Article

Analyzing capacitated networks via Boolean-based coherent pseudo-Boolean functions

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Abstract
This paper introduces a novel method for analyzing capacitated networks through the utilization of the concept of a “probability-ready expression” for a Boolean-based coherent pseudo-Boolean function. Our main concern is to assess the performance indexes of biology and ecology networks having fixed channel capacities. The technique introduced is based on constructing an exhaustive description (specifically, a value-entered Karnaugh map) for the pseudo-Boolean capacity function of the network via a generalization of the max-flow min-cut theorem. Then the function is expressed in a disjunctive-normal form (DNF) by obtaining the so-called ‘contributions’ of each entered value via standard Karnaugh maps. The technique heavily relies on the fact that the pertinent function is a coherent one, and it is self-checking since it must produce a DNF of solely uncomplemented Boolean literals. The notorious Inclusion-Exclusion (IE) Principle is ruled out as a practical means for converting the DNF of the capacity function into its probabilistic expectation (its expected value). Instead, a method is proposed for converting the DNF of the capacity function to a ‘probability-ready expression’ (PRE), which can be easily transformed, on a one-to-one basis into a probability function. Two tutorial examples demonstrate the afore-mentioned method and illustrate its computational advantages over the exhaustive state enumeration method and the IE method.

Keywords capacitated networks; map method; max-flow min-cut theorem; pseudo-Boolean function; probability-ready expression; exhaustive enumeration; inclusion-exclusion.

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1 Introduction

Biology or ecology networks might be appropriately modeled as capacitated-flow networks having independent edge capacities, that are limited real-valued random variables. Usually, the modelling of a network of this type is attained by the use of a stochastic graph G = (V, E) with V and E being sets of nodes (vertices) and branches (edges) of G, where one can distinguish a particular set $K \subseteq V$. The two extreme situations for K are those of (a) the source-to-terminal (st) case when K contains only two nodes, the source s and the destination t (to be considered herein), or (b) when K contains all nodes of the graph for which (K=V), typically depicted as the overall reliability case. Alternatively, attention sometimes diverted from probabilistic connectivity to deterministic capacity in flow networks, wherein network st capacity is computed as the maximum flow that is transmittable to the terminal node from the source node with no violation of branch capacity and under the assumption that all branches are functioning well. Under such a scheme of deterministic modeling, there is obviously deliberate implicit ignoring of the failure probabilities of both links and nodes. This paper is dealing with a composite performance index related to any network, that integrates the two afore-mentioned connectivity and capacity aspects (Rushdi, 1987b, 1988, 1990; Rushdi and Alsalami, 2020a, 2020b; 2021a, 2021b, 2021c; Alsalami and Rushdi, 2020).

The topic of a capacitated or flow network is of a paramount concern in biology and ecology sciences (Ulanowicz, 2004; Surana et al., 2005; Pascual et al., 2006; Fath et al., 2007; Hagen et al., 2012; Baguette et al., 2013; Bernabò et al., 2014; Rushdi and Hassan, 2015, 2016, 2020; VanderWaal and Ezenwa, 2016; Zhang, 2018). A classical problem of ecology and biology networks (Rushdi and Alsalami, 2021a; Rushdi and Hassan, 2015, 2016, 2020), is that of survivability (of a species), defined as the probability of successful migration of a certain organism escaping from critical source habitat patches and seeking refuge in specific destination habitat patches via heterogeneous deletable ecological corridors, possibly with uninhabitable stepping stones along the way. This problem might be reformulated in celebrated ecology and biology contexts other than that of migration, including those of: (a) dynamics of metapopulations, colonization, or invasion, (b) gene flow, (c) spread of infectious diseases, epidemics, or pandemics, and (d) energy transfer within food webs (Rushdi and Hassan, 2020). Connectivity solutions for this problem are already available, but a more powerful capacitated model for it is being sought.

This paper introduces a novel method for analyzing capacitated networks that serve as convenient models for ecology and biology networks. The paper utilizes the modern concept of a “probability-ready expression” (PRE) for a Boolean-based coherent pseudo-Boolean function that represents the network capacity function. The technique introduced herein is based on constructing an exhaustive description (specifically, a value-entered Karnaugh map or a multi-valued Karnaugh map) for the pseudo-Boolean capacity function of the network via a generalization of the max-flow min-cut theorem. Then the function is expressed in a disjunctive-normal form (DNF) by obtaining the so-called ‘contributions’ of each entered value via standard Karnaugh maps. The technique heavily relies on the fact that the pertinent function is a coherent one (enjoying properties of causality, monotonicity and relevancy). Hence, the technique is self-checking since it must produce a DNF of solely uncomplemented Boolean literals. The notorious Inclusion-Exclusion (IE) Principle is ruled out as a practical means for converting the DNF of the capacity function into its probabilistic expectation (its expected value). Instead, a method is proposed for converting the DNF of the capacity function to a ‘probability-ready expression’ (PRE), which can be easily transformed, on a one-to-one basis into a probability function. Two tutorial examples demonstrate the afore-mentioned PRE method and illustrate its computational advantages over the exhaustive state enumeration method and the IE method.

The remainder of this paper is structured as follows. Section 2 presents the underlying assumptions for our model as well as the notation used. Section 3 reviews the arithmetic and Boolean representations for a pseudo-Boolean (pseudo-switching) function that models the general capacity function for the network. Section 4 introduces the Inclusion-Exclusion (IE) Principle as an initially potential means for converting the DNF of the capacity function into its probabilistic expectation (its expected value). Section 4 shows that this principle is not only computationally intensive, but also highly error-prone, and hence dismisses it as a suitable candidate method. Section 5 introduces a method for converting the DNF of the capacity function to a 'probability-ready expression' (PRE), which can be easily transformed, on a one-to-one basis into a probability function. In Sections 6 and 7, two tutorial examples are presented to demonstrate the PRE-method. Section 8 concludes the paper.

2 Assumptions and Notation

2.1 Assumptions

(1) The physical network considered is modeled as a linear graph consisting of (a) transmission links of imperfect reliabilities and limited capacities and (b) nodes which are perfectly reliable and have unconstrained capacities.

(2) Each link in the network has two states, a successful state and an unsuccessful one. Link successes are statistically independent.

(3) Certain values are assigned to each link $(i\ ,j)$ for its reliability p_{ij} and capacity c_{ij}, where $0 \leq\ p_{ij} \leq 1$, $c_{ij} \geq 0$. The link capacity sets an upper bound on link flow in either direction.

(4) Every link in the network is directed. A bidirectional link is replaced by two directed links in antiparallel whose failures are completely dependent. These two links have equivalent reliabilities. However, they perhaps have different capacities.

2.2 Notation

n Number of branches (edges or links) in the logic diagram of the network.

$X_i, \overline{X}_i$ Indicator variables for successful and unsuccessful operation of branch i. These are binary random variables that take only one of the two discrete real values 0 and 1; $X_i = 1$ and $\overline{X}_i = 0$ if branch i is functioning, and $X_i = 0$ and $\overline{X}_i = 1$ if branch i is failed. For a bidirectional branch ij, the anti-parallel successes are the same $X_{ij} = X_{ji}$.

$S\ ,\overline{S}$ Indicator variables for successful and unsuccessful operation of the system; called system success and system failure, respectively. Successful operation can be equivalent to mere connectivity, or to the satisfaction of a certain flow requirement.

p_i, q_i Reliability and unreliability of branch i: $p_i \equiv \Pr\{X_i = 1\}, q_i \equiv \Pr\{\overline{X}_i = 1\} = 1 - p_i$. Both $p_i and q_i$ are real values in the closed real interval [0.0,1.0].

R, U Network reliability and unreliability; $R = \Pr\{S = 1\} = E\{S\}$, $U = \Pr\{\overline{S} = 1\} = 1.0 - R$, $0.0 \leq R$, $U \leq 1.0$.

c_i Flow capacity of branch i; $c_i \geq 0$.

$\boldsymbol{X}, \boldsymbol{p}, \boldsymbol{c}$ n-dimensional vectors of branch successes, reliabilities and capacities:
$\boldsymbol{X} = (X_1 X_2 \dots X_n)^T; \boldsymbol{p} = (p_1 p_2 \dots p_n)^T ; \boldsymbol{c} \equiv (c_1 c_2 \dots c_n)^T$.

T A superscript that implies the transpose of a matrix or a vector.

$\boldsymbol{X}_k$ State k of the network, denoted by a particular value of the n-dimensional vector of link successes **X**, where $k = 0,1,2, \dots, 2^n - 1$.

$C_{ij}(\boldsymbol{X})$ Capacity function of the branch (i, j) which is the maximum flow interconnection from node i to node j in state $\boldsymbol{X}$ that does not violate branch capacities, $C_{ij}(\boldsymbol{X}) \geq 0$. For an edge (i, j) : $C_{ij} = c_{ij} X_{ij}$. Since $\boldsymbol{X}$ is a switching random vector, $C_{ij}(\boldsymbol{X})$ is a discrete random variable of a probability mass function (pmf) of no more than 2^n distinct values.

$(i\ ,j)$ A directed branch or edge from node i to node j . If two or more such branches exist, they are distinguished by superscripts.

s, t Source node and terminal node

$C_{ij}(\boldsymbol{X}|1_l), C_{ij}(\boldsymbol{X}|0_l)$ The function $C_{ij}(\boldsymbol{X})$when the branch success$X_l$is set to 1 or 0, respectively. Meanings of $C_{ij}(\boldsymbol{X}|1_l, 1_m)$, etc. follow similarity.

3 Boolean-Based Coherent Pseudo-Boolean Functions

A Boolean (Switching) function $S(\boldsymbol{X})$ is a mapping $\{0,1\}^n \rightarrow \{0,1\}$, i.e., $S(\boldsymbol{X})$ is any one particular assignment of the two functional values (0 or 1) for all possible 2^n values of $\boldsymbol{X}$. By contrast, a pseudo- Boolean (pseudo-switching) function $C(\boldsymbol{X})$ is a mapping $\{0,1\}^n \rightarrow R$ where R is the field of real numbers, i.e. $C(\boldsymbol{X})$ is an assignment of a real number for each of the possible 2^n values of $\boldsymbol{X}$. Pseudo-Boolean functions play important roles for binary capacitated networks and other applications (Hammer et al., 1963; Rushdi, 1988; Foldes and Hammer, 2000a, 2000b; Rushdi and Ghaleb, 2016; Alsalami and Rushdi, 2020, 2021; Rushdi and Alsalami, 2020a, 2021a, 2021c). This section briefly characterizes coherent pseudo-Boolean functions and hints on their role in the analysis of binary flow networks.

The function $C_{ij}(\boldsymbol{X})$, which expresses the source-to-terminal capacity function of a binary flow network is characterized by the following algebraic decomposition of the function with respect to one of its input variables X_l for $l = 1,2,\ldots,n$ (Rushdi, 1987a, 1988, 1990; Alsalami and Rushdi, 2020, 2021)

$$C_{ij}(\boldsymbol{X}) = \overline{X}_l C_{ij}(\boldsymbol{X}|O_l) + X_l C_{ij}(\boldsymbol{X}|1_l)$$

$$= (1 - X_l) C_{ij}(\boldsymbol{X}|0_l) + X_l C_{ij}(\boldsymbol{X}|1_l) = C_{ij}(\boldsymbol{X}|O_l) + \left[C_{ij}(\boldsymbol{X}|1_l) - C_{ij}(\boldsymbol{X}|O_l)\right] X_l \quad (1)$$

Equation (1) can be validated through proof by perfect induction of all cases or values of $\boldsymbol{X}$, viz., $\{\boldsymbol{X}|0_l\}$ and $\{\boldsymbol{X}|1_l\}$. This decomposition relation of $C_{ij}(\boldsymbol{X})$ can be used to assert many properties of it as a pseudo-Boolean function, including, in particular,

(a) Proof of the fact that $C_{ij}(\boldsymbol{X})$ is a multi-affine function (Rushdi, 1983, 1985b; Rushdi and Ghaleb, 2015; Rushdi and Rushdi, 2017), i.e., it is an algebraic function which is a first-degree polynomial in each of its variables. This means that, if fixed values are given to any $(n-1)$ variables, the function reduces to a first-degree polynomial in the remaining variables.

(b) Demonstration of the existence of a multitude of representations for $C_{ij}(\boldsymbol{X})$ through the repeated application of (1) with respect to distinct input variables, thereby leading to an expansion tree (Rushdi and Ghaleb, 2016) for $C_{ij}(\boldsymbol{X})$. Each level of the expansion tree represents the capacity function $C_{ij}(\boldsymbol{X})$ via a variable-entered Karnaugh map (VEKM) (Rushdi, 1985a, 1987a, 2001; Rushdi and Al-Yahya, 2000, 2001a, 2001b, 2002). The two-extreme levels of the tree are of particular interest. For the zeroth level of the tree (its root), the VEKM degenerates into the initial purely-algebraic expression. For the nth level of the tree (its leaves), the VEKM degenerates into a multi-valued Karnaugh map (MVKM) (Rushdi, 2018; Rushdi and Rushdi, 2018), also called a value-entered Karnaugh map. This is a Karnaugh map of the usual Boolean combinations of the input domain, but of entries that are multiple specific real elements rather than just the two binary values $\{0,1\}$.

(c) Manifestation of system coherence of the binary flow network as properties of causality, monotonicity, and component relevancy of the capacity function $C_{ij}(\boldsymbol{X})$. Causality is expressed as

$$C_{ij}(\boldsymbol{0}) = 0, \quad (2a)$$

$$C_{ij}(\boldsymbol{1}) = \sum_{l=1}^{n} c_l \quad (2b)$$

Monotonicity means that the function $C_{ij}(\boldsymbol{X})$ is monotone non-decreasing in $\boldsymbol{X}$, and hence, the coefficient of X_l in (1) is non-negative, i.e.,

$$C_{ij}(\boldsymbol{X}|1_l) \geq C_{ij}(\boldsymbol{X}|O_l), l = 1,2, \dots, n. \, for\ all\ \boldsymbol{X} \tag{3}$$

Component relevancy means that the coefficient of $X_l$in (1) is strictly positive for some particular value(s) of $\boldsymbol{X}$, i.e.,

$$C_{ij}(\boldsymbol{X}|1_l) > C_{ij}(\boldsymbol{X}|O_l), \ \ l = 1,2, \dots, n. \, for\ at\ least\ one\ particular\ \boldsymbol{X} \tag{4}$$

(d) Deduction of the fact that $C_{ij}(\boldsymbol{X})$ can be expressed in terms of a polynomial representation, i.e., as a sum-of-products form, where the terms ‘sum’ and ‘product’ here refer to their standard genuine meanings of real addition of real products, rather than to the logical addition (ORing) of logical products (ANDed literals). Moreover, monotonicity as given by (3) asserts that the sum-of-products expression of $C_{ij}(\boldsymbol{X})$ involves only un-complemented literals X_l. Since the expectation of a (real) sum is the (real) sum of expectations, the mean (expected) value of the random function $C_{ij}(\boldsymbol{X})$, when written in a sum-of-products form, equates to;

$$E\{C_{ij}(\boldsymbol{X})\} = E\{C_{ij}\}(\boldsymbol{p}) \, , \tag{5}$$

and can be directly obtained (on a one-to-one basis) from $C_{ij}(\boldsymbol{X})$(s-o-p) by introducing the component means $p_l = \mathrm{E}\{X_l\}$ and $q_l = E\{\overline{X}_l\}$, in place of the corresponding Boolean arguments X_l, and $\overline{X}_l$. The polynomial representation has been conventionally used for handling pseudo-Boolean functions (Hammer et al., 1963; Rushdi, 1987b, 1988, 1990; Rushdi and Alsalami, 2020a, 2020b, 2021a, 2021b, 2021c; Alsalami and Rushdi, 2020).

