quad (17d)$$

These sub-functions, can together with the condition $C_{st}(\boldsymbol{X}|S_{st} = 0) = 0$, be used to fill in the map entries in Fig. 3 and Fig. 4. They can also be substituted into (14) to yield the equivalent form (4).

On the other hand, (13) can be successfully simplified by using eq. (12) and we can subsequently get an equivalent result.

There are two methods to find the disjoint paths for the network. **First method** (visually) through Karnaugh map directly in which all non-zero cells or entries of the map will be covered by disjoint loops. Each of these loops should be the largest that combines $2^i\ \{i = 0, 1, 2, 4 \ldots, n\}$as can be seen in Fig. 5.

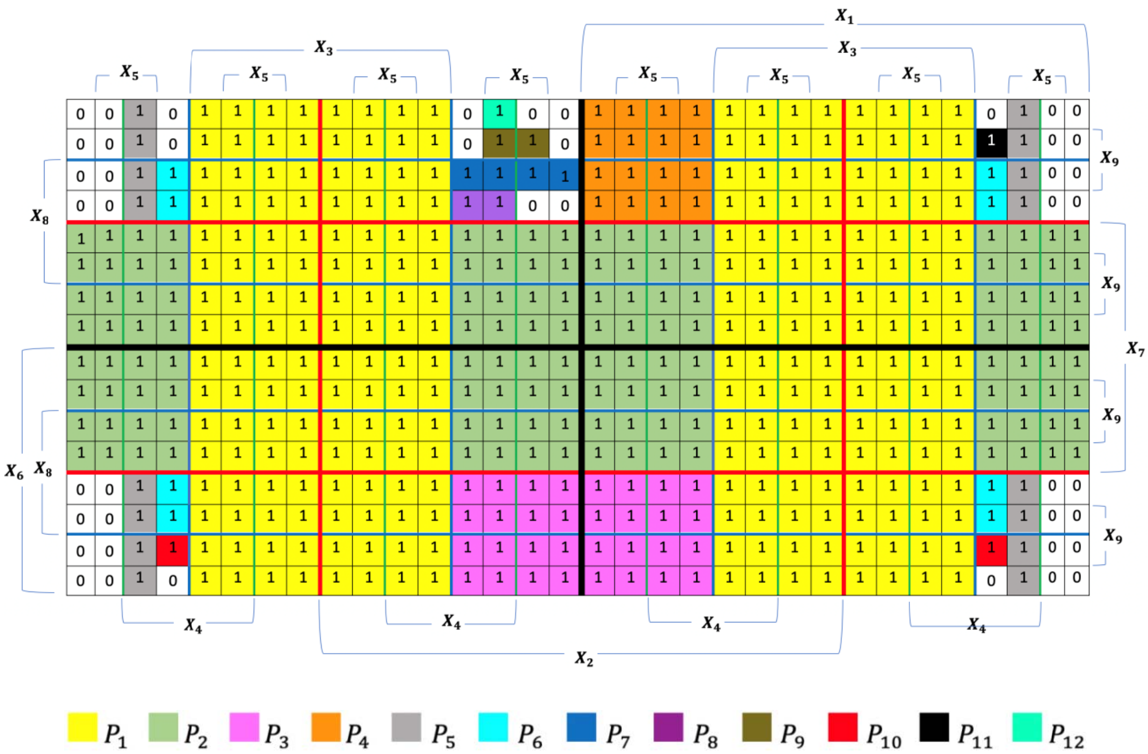


Fig. 5 Modified k-map representing the disjoint loops for the ecological network in Fig. 1.

A set of exhaustive and disjoint s-t paths for the ecological network is:

$P_1 = X_3,\ P_2 = \overline{X}_3 X_7,\quad P_3 = X_2\overline{X}_3X_6\overline{X}_7,\quad P_4 = X_1X_2\overline{X}_3\overline{X}_6\overline{X}_7,\quad P_5 = \overline{X}_2\overline{X}_3X_4X_5\overline{X}_7,$

$P_6 = \overline{X}_2\overline{X}_3X_4\overline{X}_5\overline{X}_7X_8,\quad P_7 = \overline{X}_1X_2\overline{X}_3\overline{X}_6\overline{X}_7X_8X_9,\quad P_8 = \overline{X}_1X_2\overline{X}_3X_4\overline{X}_6\overline{X}_7X_8\overline{X}_9,$

$P_9 = \overline{X}_1X_2\overline{X}_3X_5\overline{X}_6\overline{X}_7\overline{X}_8X_9,\quad P_{10} = \overline{X}_2\overline{X}_3X_4\overline{X}_5X_6\overline{X}_7\overline{X}_8X_9,\quad P_{11} = X_1\overline{X}_2\overline{X}_3X_4\overline{X}_5\overline{X}_6\overline{X}_7\overline{X}_8X_9,\ P_{12} = \overline{X}_1X_2\overline{X}_3X_4X_5\overline{X}_6\overline{X}_7\overline{X}_8\overline{X}_9$

Therefore, the sub-functions in (12) are obtained via (13) as

$$C_{st}(\boldsymbol{X}|P_1 = 1) = C_{st}(\boldsymbol{X}|1_3) = min(2X_1 + 4 + 6X_5 + 7X_6 + 8X_7 + 9X_8\,, 3X_2 + 4 + 5X_4 + 8X_7\,, 3X_2 + 4 + 6X_5 + 8X_7 + 9X_8 + 10X_9\,, 2X_1 + 4 + 5X_4 + 7X_6 + 8X_7 + 10X_9) = 11\, X_2X_6X_7 + 10\, X_1X_2\overline{X}_6X_7 + 8X_7\left(\overline{X}_2 + \overline{X}_1X_2\overline{X}_6\right) + 5X_4\left(X_5 + \overline{X}_5\left(X_8 + X_6\overline{X}_8X_9\left(X_1 + \overline{X}_1\overline{X}_2\right)\right)\right) + 4\left(1 + \overline{X}_1X_2X_4\overline{X}_5X_6\overline{X}_8X_9\right) + 3X_2\left(X_6\overline{X}_7 + \overline{X}_1\overline{X}_6X_9\left(X_8 + \overline{X}_4X_5\overline{X}_8\right)\right) + 2X_1\overline{X}_6\left(X_2\overline{X}_7 + \overline{X}_2X_4\overline{X}_5\overline{X}_8X_9\right) + X_2\overline{X}_6X_9\left(X_1X_8 + X_5\overline{X}_8\left(X_1 + \overline{X}_1X_4\right)\right)$$