Pseudo-Boolean functions are essentially equivalent to set functions, i.e., mappings of the subsets of a finite set into the real field (Foldes and Hammer, 2000b). The term pseudo-Boolean function reflects the similarity of these functions with the Boolean ones and was introduced by Hammer et al. (1963), and elucidated, explicated, and popularized by Hammer and Rudeanu (1968). In fact, a pseudo-Boolean function $C_{ij}(\boldsymbol{X})$ is Boolean if its range is contained in $\{0, 1\}$. A necessary and sufficient condition for this is that the function $C_{ij}(\boldsymbol{X})$ is equal to its square. Beside the polynomial representations of pseudo-Boolean functions, they also possess disjunctive normal forms (DNFs) as well as conjunctive normal forms (CNFs), in a striking similarity to Boolean functions (Foldes and Hammer, 2000a, 2000b). These forms are constructed in terms of the join (disjunction, ORing), meet (conjunction, ANDing) and complementation (negation) in $\{0, 1\}^n$ denoted by $\boldsymbol{X} \vee \boldsymbol{Y}$, $\boldsymbol{X} \wedge \boldsymbol{Y}$ and $\overline{\boldsymbol{X}}$, respectively, with the order relation $\boldsymbol{X} \leq \boldsymbol{Y}$ in $\{0, 1\}^n$ being defined component-wise. The symbols $\vee$and $\wedge$ also denote the maximum (max) and the minimum (min) operators in R. While the functions X and $\overline{X}$ are called Boolean literals, any function of the form $a + bX$ or $a + b\overline{X}$ (where a and b are constants and b is not equal to 0) is called a pseudo-Boolean literal. Every pseudo-Boolean literal has a unique expression $a + bX$ or $a + b\overline{X}$ such that $b > 0$. Obviously a is the minimum value of such a literal, and $a + b$ is the maximum value of it. An elementary conjunction is the greatest lower bound of one or more literals having the same minimum. Every pseudo-Boolean function f can be expressed as a finite join (disjunction) of elementary conjunctions having the same minimum a. Since the function $C_{ij}(\boldsymbol{X})$ is monotone non-decreasing in $\boldsymbol{X}$, then it can be expressed in a disjunctive normal form that contains no complemented variables. Since the function $C = C_{ij}(\boldsymbol{X})$is causal, with non-negative values, and attaining its minimum at $C_{ij}(\boldsymbol{0}) = 0$, it can be expressed as a finite disjunction of elementary conjunctions, each having a zero minimum (Rushdi and Alsalami, 2021b; 2021c), namely

$$C = \vee_{i=1}^{n_p} a_i P_i \qquad (6)$$

where $a_i P_i$ is an elementary conjunction with a zero minimum, and $P_i$depicts a standard Boolean term or product. The expression in (6) resembles a standard sum-of-products in the Boolean jargon, with the two exceptions that (a) a sum now depicts a join (or maximum) operator, and (b) a Boolean product $P_i$is now multiplied by a real weight a_i.

4 The Inclusion-Exclusion Principle

The Inclusion-Exclusion (IE) Principle allows the analyst to compute the probability of the union of n events, or equivalently the expectation of the disjunction (ORing) of n indicator variables (Rushdi and Hassan, 2016). For example, we can apply the IE Principle to the coherent capacity function (6) to obtain the following expression for its expectation

$$E\{C\} = E\{\vee_{i=1}^{n_p} a_i P_i\} = \sum_{i=1}^{n_p} E\{a_i P_i\} - \sum\sum_{1\le i<j\le n_p} E\{a_i P_i \wedge a_j P_j\} + \sum\sum\sum_{1\le i<j<k\le n_p} E\{a_i P_i \wedge a_j P_j \wedge a_k P_k\} - \ldots + (-1)^{n_p-1} E\{\wedge_{i=1}^{n_p} a_i P_i\}$$

$$= \sum_{i=1}^{n_p} E\{a_i P_i\} - \sum\sum_{1\le i<j\le n_p} E\{(a_i \wedge a_j)(P_i \wedge P_j)\} + \sum\sum\sum_{1\le i<j<k\le n_p} E\{(a_i \wedge a_j \wedge a_k)(P_i \wedge P_j \wedge P_k)\} - \ldots + (-1)^{n_p-1} E\{(\wedge_{i=1}^{n_p} a_i)(\wedge_{i=1}^{n_p} P_i)\} \qquad (7)$$

The number of terms in (7) is

$$\binom{n_p}{1} + \binom{n_p}{2} + \binom{n_p}{3} + \cdots + \binom{n_p}{n_p} = 2^{n_p} - 1 \qquad (8)$$

i.e., it is exponential in the number of disjuncted (joined) terms. This means that if we apply IE to a formula with $n_p = 7$, we get 63expectation terms, and if we apply IE to a formula with $n_p = 12$, we get 4095 expectation terms. Note that the IE formula is subject to dramatic simplifications through the application of idempotency of AND ($X_i \wedge X_i = X_i$), the application of the minimization or meet operator $(a_i \wedge a_j) = \min(a_i, a_j)$, and the imposition of term addition/cancellation. These simplifications usually resultin a substantial reduction of the final number of expectation terms. The IE formula is not our method of choice, not only because of its exponential temporal complexity, but also because of its high susceptibility to excessive round-off errors that might amount to catastrophic cancellation (Rushdi, 1986, 2010; Rushdi and Hassan, 2016; Rushdi and Amashah, 2021).

In passing, we note that if the terms P_i in equation (7) are statistically independent, then it reduces to

$$E\{C\} = \sum_{i=1}^{n_p} E\{a_i P_i\} - \sum\sum_{1\le i<j\le n_p} E\{a_i P_i\}E\{a_j P_j\} + \sum\sum\sum_{1\le i<j<k\le n_p} E\{a_i P_i\}E\{a_j P_j\}E\{a_k P_k\} - \ldots + (-1)^{n_p-1}\prod_{i=1}^{n_p} E\{a_i P_i\}$$

$$= 1 - \prod_{i=1}^{n_p}(1 - E\{a_i P_i\}) \quad (9)$$

Moreover, if the terms P_i in equation (7) are mutually disjoint, then (7) reduces to

$$E\{C\} = \sum_{i=1}^{n_p} E\{a_i P_i\} \quad (10)$$

5 Probability-Ready Expressions for the DNA of A Pseudo-Boolean Functions

To avoid the undesirable troubles of inclusion-exclusion, we convert the DNF in (6) into a disjoint one, i.e., into a probability-ready-expression (PRE) (Rushdi and Rushdi, 2017). The crucial step in this conversion is to disjoint two non-disjoint pseudo-Boolean products a_1P_1 and a_2P_2 into disjoint ones. Here a_1 and a_2 are real values, assumed to satisfy $a_1 \ge a_2$, $a_1 \wedge a_2 = a_2$, while P_1 and P_2 are Boolean products with $P_1 = T_1B$ and $P_2 = T_2B$. Here, B is a product of literals shared between P_1 and P_2 while T_1 is the product $y_1y_2 \ldots y_e$ of literals that appear in P_1 but not in P_2. We assume further that the set of literals $\{y_1, y_2, \ldots, y_e\}$(whose product constitutes the product T_1) is not empty (otherwise, the Boolean product P_2 subsumes the Boolean product P_1 and henceP_2 is absorbed in P_1, so that we might say thata_2P_2 is absorbed in a_1P_1(remember that $a_1 \ge a_2$), and, therefore, $a_1P_1 \vee a_2P_2 = a_1P_1$). We now use the IE principle to write

$$E\{a_1P_1 \vee a_2P_2\} = E\{a_1P_1\} + E\{a_2P_2\} - E\{(a_1 \wedge a_2)P_1P_2\}$$

$$= E\{a_1P_1\} + E\{a_2T_2B\} - E\{a_2T_1T_2B\}$$

$$= E\{a_1P_1\} + E\{a_2(1 - T_1)T_2B\}$$

$$= E\{a_1P_1\} + E\{a_2\overline{T_1}P_2\}$$

$$= E\{a_1P_1\} + E\{a_2(\overline{y_1} \vee y_1\overline{y_2} \vee \ldots \vee y_1y_2\ldots\overline{y_e})P_2\} \quad (11)$$

$$= E\{a_1P_1 + a_2\overline{y_1}P_2 + a_2y_1\overline{y_2}P_2 + \cdots + a_2y_1y_2\ldots y_{e-1}\overline{y_e}P_2\}$$

$$= E\{a_1P_1 \vee a_2\overline{y_1}P_2 \vee a_2y_1\overline{y_2}P_2 \vee \ldots \vee a_2y_1y_2\ldots y_{e-1}\overline{y_e}P_2\} \quad (12)$$

Equation (12) means that the expectation is preserved if we replace the non-disjoint disjunction

$$a_1P_1 \vee a_2P_2 \quad (13a)$$

by the disjoint (or PRE) one

$$a_1P_1 \vee a_2\overline{y_1}P_2 \vee a_2y_1\overline{y_2}P_2 \vee \ldots \vee a_2y_1y_2 \ldots y_{e-1}\,\overline{y_e}P_2 \quad (13b)$$

In passing, we note that if the set of literals $\{y_1, y_2, \ldots, y_e\}$is empty, then the product$T_1 = 1$ and $\overline{T_1} = 0$. This means $(a_1P_1 \vee a_2P_2)$ is replaced by $(a_1P_1 \vee a_2P_2(0) = a_1P_1)$ as required.

6 Example 1

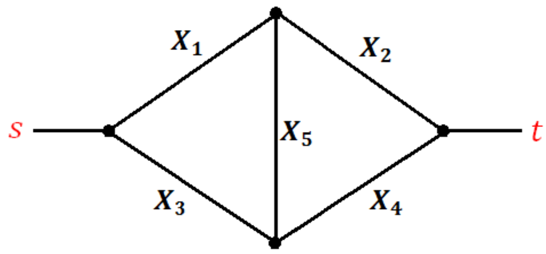


Fig. 1 A capacitated 5-branch st network with a capacity vector $c = [\,10\;\;3\;\;4\;\;4\;\;5]^T$.

To make the current exposition more specific, we revisit an example that was considered earlier in (Rushdi, 1988). This example deals with the capacity function $C_{st}(\boldsymbol{X})$ for the flow network in the present Fig. 1, which has binary links of capacities $c_1 = 10, c_2 = 3, c_3 = c_4 = 4$ and $c_5 = 5$ (in appropriate units for the commodity involved). A Karnaugh map representation of $C_{st}(\boldsymbol{X})$ is shown in Fig. 2, which might be shown to conform to the following Max-Flow Min-Cut expression

$$C_{st}(\boldsymbol{X}) = \min\{c_1X_1 + c_3X_3, c_2X_2 + c_4X_4, c_1X_1 + c_4X_4 + c_5X_5, c_2X_2 + c_3X_3 + c_5X_5\} \quad (14a)$$

$$C(X_1, X_2, X_3, X_4, X_5) = \min(10X_1 + 4X_3, 3X_2 + 4X_4, 10X_1 + 4X_4 + 5X_5, 3X_2 + 4X_3 + 5X_5) \quad (14b)$$

The function in (14) is non-zero only when system success $S = 1$.This function might be expanded through a repeated application of (1) with respect to the non-overlapping (disjoint) set of paths $\{X_1X_2, \overline{X}_1X_3X_4, X_1\overline{X}_2X_3X_4, X_1\overline{X}_2\overline{X}_3X_4X_5, \overline{X}_1X_2X_3\overline{X}_4X_5\}$(which precisely cover $S = 1$) as

$$C(X_1, X_2, X_3, X_4, X_5) = C(1,1, X_3, X_4, X_5)X_1X_2 + C(0, X_2, 1,1, X_5)\overline{X}_1X_3X_4$$

$$+ C(1,0,1,1, X_5)X_1\overline{X}_2X_3X_4 + C(1,0,0,1,1)X_1\overline{X}_2\overline{X}_3X_4X_5 + C(0,1,1,0,1)\overline{X}_1X_2X_3\overline{X}_4X_5 \quad (15)$$

In the summation of disjoint terms (15), arithmetic addition $(+)$ might be replaced with the maximum operation $(\vee)$. The pertinent coefficients in (15) are obtained as the following restrictions of (14)

$$C(1,1, X_3, X_4, X_5) = \min\{10 + 4X_3, 3 + 4X_4, 10 + 4X_4 + 5X_5, 3 + 4X_3 + 5X_5\}$$

$$= 3 + \min\{4X_4, 4X_3 + 5X_5\}$$

$$= 3 + 4X_4\left(X_3 + \overline{X}_3 X_5\right) \quad (16a)$$

$$C(0, X_2, 1,1, X_5) = \min\{4, 3X_2 + 4, 4 + 5X_5, 3X_2 + 4\} = 4 \quad (16b)$$

$$C(1,0,1,1, X_5) = \min\{14, 4, 14 + 5X_5, 4 + 5X_5\} = 4 \quad (16c)$$

$$C(1,0,0,1,1) = \min\{10, 4, 19, 5\} = 4 \quad (16d)$$

$$C(0,1,1,0,1) = \min\{4,3,5,12\} = 3 \quad (16e)$$

The results of (15) and (16) are displayed in the Karnaugh map of Fig. 3 which is equivalent to the one in Fig. 2. The expected value of $C_{st}(\boldsymbol{X})$ is therefore obtained based on a one-to-one transformation of the polynomial representation of $C_{st}(\boldsymbol{X})$, namely

$$E\{C_{st}(\boldsymbol{X})\} = \big(3 + 4p_4(p_3 + q_3 p_5)\big)p_1 p_2 + 4q_1 p_3 p_4 + 4p_1 q_2 p_3 p_4 + 4p_1 q_2 q_3 p_4 p_5 + 3q_1 p_2 p_3 q_4 p_5 \quad (17)$$

An equivalent (albeit more compact) expression was obtained via the Karnaugh-map procedure of Rushdi (1988) as

$$E\{C_{st}(\boldsymbol{X})\} = 4p_4(p_3 + p_1 q_3 p_5) + 3p_2(p_1 + q_1 p_3 q_4 p_5). \quad (18)$$

Equation (18) might be rewritten as an all-p formula by substituting each $q_i = 1 - p_i$ to obtain

$$E\{C_{st}(\boldsymbol{X})\} = 4p_3 p_4 + 4p_1 p_4 p_5 - 4p_1 p_3 p_4 p_5 + 3p_1 p_2 + 3p_2 p_3 p_5 - 3p_1 p_2 p_3 p_5 - 3p_2 p_3 p_4 p_5 + 3p_1 p_2 p_3 p_4 p_5 \quad (19)$$

We now demonstrate the novel contribution of this paper by constructing a DNF for $C_{st}(\boldsymbol{X})$ by adapting procedures of the variable-entered Karnaugh map in (Rushdi, 1985a; 1987; 2001; 2018) to the value-entered Karnaugh map in Fig. 2. The function $C_{st}(\boldsymbol{X})$ is the weighted disjunction of the values v entered in the map, where each entered value v is weighted by its 'contribution' $Co(v)$, namely

$$C_{st}(\boldsymbol{X}) = \forall_{\mathrm{v}} v Co(v) \ = 7\ Co(7)\ \vee\ 4\ Co(4)\ \vee\ 3\ Co(3) \quad (20)$$

The contribution of v is a function of $\boldsymbol{X}$represented by a standard Karnaugh map derived from the original or parent map for $C_{st}(\boldsymbol{X})$ (the one in Fig. 2). In the map for $Co(v)$, a cell is entered with 1 if its entry in the parent map is v, entered with a don't care (d) if its entry in the parent map is greater than v, and entered with 0 otherwise. Figure 4 displays the maps for $Co(3), Co(4)$ and $Co(7)$ to be used in conjunction with Fig. 2. The coverage in these maps seeks minimality rather than completeness. In fact, we ignore all-d loops (called absolutely eliminable loops) in the coverage of $Co(3)$. The final minimal DNF for $C_{st}(\boldsymbol{X})$ is given by

$$C_{st}(\boldsymbol{X}) = 7(X_1 X_2 X_3 X_4 \vee X_1 X_2 X_4 X_5) \vee 4(X_3 X_4\ \vee X_1 X_4 X_5) \vee 3(X_1 X_2 \vee X_2 X_3 X_5) \quad (21)$$

We reiterate that the operator '∨' now represents the 'max' operator over the real field R. Obtaining the expectation of $C_{st}(\boldsymbol{X})$ is not as easy as before, since it might involve the use of the notorious Inclusion-

Exclusion (IE) Principle (Rushdi, 1986, 2010; Rushdi and Hassan, 2016; Rushdi and Amashah, 2021). The IE formula with $n_p = 7$ has63expectation terms, namely

$$
\begin{aligned}
E\{C_{st}(\boldsymbol{X})\} = {} & E\{7X_1X_2X_3X_4\} + E\{7X_1X_2X_4X_5\} + E\{4X_3X_4\} + E\{4X_1X_4X_5\} + E\{3X_1X_2\} + E\{3X_2X_3X_5\} \\
& - E\{(7 \wedge 7)(X_1X_2X_3X_4)(X_1X_2X_4X_5)\} - E\{(7 \wedge 4)(X_1X_2X_3X_4)(X_3X_4)\} \\
& - E\{(7 \wedge 4)(X_1X_2X_3X_4)(X_1X_4X_5)\} - E\{(7 \wedge 3)(X_1X_2X_3X_4)(X_1X_2) \\
& - E\{(7 \wedge 3)(X_1X_2X_3X_4)(X_2X_3X_5) - E\{(7 \wedge 4)(X_1X_2X_4X_5)(X_3X_4)\}) \\
& - E\{(7 \wedge 4)(X_1X_2X_4X_5)(X_1X_4X_5)\} - E\{(7 \wedge 3)(X_1X_2X_4X_5)(X_1X_2)\} \\
& - E\{(7 \wedge 3)(X_1X_2X_4X_5)(X_2X_3X_5)\} - E\{(4 \wedge 4)(X_3X_4)(X_1X_4X_5)\} \\
& - E\{(4 \wedge 3)(X_3X_4)(X_1X_2)\} - E\{(4 \wedge 3)(X_3X_4)(X_2X_3X_5)\} - E\{(4 \wedge 3)(X_1X_4X_5)(X_1X_2)\} \\
& - E\{(4 \wedge 3)(X_1X_4X_5)(X_2X_3X_5)\} - E\{(3 \wedge 3)(X_1X_2)(X_2X_3X_5)\}
\end{aligned}
$$

$$
\begin{aligned}
& + E\{(7 \wedge 7 \wedge 4)(X_1X_2X_3X_4)(X_1X_2X_4X_5)(X_3X_4)\} + E\{(7 \wedge 7 \wedge 4)(X_1X_2X_3X_4)(X_1X_2X_4X_5)(X_1X_4X_5)\} \\
& + E\{(7 \wedge 7 \wedge 3)(X_1X_2X_3X_4)(X_1X_2X_4X_5)(X_1X_2)\} \\
& + E\{(7 \wedge 7 \wedge 3)(X_1X_2X_3X_4)(X_1X_2X_4X_5)(X_2X_3X_5)\} \\
& + E\{(7 \wedge 4 \wedge 4)(X_1X_2X_3X_4)(X_3X_4)(X_1X_4X_5)\} \\
& + E\{(7 \wedge 4 \wedge 3)(X_1X_2X_3X_4)(X_3X_4)(X_1X_2)\} \\
& + E\{(7 \wedge 4 \wedge 3)(X_1X_2X_3X_4)(X_3X_4)(X_2X_3X_5)\} \\
& + E\{(7 \wedge 4 \wedge 3)(X_1X_2X_3X_4)(X_1X_4X_5)(X_1X_2)\} \\
& + E\{(7 \wedge 4 \wedge 3)(X_1X_2X_3X_4)(X_1X_4X_5)(X_2X_3X_5)\} \\
& + E\{(7 \wedge 3 \wedge 3)(X_1X_2X_3X_4)(X_1X_2)(X_2X_3X_5)\} \\
& + E\{(7 \wedge 4 \wedge 4)(X_1X_2X_4X_5)(X_3X_4)(X_1X_4X_5)\} \\
& + E\{(7 \wedge 4 \wedge 3)(X_1X_2X_4X_5)(X_3X_4)(X_1X_2)\} \\
& + E\{(7 \wedge 4 \wedge 3)(X_1X_2X_4X_5)(X_3X_4)(X_2X_3X_5)\} \\
& + E\{(7 \wedge 4 \wedge 3)(X_1X_2X_4X_5)(X_1X_4X_5)(X_1X_2)\} \\
& + E\{(7 \wedge 4 \wedge 3)(X_1X_2X_4X_5)(X_1X_4X_5)(X_2X_3X_5)\}
\end{aligned}
$$

$$
+ E\{(7 \wedge 3 \wedge 3)(X_1X_2X_4X_5)(X_1X_2)(X_2X_3X_5)\} + E\{(4 \wedge 4 \wedge 3)(X_3X_4)(X_1X_4X_5)(X_1X_2)\}
$$

$$
\begin{aligned}
& + E\{(4 \wedge 4 \wedge 3)(X_3X_4)(X_1X_4X_5)(X_2X_3X_5)\} + E\{(4 \wedge 3 \wedge 3)(X_3X_4)(X_1X_2)(X_2X_3X_5)\} \\
& + E\{(4 \wedge 3 \wedge 3)(X_1X_4X_5)(X_1X_2)(X_2X_3X_5)\} \\
& - E\{(7 \wedge 7 \wedge 4 \wedge 4)(X_1X_2X_3X_4)(X_1X_2X_4X_5)(X_3X_4)(X_1X_4X_5)\} \\
& - E\{(7 \wedge 7 \wedge 4 \wedge 3)(X_1X_2X_3X_4)(X_1X_2X_4X_5)(X_3X_4)(X_1X_2)\} \\
& - E\{(7 \wedge 7 \wedge 4 \wedge 3)(X_1X_2X_3X_4)(X_1X_2X_4X_5)(X_3X_4)(X_2X_3X_5)\} \\
& - E\{(7 \wedge 7 \wedge 4 \wedge 3)(X_1X_2X_3X_4)(X_1X_2X_4X_5)(X_1X_4X_5)(X_1X_2)\}
\end{aligned}
$$

$$- E\{(7 \wedge 7 \wedge 4 \wedge 3)(X_1X_2X_3X_4)(X_1X_2X_4X_5)(X_1X_4X_5)(X_2X_3X_5)\}$$
$$- E\{(7 \wedge 7 \wedge 3 \wedge 3)(X_1X_2X_3X_4)(X_1X_2X_4X_5)(X_1X_2)(X_2X_3X_5)\}$$
$$- E\{(7 \wedge 4 \wedge 4 \wedge 3)(X_1X_2X_3X_4)(X_3X_4)(X_1X_4X_5)(X_1X_2)\}$$
$$- E\{(7 \wedge 4 \wedge 4 \wedge 3)(X_1X_2X_3X_4)(X_3X_4)(X_1X_4X_5)(X_2X_3X_5)\}$$
$$- E\{(7 \wedge 4 \wedge 3 \wedge 3)(X_1X_2X_3X_4)(X_3X_4)(X_1X_2)(X_2X_3X_5)\}$$
$$- E\{(7 \wedge 4 \wedge 3 \wedge 3)(X_1X_2X_3X_4)(X_1X_4X_5)(X_1X_2)(X_2X_3X_5)\}$$
$$- E\{(7 \wedge 4 \wedge 4 \wedge 3)(X_1X_2X_4X_5)(X_3X_4)(X_1X_4X_5)(X_1X_2)\}$$
$$- E\{(7 \wedge 4 \wedge 4 \wedge 3)(X_1X_2X_4X_5)(X_3X_4)(X_1X_4X_5)(X_2X_3X_5)\}$$
$$- E\{(7 \wedge 4 \wedge 3 \wedge 3)(X_1X_2X_4X_5)(X_3X_4)(X_1X_2)(X_2X_3X_5)\}$$
$$- E\{(4 \wedge 4 \wedge 3 \wedge 3)(X_3X_4)(X_1X_4X_5)(X_1X_2)(X_2X_3X_5)\}$$

$$+ E\{(7 \wedge 7 \wedge 4 \wedge 4 \wedge 3)(X_1X_2X_3X_4)(X_1X_2X_4X_5)(X_3X_4)(X_1X_4X_5)(X_1X_2)\} + E\{(7 \wedge 7 \wedge 4 \wedge 4 \wedge 3)(X_1X_2X_3X_4)(X_1X_2X_4X_5)(X_3X_4)(X_1X_4X_5)(X_2X_3X_5)\} + E\{(7 \wedge 7 \wedge 4 \wedge 3 \wedge 3)(X_1X_2X_3X_4)(X_1X_2X_4X_5)(X_3X_4)(X_1X_2)(X_2X_3X_5)\} + E\{(7 \wedge 7 \wedge 4 \wedge 3 \wedge 3)(X_1X_2X_3X_4)(X_1X_2X_4X_5)(X_1X_4X_5)(X_1X_2)(X_2X_3X_5)\} + E\{(7 \wedge 4 \wedge 4 \wedge 3 \wedge 3)(X_1X_2X_3X_4)(X_3X_4)(X_1X_4X_5)(X_1X_2)(X_2X_3X_5)\} + E\{(7 \wedge 4 \wedge 4 \wedge 3 \wedge 3)(X_1X_2X_4X_5)(X_3X_4)(X_1X_4X_5)(X_1X_2)(X_2X_3X_5)\} - E\{(7 \wedge 7 \wedge 4 \wedge 4 \wedge 3 \wedge 3)(X_1X_2X_3X_4)(X_1X_2X_4X_5)(X_3X_4)(X_1X_4X_5)(X_1X_2)(X_2X_3X_5)\} \quad (22)$$

Equation (22) can be reduced to (19) after repeated application of the meet or minimization operator $(a_i \wedge a_j) = \min(a_i, a_j)$ and the idempotency operator $(X_iX_i = X_i)$, and after extremely tedious enumerations. We now apply the disjointing procedure implied by (13) to (21) so as to gradually obtain the final expectation

$$E\{C_{st}(\boldsymbol{X})\} = E\{7(X_1X_2X_3X_4 \vee X_1X_2X_4X_5) \vee 4(X_3X_4 \vee X_1X_4X_5) \vee 3(X_1X_2 \vee X_2X_3X_5)\}$$

$$= E\{7\left(X_1X_2X_3X_4 \vee X_1X_2\overline{\boldsymbol{X}}_{\boldsymbol{3}}X_4X_5\right) \vee 4\left(X_3X_4 \vee X_1\overline{\boldsymbol{X}}_{\boldsymbol{3}}X_4X_5\right) \vee 3\left(X_1X_2 \vee \overline{\boldsymbol{X}}_{\boldsymbol{1}}X_2X_3X_5\right)\}$$

$$= E\{7\left(X_1X_2X_3X_4 \vee X_1X_2\overline{\boldsymbol{X}}_{\boldsymbol{3}}X_4X_5\right) \vee 4\left(X_3X_4(\overline{\boldsymbol{X}}_{\boldsymbol{1}} \vee \boldsymbol{X}_{\boldsymbol{1}}\overline{\boldsymbol{X}}_{\boldsymbol{2}}) \vee X_1\overline{\boldsymbol{X}}_{\boldsymbol{2}}\overline{\boldsymbol{X}}_{\boldsymbol{3}}X_4X_5\right) \vee 3\left(X_1X_2(\overline{\boldsymbol{X}}_{\boldsymbol{4}} \vee \overline{\boldsymbol{X}}_{\boldsymbol{3}}\boldsymbol{X}_{\boldsymbol{4}}\overline{\boldsymbol{X}}_{\boldsymbol{5}}) \vee \overline{\boldsymbol{X}}_{\boldsymbol{1}}X_2X_3\overline{\boldsymbol{X}}_{\boldsymbol{4}}X_5\right)\}$$

$$= E\left\{7\left(X_1X_2X_3X_4 \vee X_1X_2\overline{\boldsymbol{X}}_{\boldsymbol{3}}X_4X_5\right) \vee 4\left(\overline{\boldsymbol{X}}_{\boldsymbol{1}}X_3X_4 \vee \boldsymbol{X}_{\boldsymbol{1}}\overline{\boldsymbol{X}}_{\boldsymbol{2}}X_3X_4\right) \vee 4\left(X_1\overline{\boldsymbol{X}}_{\boldsymbol{2}}\overline{\boldsymbol{X}}_{\boldsymbol{3}}X_4X_5\right) \vee 3\left(X_1X_2\overline{\boldsymbol{X}}_{\boldsymbol{4}} \vee X_1X_2\overline{\boldsymbol{X}}_{\boldsymbol{3}}\boldsymbol{X}_{\boldsymbol{4}}\overline{\boldsymbol{X}}_{\boldsymbol{5}}\right) \vee 3\left(\overline{\boldsymbol{X}}_{\boldsymbol{1}}X_2X_3\overline{\boldsymbol{X}}_{\boldsymbol{4}}X_5\right)\right\} \quad (23)$$

$$= 7p_1p_2p_3p_4 + 7p_1p_2\boldsymbol{q_3}p_4p_5 + 4\boldsymbol{q_1}p_3p_4 + 4p_1\boldsymbol{q_2}p_3p_4 + 4p_1\boldsymbol{q_2}\boldsymbol{q_3}p_4p_5 + 3p_1p_2\boldsymbol{q_4} + 3p_1p_2\boldsymbol{q_3}p_4\boldsymbol{q_5} + 3\boldsymbol{q_1}\boldsymbol{p_2}p_3\boldsymbol{q_4}p_5 \quad (24)$$

The result in (24) can be shown to be equivalent to the earlier one in (18). Equation (23) might also be obtained from (20) if the contributions therein are interpreted as *disjoint-loop coverings* for the *asserted cells only* (See Fig. 5).