(18a)

$$\begin{aligned} C_{st}(\boldsymbol{X}|P_2 = 1) = C_{st}(\boldsymbol{X}|0_3, 1_7) &= min(2X_1 + 6X_5 + 7X_6 + 8 + 9X_8\,, 3X_2 + 5X_4 + 8\,, 3X_2 + 6X_5 + 8 + 9X_8 \\ &\quad + 10X_9\,, 2X_1 + 5X_4 + 7X_6 + 8 + 10X_9) \end{aligned}$$

$$\begin{aligned} &= 11(X_2X_6) + 10\left(X_1X_2\overline{X}_6\right) + 8\left(\overline{X}_2 + \overline{X}_1X_2\overline{X}_6\right) + 5\left(\overline{X}_4 + X_4\overline{X}_5\left(X_8 + X_6\overline{X}_8X_9\left(X_1 + \overline{X}_1\overline{X}_2\right)\right)\right) \\ &\quad + 4\left(\overline{X}_1X_2X_4\overline{X}_5X_6\overline{X}_8X_9\right) + 3\overline{X}_1X_2\overline{X}_6X_9\left(X_8 + \overline{X}_4X_5\overline{X}_8\right) + 2\left(X_1\overline{X}_2X_4\overline{X}_5\overline{X}_6\overline{X}_8X_9\right) \\ &\quad + X_2\overline{X}_6X_9\left(X_1\left(X_8 + X_5\overline{X}_8\left(1 + \overline{X}_1X_4\right)\right)\right) \end{aligned}$$

(18b)

$$\begin{aligned} C_{st}(\boldsymbol{X}|P_3 = 1) = C_{st}(\boldsymbol{X}|1_2, 0_3, 1_6, 0_7) &= min(2X_1 + 6X_5 + 7 + 9X_8\,, 3 + 5X_4\,, 3 + 6X_5 + 9X_8 + 10X_9\,, 2X_1 + 5X_4 + 7 \\ &\quad + 10X_9) \end{aligned}$$

$$= 5\left(\overline{X}_4X_5 + X_4\overline{X}_5\left(X_8 + X_1\overline{X}_8X_9\right)\right) + 4\left(\overline{X}_1X_4\overline{X}_5\overline{X}_8X_9\right) + 3$$

(18c)

$$C_{st}(\boldsymbol{X}|P_4 = 1) = C_{st}(\boldsymbol{X}|1_1, 1_2, 0_3, 0_6, 0_7) = min(2 + 6X_5 + 9X_8 , 3 + 5X_4 , 3 + 6X_5 + 9X_8 + 10X_9 , 2 + 5X_4 + 10X_9) = 5X_4\left(X_5 + \overline{X}_5X_8\right) + 2 + X_9\left(X_8 + X_5\overline{X}_8\right)$$

(18d)

$$C_{st}(\boldsymbol{X}|P_5 = 1) = C_{st}(\boldsymbol{X}|0_2, 0_3, 1_4, 1_5, 0_7)$$
$$= min(2X_1 + 6 + 7X_6 + 9X_8 , 5 , 6 + 9X_8 + 10X_9 , 2X_1 + 5 + 7X_6 + 10X_9) = 5$$

(18e)

$$C_{st}(\boldsymbol{X}|P_6 = 1) = C_{st}(\boldsymbol{X}|0_2, 0_3, 1_4, 0_5, 0_7, 1_8) = min(2X_1 + 7X_6 + 9 , 5 , 9 + 10X_9 , 2X_1 + 5 + 7X_6 + 10X_9) = 5$$

(18f)

$$C_{st}(\boldsymbol{X}|P_7 = 1) = C_{st}(\boldsymbol{X}|0_1, 1_2, 0_3, 0_6, 0_7, 1_8, 1_9) = min(6X_5 + 9 , 3 + 5X_4 , 3 + 6X_5 + 9 + 10 , 5X_4 + 10) = 5X_4 + 3$$

(18g)

$$C_{st}(\boldsymbol{X}|P_8 = 1) = C_{st}(\boldsymbol{X}|0_1, 1_2, 0_3, 1_4, 0_6, 0_7, 1_8, 0_9) = min(6X_5 + 9 , 3 + 5 , 3 + 6X_5 + 9 , 5) = 5$$

(18h)

$$C_{st}(\boldsymbol{X}|P_9 = 1) = C_{st}(\boldsymbol{X}|0_1, 1_2, 0_3, 1_5, 0_6, 0_7, 0_8, 1_9) = 3(1 + X_4)$$

(18i)

$$C_{st}(\boldsymbol{X}|P_{10} = 1) = C_{st}(\boldsymbol{X}|0_2, 0_3, 1_4, 0_5, 1_6, 0_7, 0_8, 1_9) = 5$$

(18j)

$$C_{st}(\boldsymbol{X}|P_{11} = 1) = C_{st}(\boldsymbol{X}|1_1, 0_2, 0_3, 1_4, 0_5, 0_6, 0_7, 0_8, 1_9) = 2$$

(18k)

$$C_{st}(\boldsymbol{X}|P_{12} = 1) = C_{st}(\boldsymbol{X}|0_1, 1_2, 0_3, 1_4, 1_5, 0_6, 0_7, 0_8, 0_9) = 5$$

(18l)