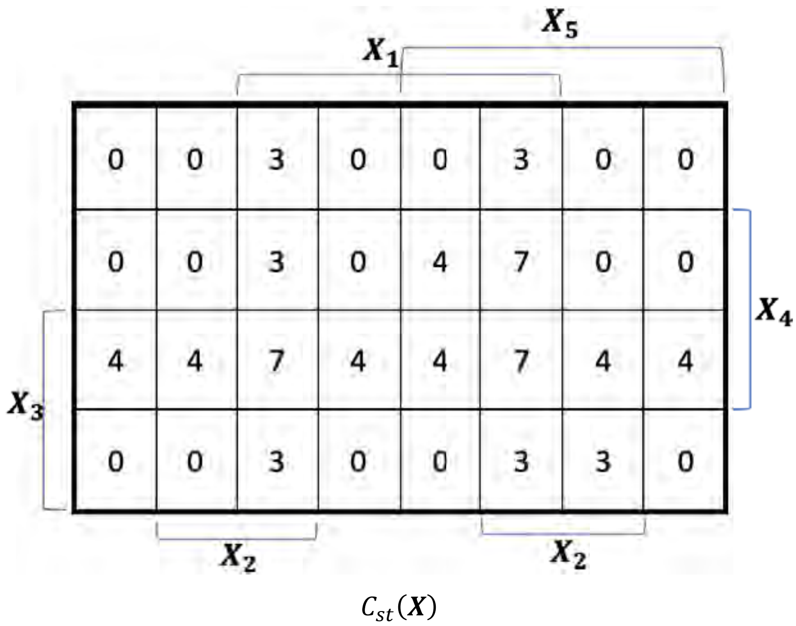


Fig. 2 A multi-valued (value-entered) Karnaugh map for the capacity function of the network in Fig. 1 when it has binary links of capacities $c_1 = 10,\ c_2 = 3, c_3 = 4, c_4 = 4, and\ c_5 = 5$.

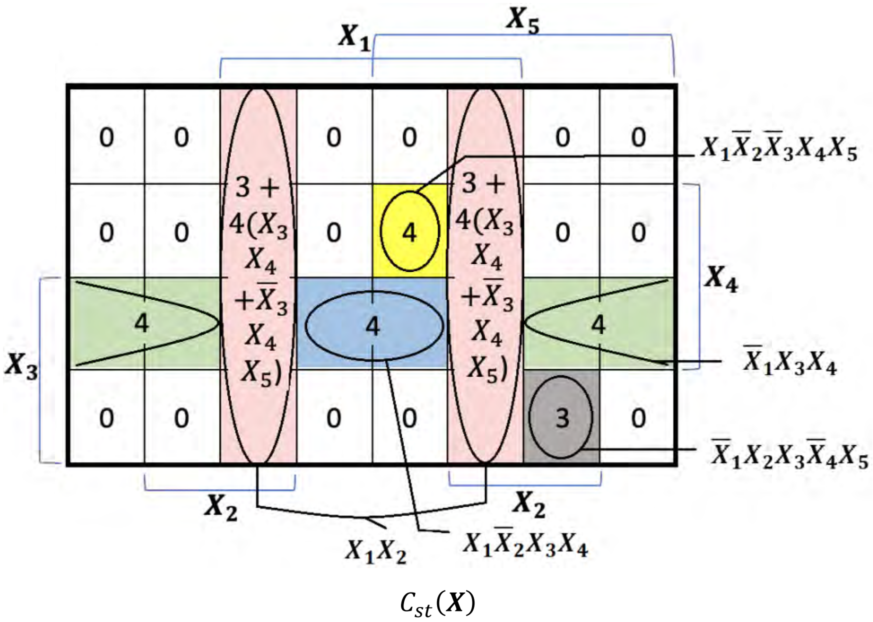
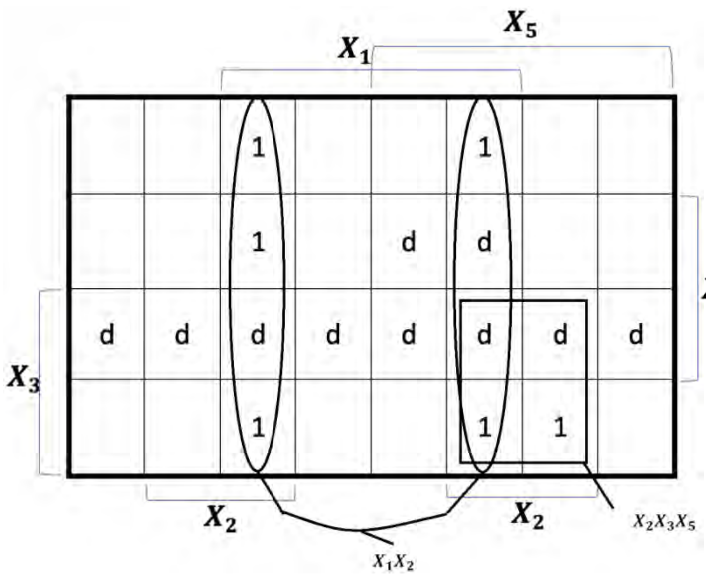
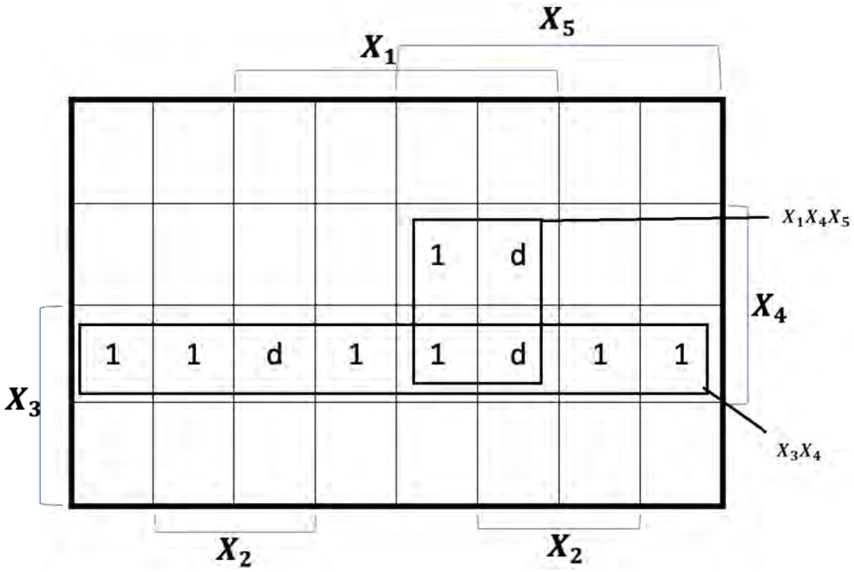


Fig. 3 A multi-valued Karnaugh map for the capacity function of Fig. 2, partitioned according to disjoint sets of paths.



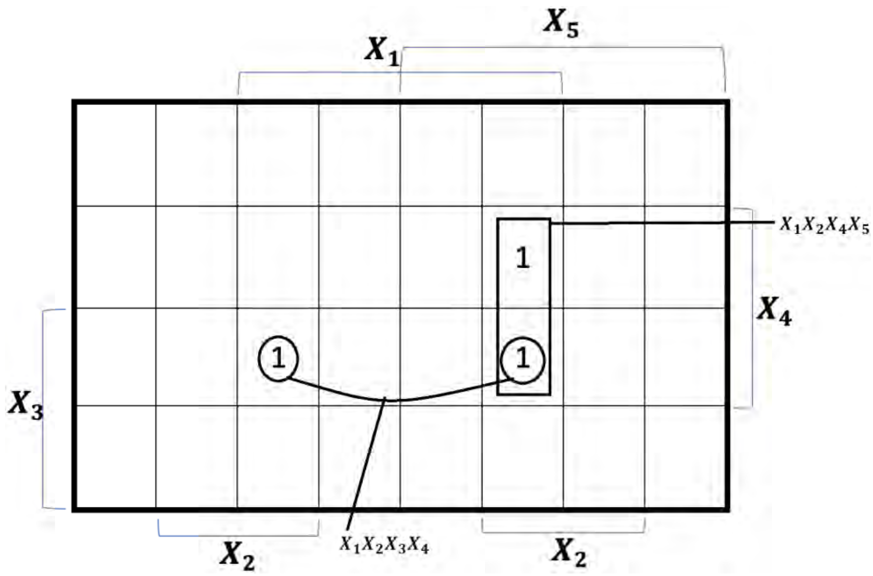
X_5
X_1
1
1
1
d
d
d
d
d
d
d
d
d
d
1
1
1
X_4
X_3
X_2
X_2
$X_2X_3X_5$
X_1X_2

$Co(3)$



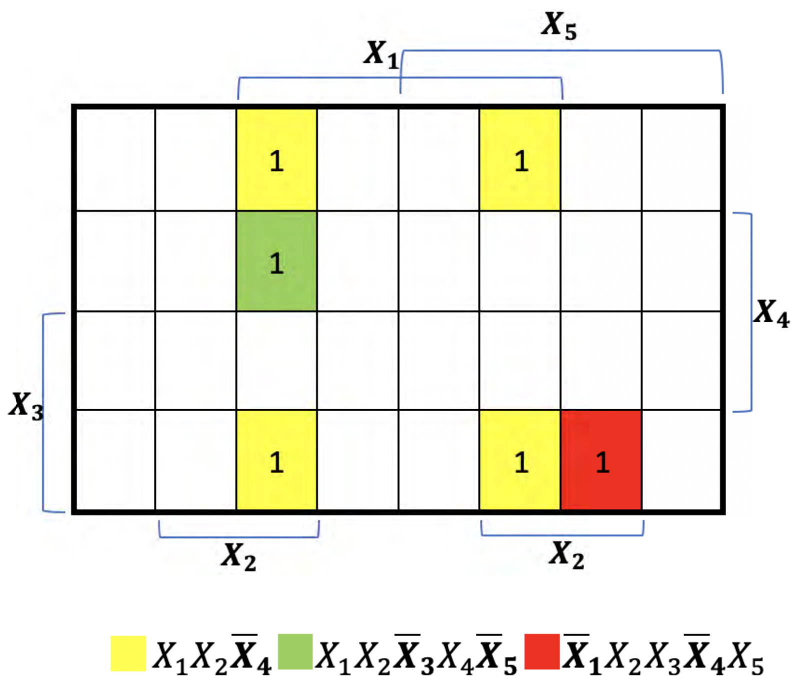
X_5
X_1
1
d
$X_1X_4X_5$
1
1
d
1
1
d
1
1
X_4
X_3
X_3X_4
X_2
X_2

$Co(4)$

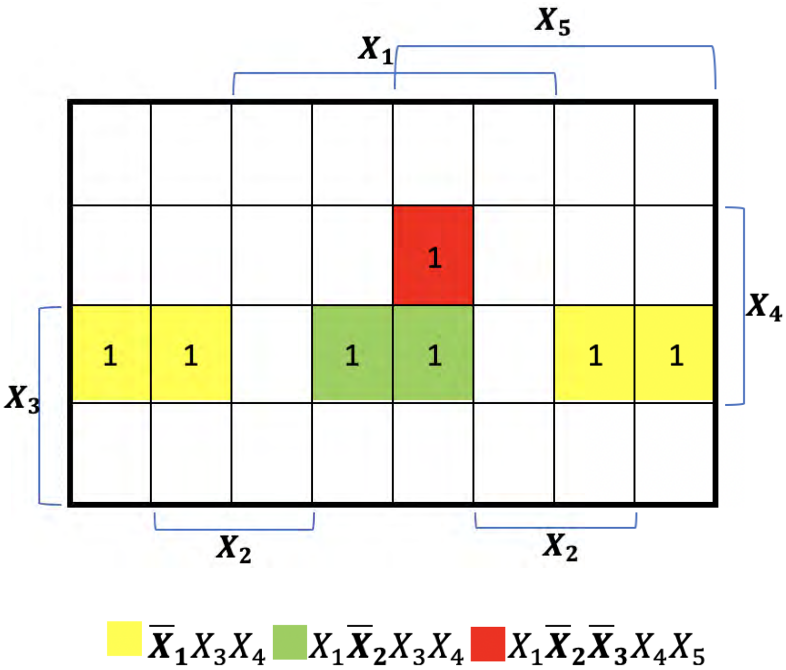


$Co(7)$

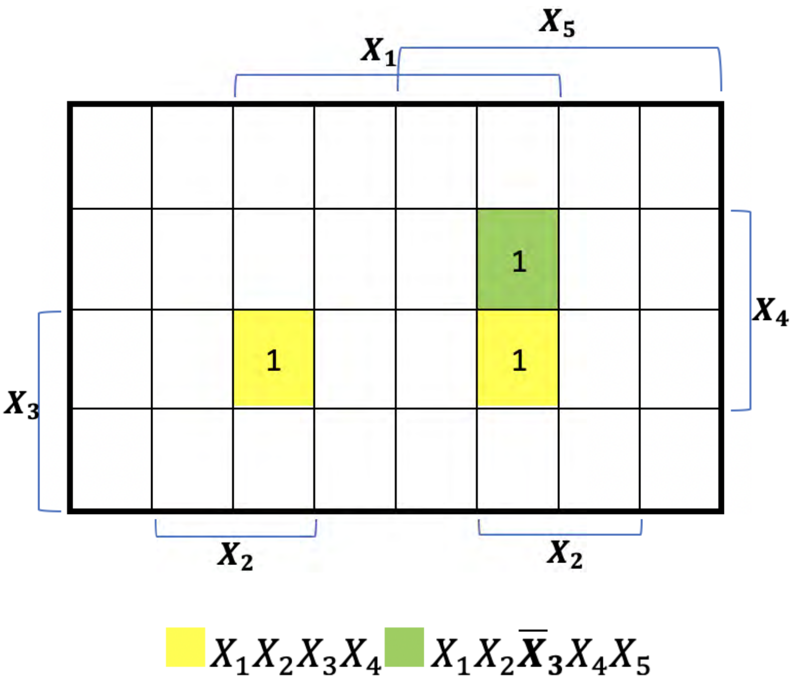
Fig. 4 Contributions of the various non-zero entries in Fig. 2. Note that absolutely eliminable loops (ones with all don't care entries) are not considered.