These sub-functions, can together with the condition $C_{st}(\boldsymbol{X}|S_{st} = 0) = 0,$ be used to fill in the map entries in Fig. 3 and Fig. 4. Moreover, they can be substituted into (12) to get the expression:

$$
\begin{aligned}
&C_{st}(\boldsymbol{X}) \\
&= X_3 \Big[11\, X_2X_6X_7 + 10\, X_1X_2\overline{X}_6X_7 + 8X_7\big(\overline{X}_2 + \overline{X}_1X_2\overline{X}_6\big) \\
&+ 5X_4 \Big(X_5 + \overline{X}_5 \Big(X_8 + X_6\overline{X}_8X_9\big(X_1 + \overline{X}_1\overline{X}_2\big)\Big)\Big) \\
&+ 4\left(1 + \overline{X}_1X_2X_4\overline{X}_5X_6\overline{X}_8X_9\right) \\
&+ 3X_2 \Big(X_6\overline{X}_7 + \overline{X}_1\overline{X}_6X_9\big(X_8 + \overline{X}_4X_5\overline{X}_8\big)\Big) \\
&+ 2X_1\overline{X}_6\big(X_2\overline{X}_7 + \overline{X}_2X_4\overline{X}_5\overline{X}_8X_9\big) \\
&+ X_2\overline{X}_6X_9 \Big(X_1X_8 + X_5\overline{X}_8\big(X_1 + \overline{X}_1X_4\big)\Big)\Big] \\
&+ \overline{X}_3\mathrm{X}_7 \Big[11(X_2X_6) + 10\big(X_1X_2\overline{X}_6\big) + 8\big(\overline{X}_2 + \overline{X}_1X_2\overline{X}_6\big) \\
&+ 5\Big(\overline{X}_4 + X_4\overline{X}_5 \Big(X_8 + X_6\overline{X}_8X_9\big(X_1 + \overline{X}_1\overline{X}_2\big)\Big)\Big) \\
&+ 4\big(\overline{X}_1X_2X_4\overline{X}_5X_6\overline{X}_8X_9\big) + 3\overline{X}_1X_2\overline{X}_6X_9\big(X_8 + \overline{X}_4X_5\overline{X}_8\big) \\
&+ 2\big(X_1\overline{X}_2X_4\overline{X}_5\overline{X}_6\overline{X}_8X_9\big) + X_2\overline{X}_6X_9 \Big(X_1 \Big(X_8 + X_5\overline{X}_8\big(1 + \overline{X}_1X_4\big)\Big)\Big)\Big] \\
&+ X_2\overline{X}_3X_6\overline{X}_7 \Big[5\Big(\overline{X}_4X_5 + X_4\overline{X}_5\big(X_8 + X_1\overline{X}_8X_9\big)\Big) + 4\big(\overline{X}_1X_4\overline{X}_5\overline{X}_8X_9\big) \\
&+ 3\Big] + X_1X_2\overline{X}_3\overline{X}_6\overline{X}_7\big[5X_4\big(X_5 + \overline{X}_5X_8\big) + 2 + X_9\big(X_8 + X_5\overline{X}_8\big)\big] \\
&+ 5\overline{X}_2\overline{X}_3X_4X_5\overline{X}_7 + 5\overline{X}_2\overline{X}_3X_4\overline{X}_5\overline{X}_7X_8 + \overline{X}_1X_2\overline{X}_3\overline{X}_6\overline{X}_7X_8X_9[5X_4 + 3] \\
&+ 5\overline{X}_1X_2\overline{X}_3X_4\overline{X}_6\overline{X}_7X_8\overline{X}_9 + \overline{X}_1X_2\overline{X}_3X_5\overline{X}_6\overline{X}_7\overline{X}_8X_9[3(1 + X_4)] \\
&+ 5\overline{X}_2\overline{X}_3X_4\overline{X}_5X_6\overline{X}_7\overline{X}_8X_9 + 2X_1\overline{X}_2\overline{X}_3X_4\overline{X}_5\overline{X}_6\overline{X}_7\overline{X}_8X_9 \\
&+ 5\overline{X}_1X_2\overline{X}_3X_4X_5\overline{X}_6\overline{X}_7\overline{X}_8\overline{X}_9
\end{aligned}
\tag{19}
$$

Expression (19) has been shown to be equivalent to expression (4) obtained by the minimal s-t cutset procedure. In fact, each of the two expressions is equivalent to the canonical representation of the Karnaugh map in Fig. 3.

Beside our manual calculations in our example, we used MATLAB to implement the max-flow min-cut algorithm that is applicable through the identification of minimal cut-sets and minimal paths in our hypothetical ecological network and consequently using the results to fill in the k-map entries in an automated way.

The second method to find the disjoint paths is by using the switching-algebraic analysis (Rushdi and Rushdi, 2017). The basic idea in disjointing in algebraic analysis is that if none of the two terms A and B in the sum $(A \vee B)$ subsumes the other and the two terms are not disjoint, then B can be disjointed with A by the relation

$$(A \vee B) = A \vee B(\overline{y_1 y_2 \dots y_e}) = A \vee B(\overline{y_1} \vee y_1\overline{y_2} \vee \dots \vee y_1 y_2 \dots y_{e-1} y_e),$$
(20)

where $\{y_1, y_2, \dots, y_e\}$ is the set of literals that appear in the term A and do not appear in the term B. Note that B is replaced by $e\ (\geq 1)$ terms that are disjoint with one another besides being disjoint with A (Rushdi and Ba-Rukab, 2005a, 2005b; Rushdi and Hassan, 2016a; Rushdi and Rushdi, 2017).

Applying disjointness, we can re-express the minimal paths (Rushdi and Hassan, 2016a) for the network in Fig. 1 in the PRE-form (as can be seen the steps in an appendix).