$Co(3)$



$Co(4)$



$Co(7)$

Fig. 5 The contributions of the various non-zero entries in Fig. 2, now interpreted as disjoint-loop coverings for the asserted cells only (See equation (23)).

7 Example 2

This example applies the map procedure and the disjointing procedure to the network shown in Fig. 6, whose branch capacities are: $\boldsymbol{c} = [\,6\;\;7\;\;4\;\;10\;\;5\;\;3\;\;4\,]^T$

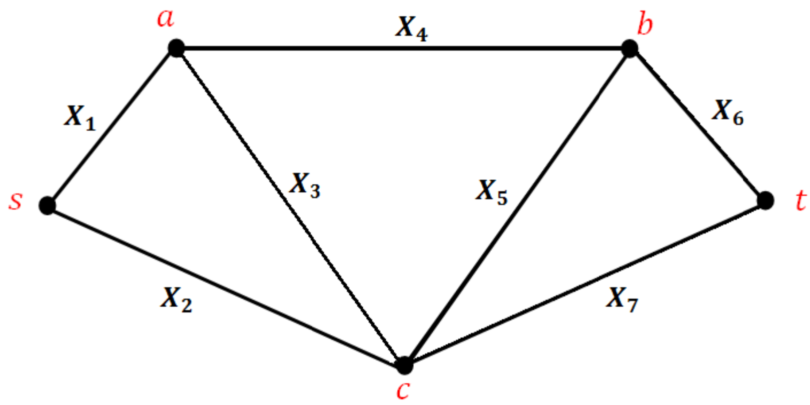


Fig. 6 A 7-branch bridge network of a capacity vector $c = [\,6\;\;7\;\;4\;\;10\;\;5\;\;3\;\;4\,]^T$.

We revisit this network that was considered earlier in Examples 1, 2, and 5 of Rushdi and Alsalami (2021c). We recall that this network deals with the capacity function $C_{st}(\boldsymbol{X})$ for the flow network in Fig. 6, which has binary links of capacities $c_1 = 6, c_2 = 7, c_3 = 4, c_4 = 10, c_5 = 5, c_6 = 3$ *and* $c_7 = 4$ (in appropriate units for the commodity involved). A multi-valued Karnaugh-map representation of $C_{st}(\boldsymbol{X})$ is shown in Fig. 7, which might be shown to conform to the following Max-Flow Min-Cut expression (25).

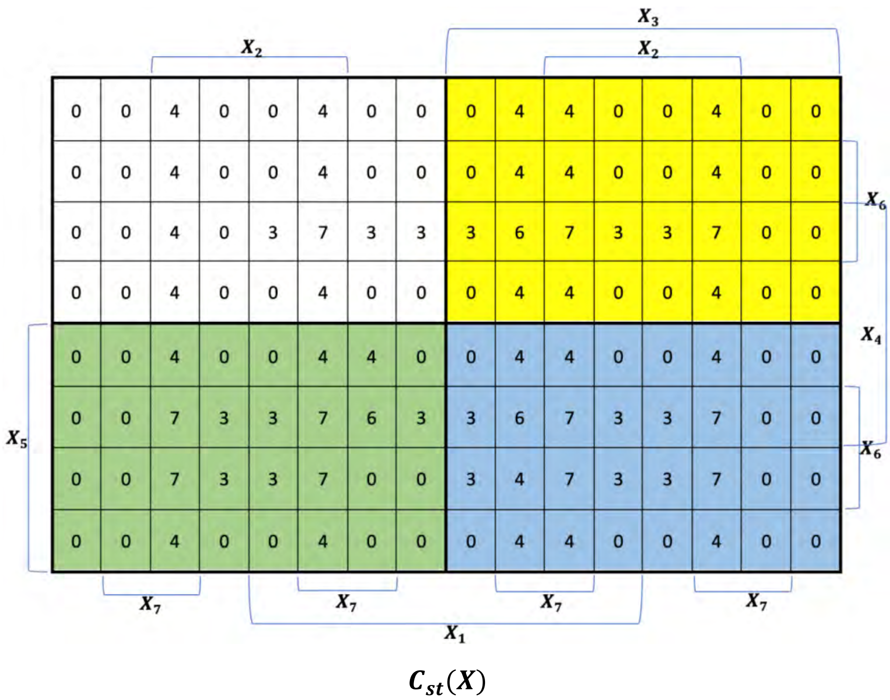


Fig. 7 Multi-valued Karnaugh map representation of the capacity pseudo-Boolean function $C_{st}(X)$ in (25).

$$C_{st}(\boldsymbol{X}) = \min\,(\,c_1X_1 + c_2X_2\,, \quad c_6X_6 + c_7X_7\,, c_4X_4 + c_3X_3 + c_2X_2\,, \quad c_4X_4 + c_5X_5 + c_7X_7\,, c_1X_1 + c_3X_3 + c_5X_5 + c_7X_7\,, \quad c_2X_2 + c_3X_3 + c_5X_5 + c_6X_6\,)$$

$$= \min(\,6X_1 + 7X_2\,,\ 3X_6 + 4X_7\,, 10X_4 + 4X_3 + 7X_2\,,\ 10X_4 + 5X_5 + 4X_7\,, 6X_1 + 4X_3 + 5X_5 + 4X_7\,, 7X_2 + 4X_3 + 5X_5 + 3X_6) \qquad (25)$$

We note that the capacity function in (25) might be expanded through a repeated application of (1) with respect to the following convenient set of non-overlapping (disjoint) paths $\{X_2X_7,\ X_1\overline{X}_2X_3X_7,\ X_2X_5X_6\overline{X}_7,\ X_1\overline{X}_2\overline{X}_3X_4X_6,\ X_1\overline{X}_2X_3X_5X_6\overline{X}_7,\ X_2X_3X_4\overline{X}_5X_6\overline{X}_7,\ X_1\overline{X}_2X_3X_4\overline{X}_5X_6\overline{X}_7,$ $X_1X_2\overline{X}_3X_4\overline{X}_5X_6\overline{X}_7,\ X_1\overline{X}_2\overline{X}_3X_4X_5\overline{X}_6X_7\,\}$ as

$$C(X_1,X_2,X_3,X_4,X_5,X_6,X_7) = C(X_1,1,X_3,X_4,X_5,X_6,1)X_2X_7 + C(1,0,1,X_4,X_5,X_6,1)\,X_1\overline{X}_2X_3X_7 + C(X_1,1,X_3,X_4,1,1,0)X_2X_5X_6\overline{X}_7 + C(1,0,0,1,X_5,1,X_7)\,X_1\overline{X}_2\overline{X}_3X_4X_6 + C(1,0,1,X_4,1,1,0)\,X_1\overline{X}_2X_3X_5X_6\overline{X}_7 + C(X_1,1,1,1,0,1,0)X_2X_3X_4\overline{X}_5X_6\overline{X}_7 + C(1,0,1,1,0,1,0)X_1\overline{X}_2X_3X_4\overline{X}_5X_6\overline{X}_7 + C(1,1,0,1,0,1,0)\,X_1X_2\overline{X}_3X_4\overline{X}_5X_6\overline{X}_7 + C(1,0,0,1,1,0,1)X_1\overline{X}_2\overline{X}_3X_4X_5\overline{X}_6X_7 \qquad (26)$$

In the disjoint arithmetic sum of products (26), arithmetic addition (+) might be replaced with the maximum operation (∨). The pertinent coefficients or subfunctions in (26) are expressed explicitly as restrictions of (25)

$$C(X_1,1,X_3,X_4,X_5,X_6,1) = \min(\,6X_1 + 7\,,\ 3X_6 + 4\,, 10X_4 + 4X_3 + 7\,,\ 10X_4 + 5X_5 + 4\,, 6X_1 + 4X_3 + 5X_5 + 4\,,\ 7 + 4X_3 + 5X_5 + 3X_6) = 4 + 3X_6\left(X_5 + X_4\overline{X}_5\big(X_1 + \overline{X}_1X_3\big)\right) \qquad (27a)$$

$$C(1,0,1,X_4,X_5,X_6,1) = \min(\,6\,,\ 3X_6 + 4\,, 10X_4 + 4\,,\ 10X_4 + 5X_5 + 4\,, 14 + 5X_5\,,\ 4 + 5X_5 + 3X_6) = 4 + 2X_4X_6 \qquad (27b)$$

$$C(X_1,1,X_3,X_4,1,1,0) = \min(\,6X_1 + 7\,,\ 3\,, 10X_4 + 4X_3 + 7\,,\ 10X_4 + 5\,, 6X_1 + 4X_3 + 5\,,\ 15 + 4X_3) = 3 \qquad (27c)$$

$$C(1,0,0,1,X_5,1,X_7) = \min(\,6\,,\ 3 + 4X_7\,, 10\,,\ 10 + 5X_5 + 4X_7\,, 6 + 5X_5 + 4X_7\,,\ 5X_5 + 3\,) = 3 + 3X_5X_7 \qquad (27d)$$

$$C(1,0,1,X_4,1,1,0) = \min(\,6\,,\ 3\,, 10X_4 + 4\,,\ 10X_4 + 5\,, 15\,,\ 12\,) = 3 \qquad (27e)$$

$$C(X_1,1,1,1,0,1,0) = \min(\,6X_1 + 7\,,\ 3\,, 21\,,\ 10\,, 6X_1 + 4\,,\ 14\,) = 3 \qquad (27f)$$

$$C(1,0,1,1,0,1,0) = \min(\,6\,,\ 3\,, 14\,,\ 10\,, 10\,,\ 7\,) = 3 \qquad (27g)$$

$$C(1,1,0,1,0,1,0) = \min(\,13\,,\ 3\,, 17\,,\ 10\,, 6\,, 10\,) = 3 \qquad (27h)$$

$$C(1,0,0,1,1,0,1) = \min(\,6\,, 4\,, 10\,,\ 19\,, 15\,,\ 5\,) = 4 \qquad (27i)$$

The results of (26) and (27) are displayed in the Karnaugh map of Fig. 8, which is equivalent to the one in Fig. 7. The expected value of $C(\boldsymbol{X}) = C_{st}(\boldsymbol{X})$ is therefore obtained based on a one-to-one transformation of the polynomial representation of $C_{st}(\boldsymbol{X})$, namely

$$E\{C_{st}(\boldsymbol{X})\} = \Big(4 + 3p_6\big(p_5 + p_4q_5(p_1 + q_1p_3)\big)\Big)p_2p_7 + (4 + 2p_4p_6)p_1q_2p_3p_7 + 3p_2p_5p_6q_7 + (3 + 3p_5p_7)p_1q_2q_3p_4p_6 + 3p_1q_2p_3p_5p_6q_7 + 3p_2p_3p_4q_5p_6q_7 + 3p_1\,q_2p_3p_4q_5p_6q_7 + 3p_1p_2q_3p_4q_5p_6q_7 + 4p_1q_2q_3p_4p_5q_6p_7 \tag{28}$$

An equivalent (albeit looking different and slightly more compact) expression might be obtained via the Karnaugh map procedure in Rushdi (1988) as

$$E\{C_{st}(\boldsymbol{X})\} = 4p_7\big(p_2 + p_1q_2(p_3(q_6 + q_4q_5p_6) + q_3p_4p_5q_6)\big) + 3p_6\Big(p_2(p_5 + p_3p_4q_5) + p_4q_5\big(p_1(q_3 + q_2)\big) + p_1q_2\big(p_3p_5 + p_4p_7(p_5 + p_3q_5)\big)\Big) + p_1q_2p_3q_4p_5p_6p_7. \tag{29}$$

Equation (29) might be rewritten as an all-p formula by substituting each $q_i = 1 - p_i$ to obtain

$$E\{C_{st}(\boldsymbol{X})\} = 4p_7\Big(p_2 + p_1(1 - p_2)\Big(p_3\big((1 - p_6) + (1 - p_5 - p_4 + p_4p_5)p_6\big) + p_4p_5(1 - p_3 - p_6 + p_3p_6)\Big)\Big) + 3p_6\Big(p_2\big(p_5 + p_3p_4(1 - p_5)\big) + p_4(1 - p_5)\Big(p_1\big((1 - p_3) + (1 - p_2)\big)\Big) + p_1(1 - p_2)\Big(p_3p_5 + p_4p_7\big(p_5 + p_3(1 - p_5)\big)\Big)\Big) + p_1p_3p_5p_6p_7\big(1 - p_2 - p_4(1 + p_2)\big) \tag{30}$$

We now construct a DNA for $C_{st}(\boldsymbol{X})$ by adapting procedures of the variable-entered Karnaugh map in (Rushdi, 1985a, 1987a, 2001; Rushdi and Al-Yahya, 2000, 2001b) to the value-entered Karnaugh map in Fig. 7. The function $C_{st}(\boldsymbol{X})$ is the weighted disjunction of the values v entered in the map, where each entered value v is weighted by its 'contribution' $Co(v)$, namely

$$C_{st}(\boldsymbol{X}) = \forall_{\mathrm{v}} vCo(v) \;=\; 7\,Co(7) \vee\; 6\,Co(6) \vee 4\,Co(4) \vee 3\,Co(3) \tag{31}$$

The contribution of v is a function of $\boldsymbol{X}$represented by a standard (conventional) Karnaugh map derived fromthe original or parent map for $C_{st}(\boldsymbol{X})$ (the one in Fig. 7). In the map for $Co(v)$, a cell is entered with 1 if its entry in the parent map is v, entered with a don't care (d) if its entry in the parent map is greater than v, and entered with 0 otherwise. Fig. 9 displays the maps for $Co(3), Co(4), Co(6)$ and $Co(7)$ to be derived from Fig. 7. The coverage in these maps seeks a minimal sum rather than a complete sum (Blake canonical form). In fact, we ignore all-d loops (called absolutely eliminable loops) in the coverage of $Co(3)\ and\ Co(6)$. The final minimal DNF for $C_{st}(\boldsymbol{X})$ is given by

$$C_{st}(\boldsymbol{X}) = 7(X_2X_5X_6X_7 \vee X_1X_2X_4X_6X_7 \vee X_2X_3X_4X_6X_7) \vee 6(X_1X_4X_5X_6X_7 \vee X_1X_3X_4X_6X_7) \vee 4(X_2X_7 \vee X_1X_3X_7 \vee X_1X_4X_5X_7) \vee 3(X_1X_4X_6 \vee X_2X_5X_6 \vee X_2X_3X_4X_6 \vee X_1X_3X_5X_6) \tag{32}$$

We reiterate that the operator '∨' now represents the 'max' operator over the real field R.Obtaining the expectation of $C_{st}(\boldsymbol{X})$ is not easy, as it might involve the use of the notorious Inclusion-Exclusion (IE) Principle, which now involves $(2^{12}-1) = 4095$ terms. Due to space limitations, we refrain from writing the IE formula in full and write just a few terms, namely

$$E\{C_{st}(\boldsymbol{X})\}$$
$$= E\{7X_2X_5X_6X_7\} + E\{7X_1X_2X_4X_6X_7\} + E\{7X_2X_3X_4X_6X_7\} + E\{6X_1X_4X_5X_6X_7\} + E\{6X_1X_3X_4X_6X_7\}$$
$$+ E\{4X_2X_7\} + E\{4X_1X_3X_7\} + E\{4X_1X_4X_5X_7\} + E\{3X_1X_4X_6\} + E\{3X_2X_5X_6\} + E\{3X_2X_3X_4X_6\}$$
$$+ E\{3X_1X_3X_5X_6\} - E\{(7 \wedge 7)(X_2X_5X_6X_7)(X_1X_2X_4X_6X_7)\} \ldots - \cdots$$
$$- E\{(7 \wedge 7 \wedge 7 \wedge 6 \wedge 6 \wedge 4 \wedge 4 \wedge 4 \wedge 3 \wedge 3 \wedge 3$$
$$\wedge 3)(X_2X_5X_6X_7)(X_1X_2X_4X_6X_7)(X_2X_3X_4X_6X_7)(X_1X_4X_5X_6X_7)(X_1X_3X_4X_6X_7)(X_2X_7)(X_1X_3X_7)(X_1X_4X_5X_7)$$
$$(X_1X_4X_6)(X_2X_5X_6)(X_2X_3X_4X_6)(X_1X_3X_5X_6)\} \quad (33)$$

Equation (33) can be reduced to (30) after repeated application of the minimization operator $(a_i \wedge a_j) =$ min (a_i, a_j) and the idempotency operator $(X_iX_i = X_i)$, and after tedious enumerations. These are some of the essential troubles with IE manipulations, which consume a lengthy time in making extensive constructions, only to mainly destroy most of them later on. We now apply the disjointing procedure proposed in Section 4 to gradually obtain the following equivalent expressions, which tend to a PRE-expectation in the last step

$$E\{C_{st}(\boldsymbol{X})\} = \mathrm{E}\{7(X_2X_5X_6X_7 \vee X_1X_2X_4X_6X_7 \vee X_2X_3X_4X_6X_7) \vee 6(X_1X_4X_5X_6X_7 \vee X_1X_3X_4X_6X_7) \vee$$
$$4(X_2X_7 \vee X_1X_3X_7 \vee X_1X_4X_5X_7) \vee 3(X_1X_4X_6 \vee X_2X_5X_6 \vee X_2X_3X_4X_6 \vee X_1X_3X_5X_6)\}$$

$$= E\{7(X_2X_5X_6X_7 \vee X_1X_2X_4\boldsymbol{\overline{X}_5}X_6X_7 \vee \boldsymbol{\overline{X}_1}X_2X_3X_4\boldsymbol{\overline{X}_5}X_6X_7) \vee 6(X_1X_4X_5X_6X_7 \vee X_1X_3X_4X_6X_7)$$
$$\vee 4(X_2X_7 \vee X_1X_3X_7 \vee X_1X_4X_5X_7) \vee 3(X_1X_4X_6 \vee X_2X_5X_6 \vee X_2X_3X_4X_6 \vee X_1X_3X_5X_6)\}$$

$$= E\{7(X_2X_5X_6X_7 \vee X_1X_2X_4\boldsymbol{\overline{X}_5}X_6X_7 \vee \boldsymbol{\overline{X}_1}X_2X_3X_4\boldsymbol{\overline{X}_5}X_6X_7) \vee 6\big(X_1X_4X_5X_6X_7 \vee X_1X_3X_4\boldsymbol{\overline{X}_5}X_6X_7\big)$$
$$\vee 4(X_2X_7 \vee X_1X_3X_7 \vee X_1X_4X_5X_7) \vee 3(X_1X_4X_6 \vee X_2X_5X_6 \vee X_2X_3X_4X_6 \vee X_1X_3X_5X_6)\}$$

$$= E\{7(X_2X_5X_6X_7 \vee X_1X_2X_4\boldsymbol{\overline{X}_5}X_6X_7 \vee \boldsymbol{\overline{X}_1}X_2X_3X_4\boldsymbol{\overline{X}_5}X_6X_7) \vee 6\big(X_1\boldsymbol{\overline{X}_2}X_4X_5X_6X_7 \vee X_1\boldsymbol{\overline{X}_2}X_3X_4\boldsymbol{\overline{X}_5}X_6X_7\big)$$
$$\vee 4(X_2X_7 \vee X_1X_3X_7 \vee X_1X_4X_5X_7) \vee 3(X_1X_4X_6 \vee X_2X_5X_6 \vee X_2X_3X_4X_6 \vee X_1X_3X_5X_6)\}$$

$$= E\{7(X_2X_5X_6X_7 \vee X_1X_2X_4\boldsymbol{\overline{X}_5}X_6X_7 \vee \boldsymbol{\overline{X}_1}X_2X_3X_4\boldsymbol{\overline{X}_5}X_6X_7) \vee 6\big(X_1\boldsymbol{\overline{X}_2}X_4X_5X_6X_7 \vee X_1\boldsymbol{\overline{X}_2}X_3X_4\boldsymbol{\overline{X}_5}X_6X_7\big)$$
$$\vee 4\big(X_2X_7(\boldsymbol{\overline{X}_6} \vee \boldsymbol{\overline{X}_5X_6}) \vee X_1\boldsymbol{\overline{X}_2}X_3X_7 \vee X_1\boldsymbol{\overline{X}_2}\boldsymbol{\overline{X}_3}X_4X_5X_7\big)$$
$$\vee 3(X_1X_4X_6 \vee X_2X_5X_6 \vee X_2X_3X_4X_6 \vee X_1X_3X_5X_6)\}$$

$$= E\{7(X_2X_5X_6X_7 \vee X_1X_2X_4\boldsymbol{\overline{X}_5}X_6X_7 \vee \boldsymbol{\overline{X}_1}X_2X_3X_4\boldsymbol{\overline{X}_5}X_6X_7) \vee 6\big(X_1\boldsymbol{\overline{X}_2}X_4X_5X_6X_7 \vee X_1\boldsymbol{\overline{X}_2}X_3X_4\boldsymbol{\overline{X}_5}X_6X_7\big)$$
$$\vee 4\big(X_2X_7(\boldsymbol{\overline{X}_6} \vee \boldsymbol{\overline{X}_5X_6}(\boldsymbol{\overline{X}_4} \vee \boldsymbol{\overline{X}_1\overline{X}_3X_4})) \vee X_1\boldsymbol{\overline{X}_2}X_3X_7 \vee X_1\boldsymbol{\overline{X}_2}\boldsymbol{\overline{X}_3}X_4X_5\boldsymbol{\overline{X}_6}X_7\big)$$
$$\vee 3(X_1X_4X_6 \vee X_2X_5X_6 \vee X_2X_3X_4X_6 \vee X_1X_3X_5X_6)\}$$

$$= E\{7(X_2X_5X_6X_7 \vee X_1X_2X_4\overline{\boldsymbol{X}}_{\mathbf{5}}X_6X_7 \vee \overline{\boldsymbol{X}}_{\mathbf{1}}X_2X_3X_4\overline{\boldsymbol{X}}_{\mathbf{5}}X_6X_7) \vee 6(X_1\overline{\boldsymbol{X}}_{\mathbf{2}}X_4X_5X_6X_7 \vee X_1\overline{\boldsymbol{X}}_{\mathbf{2}}X_3X_4\overline{\boldsymbol{X}}_{\mathbf{5}}X_6X_7)$$
$$\vee 4(X_2X_7(\overline{\boldsymbol{X}}_{\mathbf{6}} \vee \overline{\boldsymbol{X}}_{\mathbf{5}}\boldsymbol{X}_{\mathbf{6}}(\overline{\boldsymbol{X}}_{\mathbf{4}} \vee \overline{\boldsymbol{X}}_{\mathbf{1}}\overline{\boldsymbol{X}}_{\mathbf{3}}\boldsymbol{X}_{\mathbf{4}})) \vee X_1\overline{\boldsymbol{X}}_{\mathbf{2}}X_3X_7(\overline{\boldsymbol{X}}_{\mathbf{6}} \vee \overline{\boldsymbol{X}}_{\mathbf{4}}\boldsymbol{X}_{\mathbf{6}} \vee \boldsymbol{X}_{\mathbf{4}}\overline{\boldsymbol{X}}_{\mathbf{5}}\boldsymbol{X}_{\mathbf{6}}(\mathbf{0}))$$
$$\vee X_1\overline{\boldsymbol{X}}_{\mathbf{2}}\overline{\boldsymbol{X}}_{\mathbf{3}}X_4X_5\overline{\boldsymbol{X}}_{\mathbf{6}}X_7)$$
$$\vee 3(X_1X_4X_6 \vee X_2X_5X_6(\overline{\boldsymbol{X}}_{\mathbf{4}} \vee \overline{\boldsymbol{X}}_{\mathbf{1}}\boldsymbol{X}_{\mathbf{4}}) \vee \overline{\boldsymbol{X}}_{\mathbf{1}}X_2X_3X_4\overline{\boldsymbol{X}}_{\mathbf{5}}X_6 \vee X_1\overline{\boldsymbol{X}}_{\mathbf{2}}X_3\overline{\boldsymbol{X}}_{\mathbf{4}}X_5X_6)\}$$

$$= E\{7(X_2X_5X_6X_7 \vee X_1X_2X_4\overline{\boldsymbol{X}}_{\mathbf{5}}X_6X_7 \vee \overline{\boldsymbol{X}}_{\mathbf{1}}X_2X_3X_4\overline{\boldsymbol{X}}_{\mathbf{5}}X_6X_7) \vee 6(X_1\overline{\boldsymbol{X}}_{\mathbf{2}}X_4X_5X_6X_7 \vee X_1\overline{\boldsymbol{X}}_{\mathbf{2}}X_3X_4\overline{\boldsymbol{X}}_{\mathbf{5}}X_6X_7)$$
$$\vee 4(X_2X_7(\overline{\boldsymbol{X}}_{\mathbf{6}} \vee \overline{\boldsymbol{X}}_{\mathbf{5}}\boldsymbol{X}_{\mathbf{6}}(\overline{\boldsymbol{X}}_{\mathbf{4}} \vee \overline{\boldsymbol{X}}_{\mathbf{1}}\overline{\boldsymbol{X}}_{\mathbf{3}}\boldsymbol{X}_{\mathbf{4}})) \vee X_1\overline{\boldsymbol{X}}_{\mathbf{2}}X_3X_7(\overline{\boldsymbol{X}}_{\mathbf{6}} \vee \overline{\boldsymbol{X}}_{\mathbf{4}}\boldsymbol{X}_{\mathbf{6}}) \vee X_1\overline{\boldsymbol{X}}_{\mathbf{2}}\overline{\boldsymbol{X}}_{\mathbf{3}}X_4X_5\overline{\boldsymbol{X}}_{\mathbf{6}}X_7)$$
$$\vee 3(X_1X_4X_6(\overline{\boldsymbol{X}}_{\mathbf{7}} \vee \overline{\boldsymbol{X}}_{\mathbf{2}}\overline{\boldsymbol{X}}_{\mathbf{3}}\overline{\boldsymbol{X}}_{\mathbf{5}}\boldsymbol{X}_{\mathbf{7}} \vee \boldsymbol{X}_{\mathbf{2}}\overline{\boldsymbol{X}}_{\mathbf{5}}\boldsymbol{X}_{\mathbf{7}}(\mathbf{0})) \vee X_2X_5X_6(\overline{\boldsymbol{X}}_{\mathbf{4}}\overline{\boldsymbol{X}}_{\mathbf{7}} \vee \overline{\boldsymbol{X}}_{\mathbf{1}}\boldsymbol{X}_{\mathbf{4}}\overline{\boldsymbol{X}}_{\mathbf{7}})$$
$$\vee \overline{\boldsymbol{X}}_{\mathbf{1}}X_2X_3X_4\overline{\boldsymbol{X}}_{\mathbf{5}}X_6\overline{\boldsymbol{X}}_{\mathbf{7}} \vee X_1\overline{\boldsymbol{X}}_{\mathbf{2}}X_3\overline{\boldsymbol{X}}_{\mathbf{4}}X_5X_6\overline{\boldsymbol{X}}_{\mathbf{7}})\}$$

$$= E\{7(X_2X_5X_6X_7 \vee X_1X_2X_4\overline{\boldsymbol{X}}_{\mathbf{5}}X_6X_7 \vee \overline{\boldsymbol{X}}_{\mathbf{1}}X_2X_3X_4\overline{\boldsymbol{X}}_{\mathbf{5}}X_6X_7) \vee 6(X_1\overline{\boldsymbol{X}}_{\mathbf{2}}X_4X_5X_6X_7 \vee X_1\overline{\boldsymbol{X}}_{\mathbf{2}}X_3X_4\overline{\boldsymbol{X}}_{\mathbf{5}}X_6X_7) \vee 4(X_2X_7(\overline{\boldsymbol{X}}_{\mathbf{6}} \vee \overline{\boldsymbol{X}}_{\mathbf{5}}\boldsymbol{X}_{\mathbf{6}}(\overline{\boldsymbol{X}}_{\mathbf{4}} \vee \overline{\boldsymbol{X}}_{\mathbf{1}}\overline{\boldsymbol{X}}_{\mathbf{3}}\boldsymbol{X}_{\mathbf{4}})) \vee X_1\overline{\boldsymbol{X}}_{\mathbf{2}}X_3X_7(\overline{\boldsymbol{X}}_{\mathbf{6}} \vee \overline{\boldsymbol{X}}_{\mathbf{4}}\boldsymbol{X}_{\mathbf{6}}) \vee X_1\overline{\boldsymbol{X}}_{\mathbf{2}}\overline{\boldsymbol{X}}_{\mathbf{3}}X_4X_5\overline{\boldsymbol{X}}_{\mathbf{6}}X_7) \vee 3(X_1X_4X_6(\overline{\boldsymbol{X}}_{\mathbf{7}} \vee \overline{\boldsymbol{X}}_{\mathbf{2}}\overline{\boldsymbol{X}}_{\mathbf{3}}\overline{\boldsymbol{X}}_{\mathbf{5}}\boldsymbol{X}_{\mathbf{7}}) \vee X_2X_5X_6(\overline{\boldsymbol{X}}_{\mathbf{4}}\overline{\boldsymbol{X}}_{\mathbf{7}} \vee \overline{\boldsymbol{X}}_{\mathbf{1}}\boldsymbol{X}_{\mathbf{4}}\overline{\boldsymbol{X}}_{\mathbf{7}}) \vee \overline{\boldsymbol{X}}_{\mathbf{1}}X_2X_3X_4\overline{\boldsymbol{X}}_{\mathbf{5}}X_6\overline{\boldsymbol{X}}_{\mathbf{7}} \vee X_1\overline{\boldsymbol{X}}_{\mathbf{2}}X_3\overline{\boldsymbol{X}}_{\mathbf{4}}X_5X_6\overline{\boldsymbol{X}}_{\mathbf{7}})\}. \quad (34)$$