A set of exhaustive and minimal s-t paths for the ecological networkis (Rushdi and Hassan, 2016a):

$\{P_1 = X_3, P_2 = X_7, P_3 = X_2X_6, P_4 = X_1X_2, P_5 = X_4X_5, P_6 = X_4X_8, P_7 = X_2X_5X_9,$

$P_8 = X_2X_8X_9, P_9 = X_4X_6X_9, P_{10} = X_1X_4X_9\}.$

This set might be replaced by another exhaustive and disjoint (and hence, non-minimal) set of paths, whose disjunction constitutes the probability-ready-expression reproduced from Equation (46) in Rushdi and Hassan (2016a), namely

$$\begin{aligned} S_{PRE} = {} & X_3 \vee X_7\overline{X}_3 \vee X_2X_6\overline{X}_3\overline{X}_7 \vee X_1X_2\overline{X}_3\overline{X}_7\overline{X}_6 \vee X_4X_5\overline{X}_3\overline{X}_7\overline{X}_2 \vee X_4X_5\overline{X}_3\overline{X}_7X_2\overline{X}_6\overline{X}_1 \\ & \vee X_4X_8\overline{X}_3\overline{X}_7\overline{X}_2\overline{X}_5 \vee X_4X_8\overline{X}_3\overline{X}_7X_2\overline{X}_6\overline{X}_1\overline{X}_5 \vee X_2X_5X_9\overline{X}_3\overline{X}_7\overline{X}_6\overline{X}_1\overline{X}_4 \\ & \vee X_2X_8X_9\overline{X}_3\overline{X}_7\overline{X}_6\overline{X}_1\overline{X}_4\overline{X}_5 \vee X_4X_6X_9\overline{X}_3\overline{X}_7\overline{X}_2\overline{X}_5\overline{X}_8 \vee X_1X_4X_9\overline{X}_3\overline{X}_7\overline{X}_2\overline{X}_5\overline{X}_8\overline{X}_6 \end{aligned}$$

(21)

So, each term in the expression (21) represents disjoint path for the ecological network in Fig. 1. Therefore, the sub-functions in (12) are obtained via (13) and hence these sub-functions, can together with the condition $C_{st}(\boldsymbol{X}|S_{st} = 0) = 0$, be used to fill in the map entries in Fig. 3 and Fig. 4. Moreover, they can be substituted into (12) to get the expression:

$$
\begin{aligned}
C_{st}(\boldsymbol{X}) = X_3 \Big[& 11\, X_2 X_6 X_7 + 10\, X_1 X_2 \overline{X}_6 X_7 + 8 X_7 \left(\overline{X}_2 + \overline{X}_1 X_2 \overline{X}_6\right) \\
& + 5 X_4 \left(X_5 + \overline{X}_5 \left(X_8 + X_6 \overline{X}_8 X_9 \left(X_1 + \overline{X}_1 \overline{X}_2\right)\right)\right) + 4 \left(1 + \overline{X}_1 X_2 X_4 \overline{X}_5 X_6 \overline{X}_8 X_9\right) \\
& + 3 X_2 \left(X_6 \overline{X}_7 + \overline{X}_1 \overline{X}_6 X_9 \left(X_8 + \overline{X}_4 X_5 \overline{X}_8\right)\right) + 2 X_1 \overline{X}_6 \left(X_2 \overline{X}_7 + \overline{X}_2 X_4 \overline{X}_5 \overline{X}_8 X_9\right) \\
& + X_2 \overline{X}_6 X_9 \left(X_1 X_8 + X_5 \overline{X}_8 \left(X_1 + \overline{X}_1 X_4\right)\right)\Big] \\
& + X_7 \overline{X}_3 \Big[11 (X_2 X_6) + 10 \left(X_1 X_2 \overline{X}_6\right) + 8 \left(\overline{X}_2 + \overline{X}_1 X_2 \overline{X}_6\right) \\
& + 5 \left(\overline{X}_4 + X_4 \overline{X}_5 \left(X_8 + X_6 \overline{X}_8 X_9 \left(X_1 + \overline{X}_1 \overline{X}_2\right)\right)\right) + 4 \left(\overline{X}_1 X_2 X_4 \overline{X}_5 X_6 \overline{X}_8 X_9\right) \\
& + 3 \overline{X}_1 X_2 \overline{X}_6 X_9 \left(X_8 + \overline{X}_4 X_5 \overline{X}_8\right) + 2 \left(X_1 \overline{X}_2 X_4 \overline{X}_5 \overline{X}_6 \overline{X}_8 X_9\right) \\
& + X_2 \overline{X}_6 X_9 \left(X_1 \left(X_8 + X_5 \overline{X}_8 \left(1 + \overline{X}_1 X_4\right)\right)\right)\Big] \\
& + X_2 X_6 \overline{X}_3 \overline{X}_7 \left[5 \left(\overline{X}_4 X_5 + X_4 \overline{X}_5 \left(X_8 + X_1 \overline{X}_8 X_9\right)\right) + 4 \left(\overline{X}_1 X_4 \overline{X}_5 \overline{X}_8 X_9\right) + 3\right] \\
& + X_1 X_2 \overline{X}_3 \overline{X}_7 \overline{X}_6 \left[5 X_4 \left(X_5 + \overline{X}_5 X_8\right) + 2 + X_9 \left(X_8 + X_5 \overline{X}_8\right)\right] + 5 X_4 X_5 \overline{X}_3 \overline{X}_7 \overline{X}_2 \\
& + X_4 X_5 \overline{X}_3 \overline{X}_7 X_2 \overline{X}_6 \overline{X}_1 \left[5 + 3 X_8 X_9 + \overline{X}_8 X_9\right] + 5 X_4 X_8 \overline{X}_3 \overline{X}_7 \overline{X}_2 \overline{X}_5 \\
& + X_4 X_8 \overline{X}_3 \overline{X}_7 X_2 \overline{X}_6 \overline{X}_1 \overline{X}_5 [5 + 3 X_9] + 3 X_2 X_5 X_9 \overline{X}_3 \overline{X}_7 \overline{X}_6 \overline{X}_1 \overline{X}_4 \\
& + 3 X_2 X_8 X_9 \overline{X}_3 \overline{X}_7 \overline{X}_6 \overline{X}_1 \overline{X}_4 \overline{X}_5 + 5 X_4 X_6 X_9 \overline{X}_3 \overline{X}_7 \overline{X}_2 \overline{X}_5 \overline{X}_8 + 2 X_1 X_4 X_9 \overline{X}_3 \overline{X}_7 \overline{X}_2 \overline{X}_5 \overline{X}_8 \overline{X}_6
\end{aligned} \tag{22}
$$

Expression (22) can be shown to be an equivalent to expression (4) obtained by the minimal s-t cutset procedure and expression (19) obtained visually through Karnaugh map directly.

6 Conclusions

This paper presents a tutorial exposition of various methods for analyzing a capacitated or flow network with an ecological perspective, i.e., the problem of evaluating the survivability of a specific species, the probability of its avoiding local extinction by migration between habitat patches via imperfect heterogeneous corridors when there are definitely several destination habitat patches with the paths to them from the critical habitat patch sharing some edges (corridors) in common.

These methods include Karnaugh maps, which are crucial in providing not only the visual insight necessary to write better future software but also adequate means of verifying such software and a generalization of the max-flow min-cut theorem, which is assisted by the identification of minimal cut-sets and minimal paths in the ecological flow networks.

APPENDIX

The minimum paths for the network are:

$P_1 = X_3\,, P_2 = X_7\,, P_3 = X_2X_6\,, P_4 = X_1X_2,\ P_5 = X_4X_5\,, P_6 = X_4X_8\,, P_7 = X_2X_5X_9\,,$

$P_8 = X_2X_8X_9\,, P_9 = X_4X_6X_9\,, P_{10} = X_1X_4X_9$

$S = X_3 \vee X_7 \vee X_2X_6 \vee X_1X_2 \vee X_4X_5 \vee X_4X_8 \vee X_2X_5X_9 \vee X_2X_8X_9 \vee X_4X_6X_9 \vee X_1X_4X_9$

$S_{PRE(1)} = X_3 \vee X_7 \vee X_2X_6 \vee X_1X_2 \vee X_4X_5 \vee X_4X_8 \vee X_2X_5X_9 \vee X_2X_8X_9 \vee X_4X_6X_9 \vee X_1X_4X_9$

$$S_{PRE(2)} = X_3 \vee X_7\bar{X}_3 \vee X_2X_6\bar{X}_3 \vee X_1X_2\bar{X}_3 \vee X_4X_5\bar{X}_3 \vee X_4X_8\bar{X}_3 \vee X_2X_5X_9\bar{X}_3 \vee X_2X_8X_9\bar{X}_3 \vee X_4X_6X_9\bar{X}_3 \vee X_1X_4X_9\bar{X}_3$$

$$S_{PRE(3)} = X_3 \vee X_7\bar{X}_3 \vee X_2X_6\bar{X}_3\bar{X}_7 \vee X_1X_2\bar{X}_3\bar{X}_7 \vee X_4X_5\bar{X}_3\bar{X}_7 \vee X_4X_8\bar{X}_3\bar{X}_7 \vee X_2X_5X_9\bar{X}_3\bar{X}_7 \vee X_2X_8X_9\bar{X}_3\bar{X}_7 \vee X_4X_6X_9\bar{X}_3\bar{X}_7 \vee X_1X_4X_9\bar{X}_3\bar{X}_7$$

$$S_{PRE(4)} = X_3 \vee X_7\bar{X}_3 \vee X_2X_6\bar{X}_3\bar{X}_7 \vee X_1X_2\bar{X}_3\bar{X}_7(\bar{X}_2 \vee X_2\bar{X}_6) \vee X_4X_5\bar{X}_3\bar{X}_7(\bar{X}_2 \vee X_2\bar{X}_6) \vee X_4X_8\bar{X}_3\bar{X}_7(\bar{X}_2 \vee X_2\bar{X}_6) \vee X_2X_5X_9\bar{X}_3\bar{X}_7(\bar{X}_2 \vee X_2\bar{X}_6) \vee X_2X_8X_9\bar{X}_3\bar{X}_7(\bar{X}_2 \vee X_2\bar{X}_6) \vee X_4X_6X_9\bar{X}_3\bar{X}_7(\bar{X}_2 \vee X_2\bar{X}_6) \vee X_1X_4X_9\bar{X}_3\bar{X}_7(\bar{X}_2 \vee X_2\bar{X}_6)$$

Simplify and absorption of all possible terms in their subsumed term:

$$S_{PRE(4)} = X_3 \vee X_7\bar{X}_3 \vee X_2X_6\bar{X}_3\bar{X}_7 \vee X_1X_2\bar{X}_3\bar{X}_7\bar{X}_6 \vee X_4X_5\bar{X}_3\bar{X}_7\bar{X}_2 \vee X_4X_5\bar{X}_3\bar{X}_7X_2\bar{X}_6 \vee X_4X_8\bar{X}_3\bar{X}_7\bar{X}_2 \vee X_4X_8\bar{X}_3\bar{X}_7X_2\bar{X}_6 \vee X_2X_5X_9\bar{X}_3\bar{X}_7\bar{X}_6 \vee X_2X_8X_9\bar{X}_3\bar{X}_7\bar{X}_6 \vee X_4X_6X_9\bar{X}_3\bar{X}_7\bar{X}_2 \vee X_1X_4X_9\bar{X}_3\bar{X}_7\bar{X}_2$$

$$S_{PRE(5)} = X_3 \vee X_7\bar{X}_3 \vee X_2X_6\bar{X}_3\bar{X}_7 \vee X_1X_2\bar{X}_3\bar{X}_7\bar{X}_6 \vee X_4X_5\bar{X}_3\bar{X}_7\bar{X}_2(\bar{X}_1 \vee X_1\bar{X}_2) \vee X_4X_5\bar{X}_3\bar{X}_7X_2\bar{X}_6(\bar{X}_1 \vee X_1\bar{X}_2) \vee X_4X_8\bar{X}_3\bar{X}_7\bar{X}_2(\bar{X}_1 \vee X_1\bar{X}_2) \vee X_4X_8\bar{X}_3\bar{X}_7X_2\bar{X}_6(\bar{X}_1 \vee X_1\bar{X}_2) \vee X_2X_5X_9\bar{X}_3\bar{X}_7\bar{X}_6(\bar{X}_1 \vee X_1\bar{X}_2) \vee X_2X_8X_9\bar{X}_3\bar{X}_7\bar{X}_6(\bar{X}_1 \vee X_1\bar{X}_2) \vee X_4X_6X_9\bar{X}_3\bar{X}_7\bar{X}_2(\bar{X}_1 \vee X_1\bar{X}_2) \vee X_1X_4X_9\bar{X}_3\bar{X}_7\bar{X}_2(\bar{X}_1 \vee X_1\bar{X}_2)$$