Finally, we obtain a compact probability-domain expression by a one-to-one transformation of (34)

$$E\{C_{st}(\boldsymbol{X})\} = 7p_2p_5p_6p_7 + 7p_1p_2p_4\boldsymbol{q}_{\mathbf{5}}p_6p_7 + 7\boldsymbol{q}_{\mathbf{1}}p_2p_3p_4\boldsymbol{q}_{\mathbf{5}}p_6p_7 + 6p_1\boldsymbol{q}_{\mathbf{2}}p_4p_5p_6p_7 + 6p_1\boldsymbol{q}_{\mathbf{2}}p_3p_4\boldsymbol{q}_{\mathbf{5}}p_6p_7 + 4p_2\boldsymbol{q}_{\mathbf{6}}p_7 + 4p_2\boldsymbol{q}_{\mathbf{4}}\boldsymbol{q}_{\mathbf{5}}p_6p_7 + 4\boldsymbol{q}_{\mathbf{1}}p_2\boldsymbol{q}_{\mathbf{3}}p_4\boldsymbol{q}_{\mathbf{5}}p_6p_7 + 4p_1\boldsymbol{q}_{\mathbf{2}}p_3\boldsymbol{q}_{\mathbf{6}}p_7 + 4p_1\boldsymbol{q}_{\mathbf{2}}p_3\boldsymbol{q}_{\mathbf{4}}p_6p_7 + 4p_1\boldsymbol{q}_{\mathbf{2}}\boldsymbol{q}_{\mathbf{3}}p_4p_5\boldsymbol{q}_{\mathbf{6}}p_7 + 3p_1p_4p_6\boldsymbol{q}_{\mathbf{7}} + 3p_1\boldsymbol{q}_{\mathbf{2}}\boldsymbol{q}_{\mathbf{3}}p_4\boldsymbol{q}_{\mathbf{5}}p_6p_7 + 3p_2\boldsymbol{q}_{\mathbf{4}}p_5p_6\boldsymbol{q}_{\mathbf{7}} + 3\boldsymbol{q}_{\mathbf{1}}p_2p_4p_5p_6\boldsymbol{q}_{\mathbf{7}} + 3\boldsymbol{q}_{\mathbf{1}}p_2p_3p_4\boldsymbol{q}_{\mathbf{5}}p_6\boldsymbol{q}_{\mathbf{7}} + 3p_1\boldsymbol{q}_{\mathbf{2}}p_3\boldsymbol{q}_{\mathbf{4}}p_5p_6\boldsymbol{q}_{\mathbf{7}} \quad (35)$$

The result in (35) can be shown to be equivalent to the earlier one in (29). Figure 10 shows that (35) corresponds to a disjoint-loop covering for the map in Fig. 7. In fact, (35) might be obtained from (31) if the contributions therein are interpreted as a *disjoint-loop covering* for the *asserted cells only* (See Fig. 11).

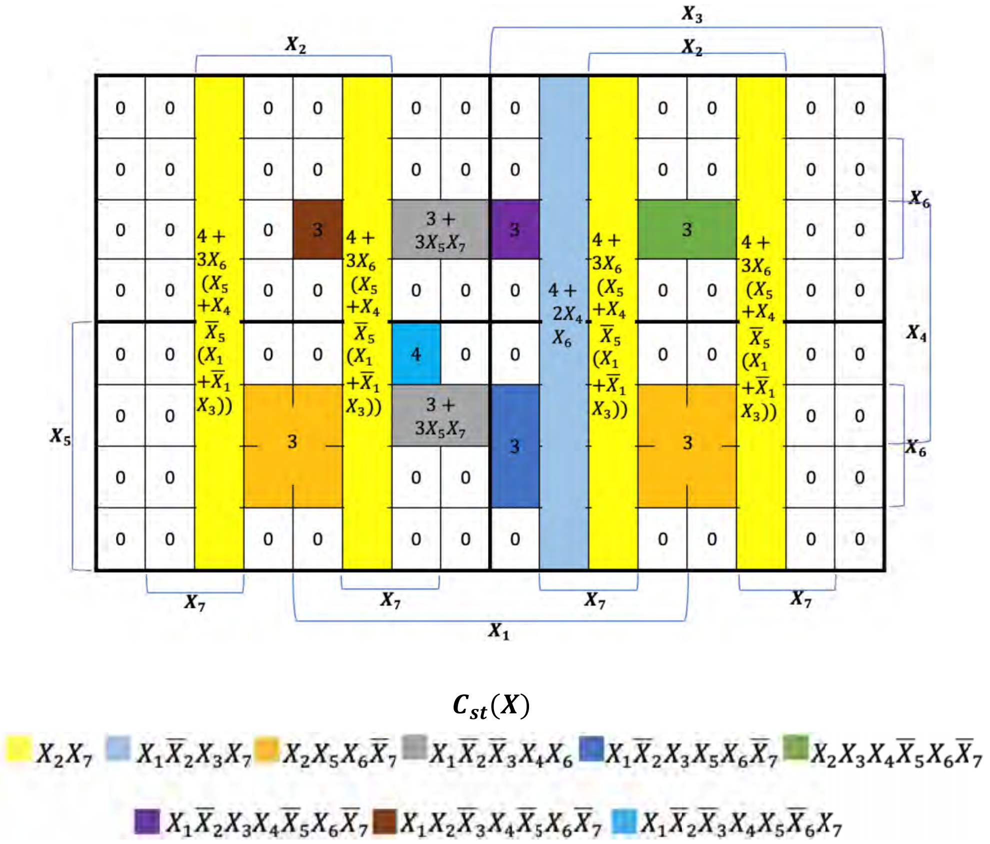
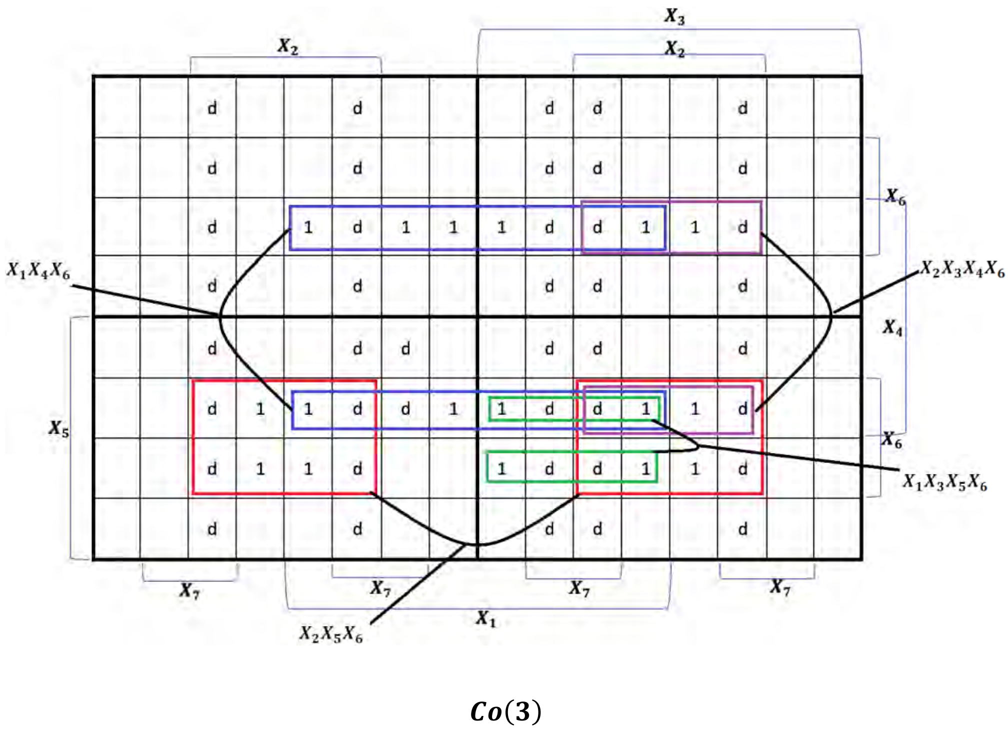
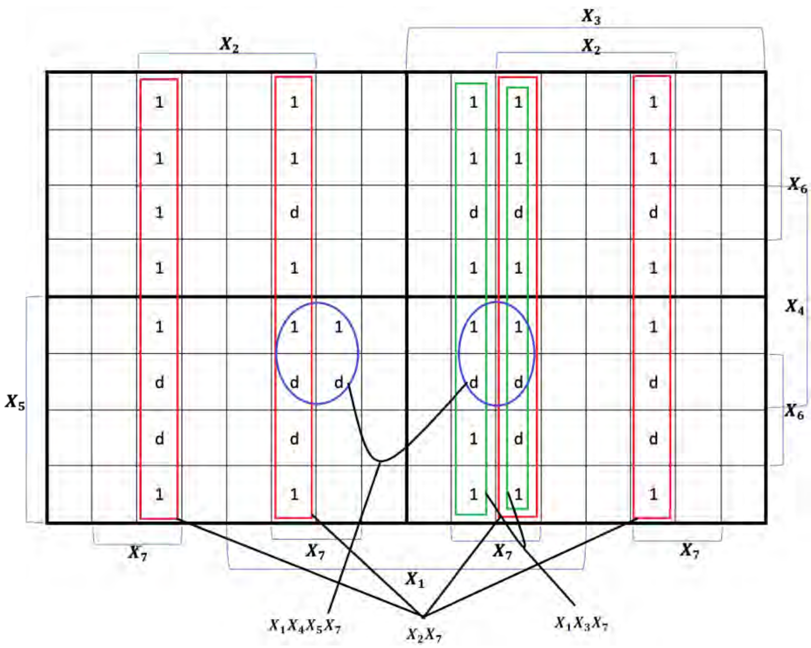


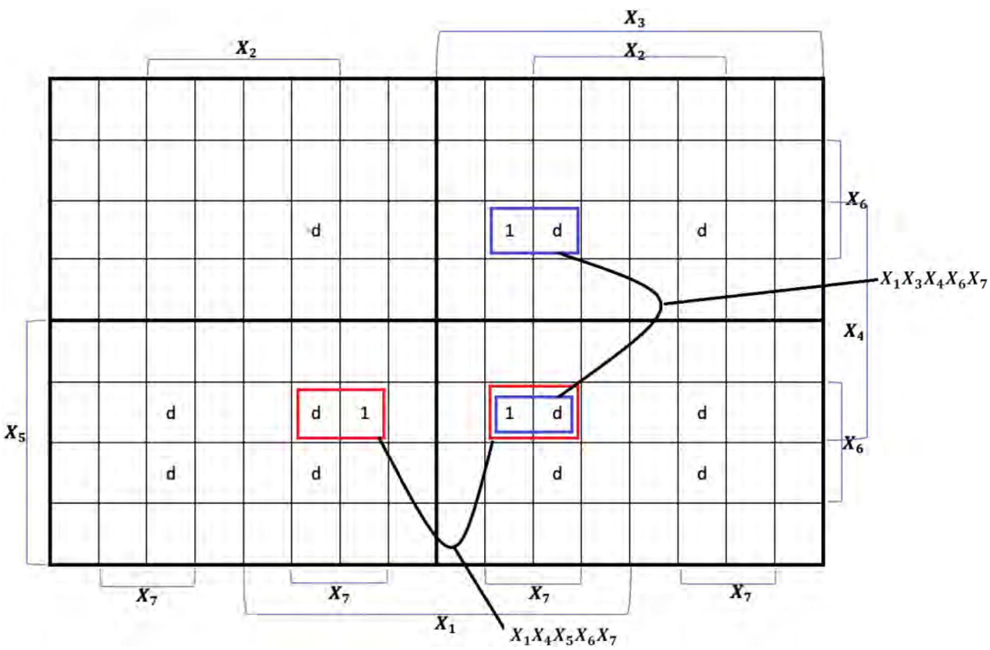
Fig. 8 A Karnaugh map for the capacity function of Fig. 7, partitioned according to the given set of disjoint paths shown.