Simplify and absorption of all possible terms in their subsumed term:

$$S_{PRE(5)} = X_3 \vee X_7\bar{X}_3 \vee X_2X_6\bar{X}_3\bar{X}_7 \vee X_1X_2\bar{X}_3\bar{X}_7\bar{X}_6 \vee X_4X_5\bar{X}_3\bar{X}_7\bar{X}_2\bar{X}_1 \vee X_4X_5\bar{X}_3\bar{X}_7\bar{X}_2X_1 \vee X_4X_5\bar{X}_3\bar{X}_7X_2\bar{X}_6\bar{X}_1 \vee X_4X_8\bar{X}_3\bar{X}_7\bar{X}_2\bar{X}_1 \vee X_4X_8\bar{X}_3\bar{X}_7\bar{X}_2X_1 \vee X_4X_8\bar{X}_3\bar{X}_7X_2\bar{X}_6\bar{X}_1 \vee X_2X_5X_9\bar{X}_3\bar{X}_7\bar{X}_6\bar{X}_1 \vee X_2X_8X_9\bar{X}_3\bar{X}_7\bar{X}_6\bar{X}_1 \vee X_4X_6X_9\bar{X}_3\bar{X}_7\bar{X}_2\bar{X}_1 \vee X_4X_6X_9\bar{X}_3\bar{X}_7\bar{X}_2X_1 \vee X_1X_4X_9\bar{X}_3\bar{X}_7\bar{X}_2$$

$$S_{PRE(5)} = X_3 \vee X_7\overline{X}_3 \vee X_2X_6\overline{X}_3\overline{X}_7 \vee X_1X_2\overline{X}_3\overline{X}_7\overline{X}_6 \vee X_4X_5\overline{X}_3\overline{X}_7\overline{X}_2 \vee X_4X_5\overline{X}_3\overline{X}_7X_2\overline{X}_6\overline{X}_1 \vee X_4X_8\overline{X}_3\overline{X}_7\overline{X}_2$$
$$\vee X_4X_8\overline{X}_3\overline{X}_7X_2\overline{X}_6\overline{X}_1 \vee X_2X_5X_9\overline{X}_3\overline{X}_7\overline{X}_6\overline{X}_1 \vee X_2X_8X_9\overline{X}_3\overline{X}_7\overline{X}_6\overline{X}_1$$
$$\vee X_4X_6X_9\overline{X}_3\overline{X}_7\overline{X}_2 \vee X_1X_4X_9\overline{X}_3\overline{X}_7\overline{X}_2$$

$$S_{PRE(6)} = X_3 \vee X_7\overline{X}_3 \vee X_2X_6\overline{X}_3\overline{X}_7 \vee X_1X_2\overline{X}_3\overline{X}_7\overline{X}_6 \vee X_4X_5\overline{X}_3\overline{X}_7\overline{X}_2 \vee X_4X_5\overline{X}_3\overline{X}_7X_2\overline{X}_6\overline{X}_1$$
$$\vee X_4X_8\overline{X}_3\overline{X}_7\overline{X}_2(\overline{X}_4 \vee X_4\overline{X}_5) \vee X_4X_8\overline{X}_3\overline{X}_7X_2\overline{X}_6\overline{X}_1(\overline{X}_4 \vee X_4\overline{X}_5) \vee X_2X_5X_9\overline{X}_3\overline{X}_7\overline{X}_6\overline{X}_1(\overline{X}_4$$
$$\vee X_4\overline{X}_5) \vee X_2X_8X_9\overline{X}_3\overline{X}_7\overline{X}_6\overline{X}_1(\overline{X}_4 \vee X_4\overline{X}_5) \vee X_4X_6X_9\overline{X}_3\overline{X}_7\overline{X}_2(\overline{X}_4$$
$$\vee X_4\overline{X}_5) \vee X_1X_4X_9\overline{X}_3\overline{X}_7\overline{X}_2(\overline{X}_4 \vee X_4\overline{X}_5)$$

Simplify and subsumes all required terms:

$$S_{PRE(6)} = X_3 \vee X_7\overline{X}_3 \vee X_2X_6\overline{X}_3\overline{X}_7 \vee X_1X_2\overline{X}_3\overline{X}_7\overline{X}_6 \vee X_4X_5\overline{X}_3\overline{X}_7\overline{X}_2 \vee X_4X_5\overline{X}_3\overline{X}_7X_2\overline{X}_6\overline{X}_1$$
$$\vee X_4X_8\overline{X}_3\overline{X}_7\overline{X}_2\overline{X}_5 \vee X_4X_8\overline{X}_3\overline{X}_7X_2\overline{X}_6\overline{X}_1\overline{X}_5 \vee X_2X_5X_9\overline{X}_3\overline{X}_7\overline{X}_6\overline{X}_1\overline{X}_4$$
$$\vee X_2X_8X_9\overline{X}_3\overline{X}_7\overline{X}_6\overline{X}_1\overline{X}_4 \vee X_2X_8X_9\overline{X}_3\overline{X}_7\overline{X}_6\overline{X}_1X_4\overline{X}_5$$
$$\vee X_4X_6X_9\overline{X}_3\overline{X}_7\overline{X}_2\overline{X}_5 \vee X_1X_4X_9\overline{X}_3\overline{X}_7\overline{X}_2\overline{X}_5$$

$$S_{PRE(7)} = X_3 \vee X_7\overline{X}_3 \vee X_2X_6\overline{X}_3\overline{X}_7 \vee X_1X_2\overline{X}_3\overline{X}_7\overline{X}_6 \vee X_4X_5\overline{X}_3\overline{X}_7\overline{X}_2 \vee X_4X_5\overline{X}_3\overline{X}_7X_2\overline{X}_6\overline{X}_1$$
$$\vee X_4X_8\overline{X}_3\overline{X}_7\overline{X}_2\overline{X}_5 \vee X_4X_8\overline{X}_3\overline{X}_7X_2\overline{X}_6\overline{X}_1\overline{X}_5 \vee X_2X_5X_9\overline{X}_3\overline{X}_7\overline{X}_6\overline{X}_1\overline{X}_4(\overline{X}_4 \vee X_4\overline{X}_8)$$
$$\vee X_2X_8X_9\overline{X}_3\overline{X}_7\overline{X}_6\overline{X}_1\overline{X}_4(\overline{X}_4 \vee X_4\overline{X}_8) \vee X_2X_8X_9\overline{X}_3\overline{X}_7\overline{X}_6\overline{X}_1X_4\overline{X}_5(\overline{X}_4 \vee X_4\overline{X}_8)$$
$$\vee X_4X_6X_9\overline{X}_3\overline{X}_7\overline{X}_2\overline{X}_5(\overline{X}_4 \vee X_4\overline{X}_8) \vee X_1X_4X_9\overline{X}_3\overline{X}_7\overline{X}_2\overline{X}_5(\overline{X}_4 \vee X_4\overline{X}_8)$$

Simplify and subsumes all required terms:

$$S_{PRE(7)} = X_3 \vee X_7\overline{X}_3 \vee X_2X_6\overline{X}_3\overline{X}_7 \vee X_1X_2\overline{X}_3\overline{X}_7\overline{X}_6 \vee X_4X_5\overline{X}_3\overline{X}_7\overline{X}_2 \vee X_4X_5\overline{X}_3\overline{X}_7X_2\overline{X}_6\overline{X}_1$$
$$\vee X_4X_8\overline{X}_3\overline{X}_7\overline{X}_2\overline{X}_5 \vee X_4X_8\overline{X}_3\overline{X}_7X_2\overline{X}_6\overline{X}_1\overline{X}_5 \vee X_2X_5X_9\overline{X}_3\overline{X}_7\overline{X}_6\overline{X}_1\overline{X}_4$$
$$\vee X_2X_8X_9\overline{X}_3\overline{X}_7\overline{X}_6\overline{X}_1\overline{X}_4 \vee X_4X_6X_9\overline{X}_3\overline{X}_7\overline{X}_2\overline{X}_5\overline{X}_8 \vee X_1X_4X_9\overline{X}_3\overline{X}_7\overline{X}_2\overline{X}_5\overline{X}_8$$

$$S_{PRE(8)} = X_3 \vee X_7\overline{X}_3 \vee X_2X_6\overline{X}_3\overline{X}_7 \vee X_1X_2\overline{X}_3\overline{X}_7\overline{X}_6 \vee X_4X_5\overline{X}_3\overline{X}_7\overline{X}_2 \vee X_4X_5\overline{X}_3\overline{X}_7X_2\overline{X}_6\overline{X}_1$$
$$\vee X_4X_8\overline{X}_3\overline{X}_7\overline{X}_2\overline{X}_5 \vee X_4X_8\overline{X}_3\overline{X}_7X_2\overline{X}_6\overline{X}_1\overline{X}_5 \vee X_2X_5X_9\overline{X}_3\overline{X}_7\overline{X}_6\overline{X}_1\overline{X}_4$$
$$\vee X_2X_8X_9\overline{X}_3\overline{X}_7\overline{X}_6\overline{X}_1\overline{X}_4(\overline{X}_2 \vee X_2\overline{X}_5 \vee X_2X_5\overline{X}_9)$$
$$\vee X_4X_6X_9\overline{X}_3\overline{X}_7\overline{X}_2\overline{X}_5\overline{X}_8(\overline{X}_2 \vee X_2\overline{X}_5 \vee X_2X_5\overline{X}_9) \vee X_1X_4X_9\overline{X}_3\overline{X}_7\overline{X}_2\overline{X}_5\overline{X}_8(\overline{X}_2 \vee X_2\overline{X}_5$$
$$\vee X_2X_5\overline{X}_9)$$

Simplify and subsumes all required terms:

$$S_{PRE(8)} = X_3 \vee X_7\overline{X}_3 \vee X_2X_6\overline{X}_3\overline{X}_7 \vee X_1X_2\overline{X}_3\overline{X}_7\overline{X}_6 \vee X_4X_5\overline{X}_3\overline{X}_7\overline{X}_2 \vee X_4X_5\overline{X}_3\overline{X}_7X_2\overline{X}_6\overline{X}_1$$
$$\vee X_4X_8\overline{X}_3\overline{X}_7\overline{X}_2\overline{X}_5 \vee X_4X_8\overline{X}_3\overline{X}_7X_2\overline{X}_6\overline{X}_1\overline{X}_5 \vee X_2X_5X_9\overline{X}_3\overline{X}_7\overline{X}_6\overline{X}_1\overline{X}_4$$
$$\vee X_2X_8X_9\overline{X}_3\overline{X}_7\overline{X}_6\overline{X}_1\overline{X}_4\overline{X}_5 \vee X_4X_6X_9\overline{X}_3\overline{X}_7\overline{X}_2\overline{X}_5\overline{X}_8 \vee X_1X_4X_9\overline{X}_3\overline{X}_7\overline{X}_2\overline{X}_5\overline{X}_8$$

$$\begin{aligned}S_{PRE(9)} = {} & X_3 \vee X_7\overline{X}_3 \vee X_2X_6\overline{X}_3\overline{X}_7 \vee X_1X_2\overline{X}_3\overline{X}_7\overline{X}_6 \vee X_4X_5\overline{X}_3\overline{X}_7\overline{X}_2 \vee X_4X_5\overline{X}_3\overline{X}_7X_2\overline{X}_6\overline{X}_1 \\ & \vee X_4X_8\overline{X}_3\overline{X}_7\overline{X}_2\overline{X}_5 \vee X_4X_8\overline{X}_3\overline{X}_7X_2\overline{X}_6\overline{X}_1\overline{X}_5 \vee X_2X_5X_9\overline{X}_3\overline{X}_7\overline{X}_6\overline{X}_1\overline{X}_4 \\ & \vee X_2X_8X_9\overline{X}_3\overline{X}_7\overline{X}_6\overline{X}_1\overline{X}_4\overline{X}_5 \vee X_4X_6X_9\overline{X}_3\overline{X}_7\overline{X}_2\overline{X}_5\overline{X}_8(\overline{X}_2 \vee X_2\overline{X}_8 \\ & \vee X_2X_8\overline{X}_9) \vee X_1X_4X_9\overline{X}_3\overline{X}_7\overline{X}_2\overline{X}_5\overline{X}_8(\overline{X}_2 \vee X_2\overline{X}_8 \vee X_2X_8\overline{X}_9)\end{aligned}$$