X_3
X_2
X_2
X_6
X_4
X_6
X_5
X_7
X_7
X_7
X_7
X_1
$X_1X_4X_5X_7$
X_2X_7
$X_1X_3X_7$

$Co(4)$



X_3
X_2
X_2
X_6
$X_1X_3X_4X_6X_7$
X_4
X_6
X_5
X_7
X_7
X_7
X_7
X_1
$X_1X_4X_5X_6X_7$

$Co(6)$

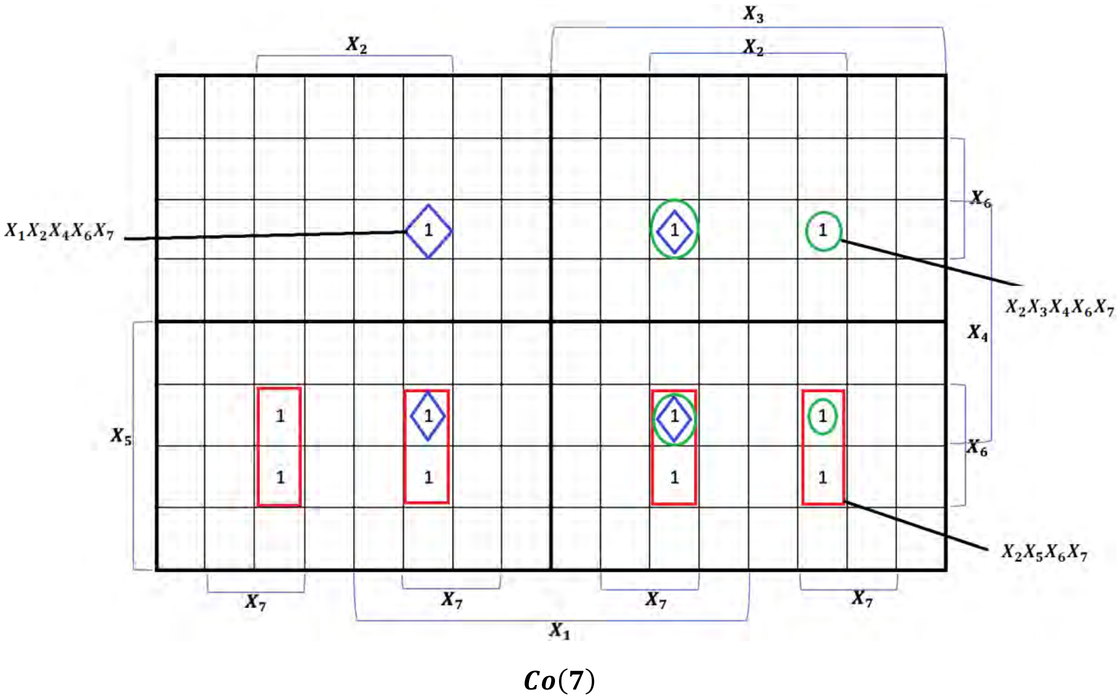


Fig. 9 Contributions of the various non-zero entries in Fig. 7, ignoring absolutely-eliminable loops.

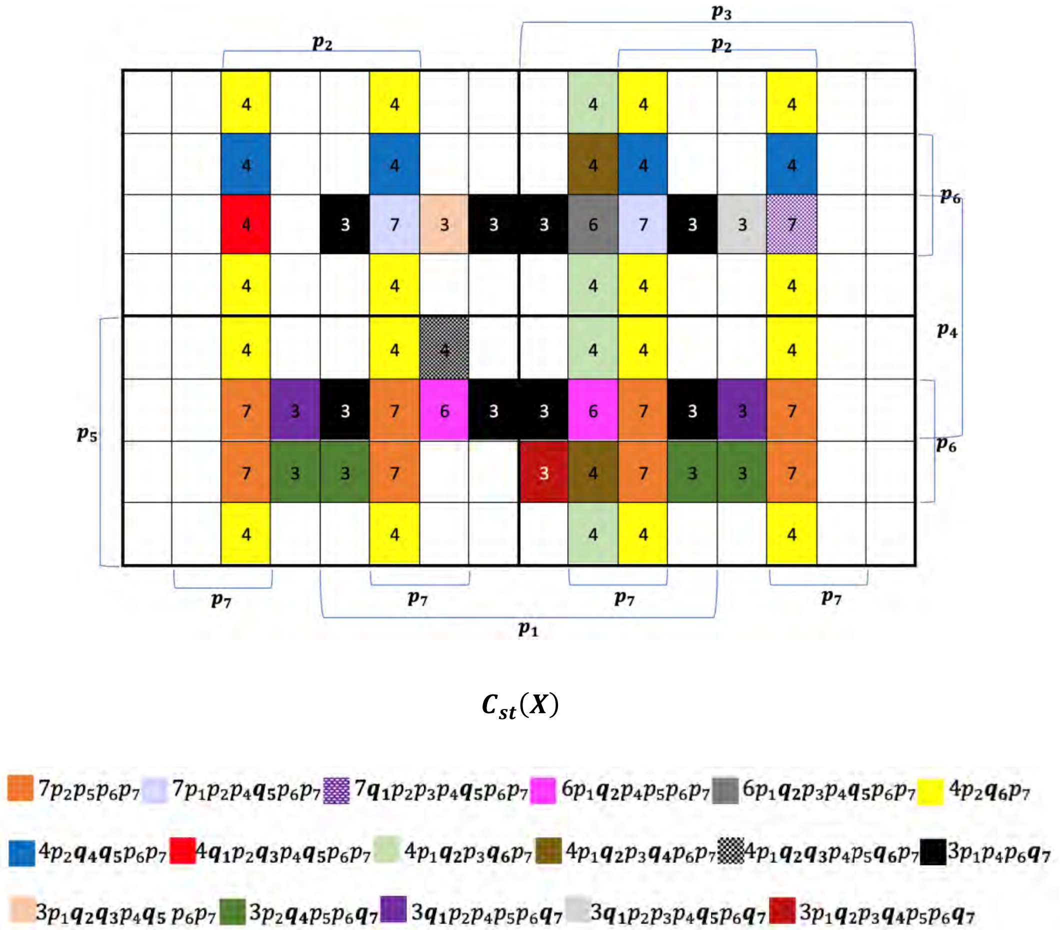
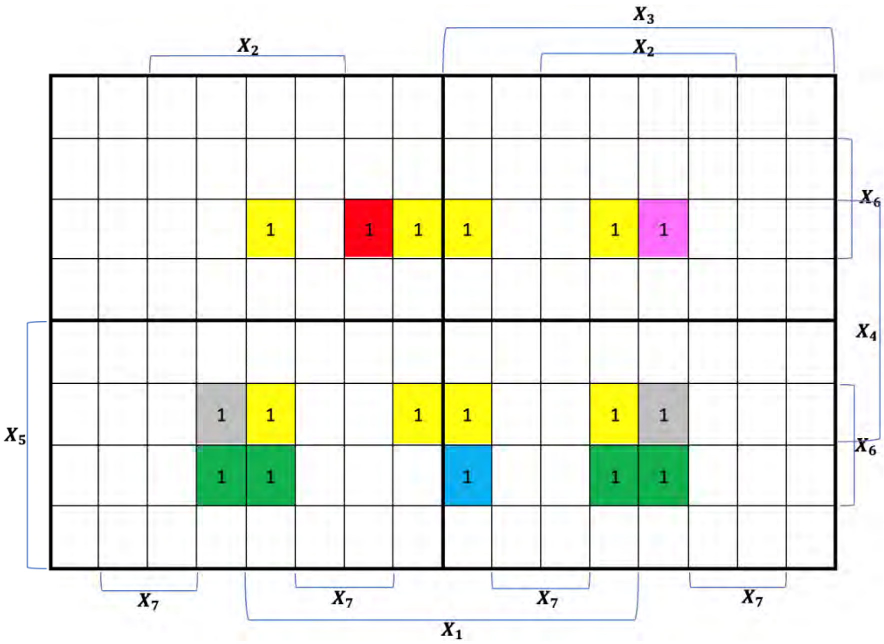
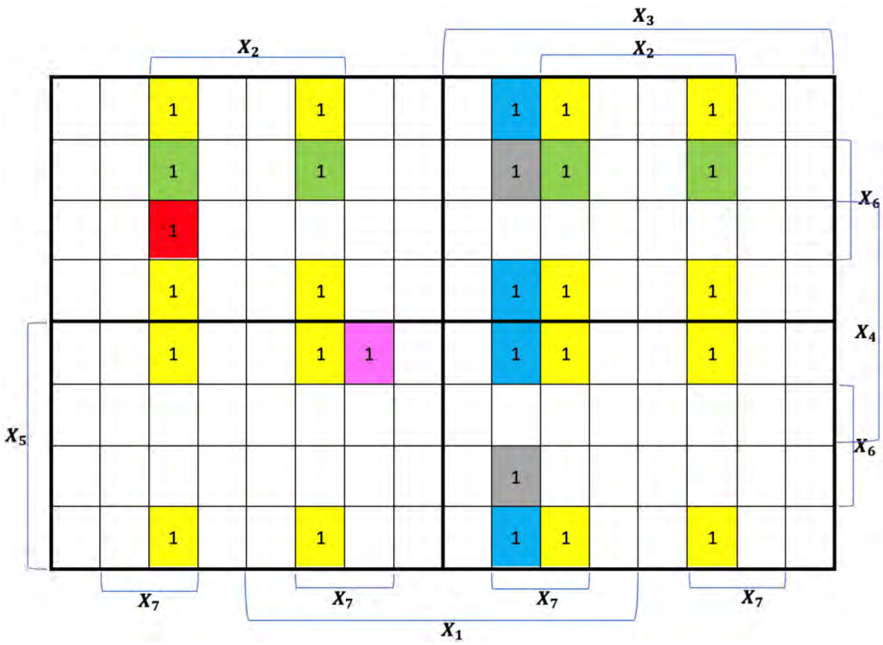


Fig. 10 A Karnaugh map (probability map) representing equation (35).



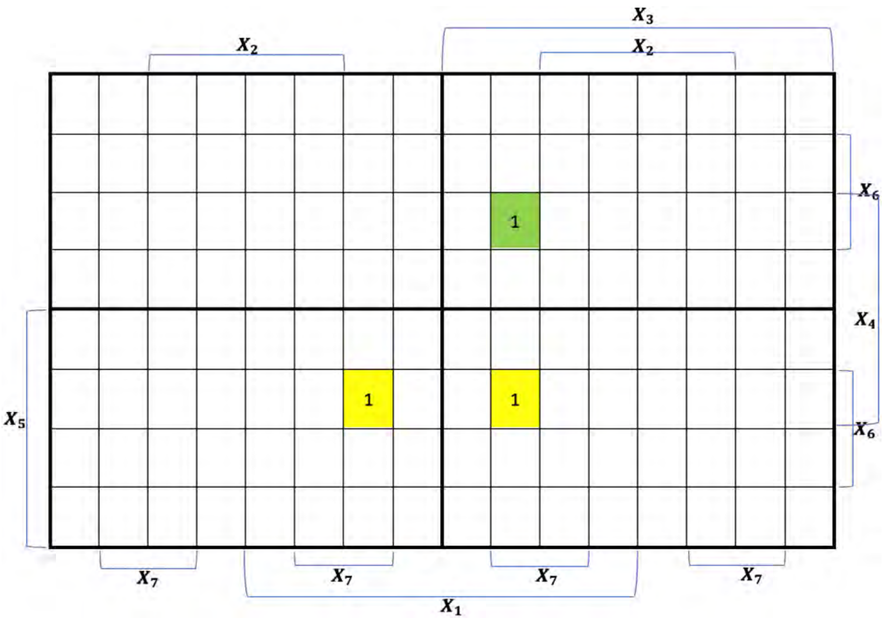
$X_1X_4X_6\bar{X}_7$ $X_1\bar{X}_2\bar{X}_3X_4\bar{X}_5X_6X_7$ $X_2\bar{X}_4X_5X_6\bar{X}_7$
$\bar{X}_1X_2X_4X_5X_6\bar{X}_7$ $\bar{X}_1X_2X_3X_4\bar{X}_5X_6\bar{X}_7$ $X_1\bar{X}_2X_3\bar{X}_4X_5X_6\bar{X}_7$

$Co(3)$



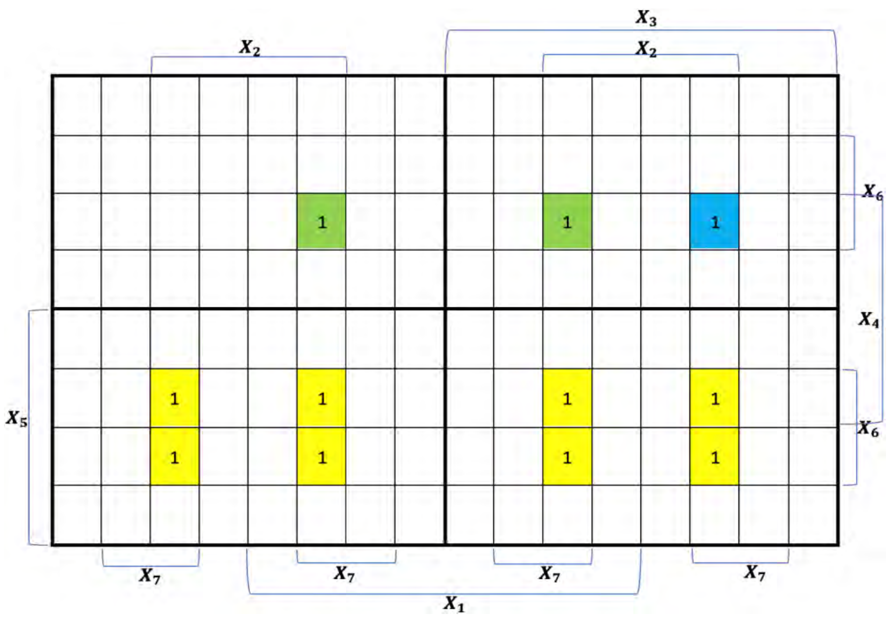
$X_2\bar{X}_6X_7$ $X_2\bar{X}_4\bar{X}_5X_6X_7$ $\bar{X}_1X_2\bar{X}_3X_4\bar{X}_5X_6X_7$
$X_1\bar{X}_2X_3\bar{X}_6X_7$ $X_1\bar{X}_2X_3\bar{X}_4X_6X_7$ $X_1\bar{X}_2\bar{X}_3X_4X_5\bar{X}_6X_7$

$Co(4)$



$X_1\bar{X}_2X_4X_5X_6X_7$ $X_1\bar{X}_2X_3X_4\bar{X}_5X_6X_7$

$Co(6)$



$X_2X_5X_6X_7$ $X_1X_2X_4\bar{X}_5X_6X_7$ $\bar{X}_1X_2X_3X_4\bar{X}_5X_6X_7$

$Co(7)$

Fig. 11 The contributions of the various non-zero entries in Fig. 7, now interpreted as disjoint-loop coverings for the asserted cells only (See equation (34)).

8 Conclusions

This paper introduces a novel method for analyzing capacitated flow networks through the utilization of the concept of a "probability-ready expression" for a Boolean-based coherent pseudo-Boolean function. This function serves as the capacity function of the pertinent flow network, which might be a biology or ecology network, in which the flowing 'commodity' might be migrating species, infectious pathogens, energy, nutrients, or genes. The method introduced is very useful since it generalizes Boolean techniques and tools for handling pseudo-Boolean functions.

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