Simplify and subsumes all required terms:

$$\begin{aligned}S_{PRE(9)} = {} & X_3 \vee X_7\overline{X}_3 \vee X_2X_6\overline{X}_3\overline{X}_7 \vee X_1X_2\overline{X}_3\overline{X}_7\overline{X}_6 \vee X_4X_5\overline{X}_3\overline{X}_7\overline{X}_2 \vee X_4X_5\overline{X}_3\overline{X}_7X_2\overline{X}_6\overline{X}_1 \\ & \vee X_4X_8\overline{X}_3\overline{X}_7\overline{X}_2\overline{X}_5 \vee X_4X_8\overline{X}_3\overline{X}_7X_2\overline{X}_6\overline{X}_1\overline{X}_5 \vee X_2X_5X_9\overline{X}_3\overline{X}_7\overline{X}_6\overline{X}_1\overline{X}_4 \\ & \vee X_2X_8X_9\overline{X}_3\overline{X}_7\overline{X}_6\overline{X}_1\overline{X}_4\overline{X}_5 \vee X_4X_6X_9\overline{X}_3\overline{X}_7\overline{X}_2\overline{X}_5\overline{X}_8 \vee X_1X_4X_9\overline{X}_3\overline{X}_7\overline{X}_2\overline{X}_5\overline{X}_8\end{aligned}$$

$$\begin{aligned}S_{PRE(9)} = {} & X_3 \vee X_7\overline{X}_3 \vee X_2X_6\overline{X}_3\overline{X}_7 \vee X_1X_2\overline{X}_3\overline{X}_7\overline{X}_6 \vee X_4X_5\overline{X}_3\overline{X}_7\overline{X}_2 \vee X_4X_5\overline{X}_3\overline{X}_7X_2\overline{X}_6\overline{X}_1 \\ & \vee X_4X_8\overline{X}_3\overline{X}_7\overline{X}_2\overline{X}_5 \vee X_4X_8\overline{X}_3\overline{X}_7X_2\overline{X}_6\overline{X}_1\overline{X}_5 \vee X_2X_5X_9\overline{X}_3\overline{X}_7\overline{X}_6\overline{X}_1\overline{X}_4 \\ & \vee X_2X_8X_9\overline{X}_3\overline{X}_7\overline{X}_6\overline{X}_1\overline{X}_4\overline{X}_5 \vee X_4X_6X_9\overline{X}_3\overline{X}_7\overline{X}_2\overline{X}_5\overline{X}_8 \vee X_1X_4X_9\overline{X}_3\overline{X}_7\overline{X}_2\overline{X}_5\overline{X}_8(\overline{X}_4 \vee X_4\overline{X}_6 \\ & \vee X_4X_6\overline{X}_9)\end{aligned}$$

Simplify and subsumes all required terms:

$$\begin{aligned}S_{PRE(9)} = {} & X_3 \vee X_7\overline{X}_3 \vee X_2X_6\overline{X}_3\overline{X}_7 \vee X_1X_2\overline{X}_3\overline{X}_7\overline{X}_6 \vee X_4X_5\overline{X}_3\overline{X}_7\overline{X}_2 \vee X_4X_5\overline{X}_3\overline{X}_7X_2\overline{X}_6\overline{X}_1 \\ & \vee X_4X_8\overline{X}_3\overline{X}_7\overline{X}_2\overline{X}_5 \vee X_4X_8\overline{X}_3\overline{X}_7X_2\overline{X}_6\overline{X}_1\overline{X}_5 \vee X_2X_5X_9\overline{X}_3\overline{X}_7\overline{X}_6\overline{X}_1\overline{X}_4 \\ & \vee X_2X_8X_9\overline{X}_3\overline{X}_7\overline{X}_6\overline{X}_1\overline{X}_4\overline{X}_5 \vee X_4X_6X_9\overline{X}_3\overline{X}_7\overline{X}_2\overline{X}_5\overline{X}_8 \vee X_1X_4X_9\overline{X}_3\overline{X}_7\overline{X}_2\overline{X}_5\overline{X}_8\overline{X}_6\end{aligned}$$

$$\begin{aligned}S_{PRE(T)} = {} & X_3 \vee X_7\overline{X}_3 \vee X_2X_6\overline{X}_3\overline{X}_7 \vee X_1X_2\overline{X}_3\overline{X}_7\overline{X}_6 \vee X_4X_5\overline{X}_3\overline{X}_7\overline{X}_2 \vee X_4X_5\overline{X}_3\overline{X}_7X_2\overline{X}_6\overline{X}_1 \\ & \vee X_4X_8\overline{X}_3\overline{X}_7\overline{X}_2\overline{X}_5 \vee X_4X_8\overline{X}_3\overline{X}_7X_2\overline{X}_6\overline{X}_1\overline{X}_5 \vee X_2X_5X_9\overline{X}_3\overline{X}_7\overline{X}_6\overline{X}_1\overline{X}_4 \\ & \vee X_2X_8X_9\overline{X}_3\overline{X}_7\overline{X}_6\overline{X}_1\overline{X}_4\overline{X}_5 \vee X_4X_6X_9\overline{X}_3\overline{X}_7\overline{X}_2\overline{X}_5\overline{X}_8 \vee X_1X_4X_9\overline{X}_3\overline{X}_7\overline{X}_2\overline{X}_5\overline{X}_8\overline{X}_6\end{aligned}$$

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