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Dynamical behavior of an eco-epidemiological model incorporating Holling type-II functional response with prey refuge and constant prey harvesting

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Abstract
In the present investigation, we examine the consequences of a predator-prey model that includes a constant harvesting technique in a population of susceptible prey. A particular kind of flipping functional approach is present in our proposed prey-predator system; in this response, the predator consumes on susceptible and sick prey, however it shifts its attention to a new sort of prey when its supply of that kind of prey decreases. By employing boundedness, positivity, equilibrium analysis and stability analysis, the essential mathematical characteristics of the model are explored. Attention is directed on the prey refuge in further explorations of the Hopf bifurcation close to the coexistence equilibrium point. This paper's unique contribution is that it examines the dynamics of predator-prey systems from an eco-epidemiological perspective while simultaneously considering the impacts of prey refuge and constant-rate harvesting. To ascertain the critical values of the bifurcation parameters, if present, and to validate the primary findings, numerical simulations are executed. Our numerical simulations reveal that the presence of prey refuge increases the severity of sickness, which is how the three-species eco-epidemiological system creates chaos. However, we find that both the prey refuge and harvesting can manage the resulting chaotic dynamics.

Keywords eco-epidemic model; susceptible prey; infected prey; predator; Hopf bifurcation; harvesting; prey refuge.

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1 Introduction

There is consensus that infections affect both natural systems and population density. In order to better understand the eco-system, it is crucial to study how epidemiological factors affect the dynamics of prey-predator interactions. The area of study on mathematical modelling of epidemics has been greatly influenced by the pioneering work of Kermack and McKendrick (1927) on SIRS systems. Anderson and May (1981) were the first to investigate the invasion, persistence, and spread of infectious diseases by using an

$$J_{E_2}=\begin{bmatrix} 1-2x-P_1-\frac{B_1(1-m)A_2z}{(A_2+(1-m)x)^2} & -\frac{\theta_1 x}{A_1+x} & -\frac{B_1(1-m)x}{A_2+(1-m)x} \\ 0 & \frac{\theta_1 x}{A_1+x}-\frac{B_2A_2z}{(A_2+y)^2}-M_1 & 0 \\ \frac{C_1(1-m)zA_2}{(A_2+(1-m)x)^2} & \frac{C_2 z}{A_2} & 0 \end{bmatrix} \quad (4)$$

So, $\lambda_1=\frac{\theta_1 x}{A_1+x}-\frac{B_2A_2z}{(A_2+y)^2}-M_1$ is one of the eigen values, while the remaining two are the roots of the equation $\lambda^2-l_{11}\lambda-l_{13}l_{31}=0$. Thus, in case when $l_{11}<0$ and $l_{13}l_{31}<0$, the two eigen values necessarily possess a negative real portion. But one eigen value is $\lambda_1<0$ if condition mentioned in Theorem 3 hold.

Theorem 6. The equilibrium point $E_2(x^*,y^*,z^*)$ is locally asymptotically stable if $D_1>0, D_3>0$ and $\Delta>0$.

Proof:

At $E_3(x^*,y^*,z^*)$ $J_{E_2}=\begin{bmatrix} -x+\frac{\theta_1 xy}{(A_1+x)^2}+\frac{B_1(1-m)^2xz}{(A_2+(1-m)x)^2} & -\frac{\theta_1 x}{A_1+x} & -\frac{B_1(1-m)x}{A_2+(1-m)x} \\ \frac{\theta_1 A_1 y}{(A_1+x)^2} & \frac{B_2yz}{(A_2+y)^2} & -\frac{B_2 y}{(A_2+y)} \\ \frac{C_1(1-m)zA_2}{(A_2+(1-m)x)^2} & \frac{C_2A_2z}{(A_2+y)^2} & 0 \end{bmatrix}$ (5)

The laten equation of above community matrix is $\lambda^3+D_1\lambda^2+D_2\lambda+D_3=0$

where $D_1=-(l_{11}+l_{22})$, $D_2=l_{11}l_{22}-l_{12}l_{21}-l_{13}l_{31}-l_{23}l_{32}$ and $D_3=l_{32}(l_{11}l_{23}-l_{13}l_{21})+l_{31}(l_{13}l_{22}-l_{12}l_{23})$. Consider $\Delta=D_1D_2-D_3$ and so by Routh-Hurwitz criterion all the roots of above latent equation have negative real parts if $D_1>0, D_3>0$ and $\Delta>0$. Thus, the positive equilibrium point E_3 is locally asymptotically stable.

5 Hopf Bifurcation At Positive Equilibrium Point E_3

In this subsection, we explore whether the system dynamics vary with respect to m through Hopf bifurcation analysis.

Theorem 7. When the prey refuge m exceeds a threshold value, the dynamical system experiences a Hopf bifurcation over the interior equilibrium E_3. Hopf bifurcation develops at $m=m^*$ if the following two criteria are met concurrently $H_1(m^*)>0, H_3(m^*)>0$ and $H_1(m^*)H_2(m^*)=H_3(m^*)$ with

$H_1\left(m^*\right)H_2'\left(m^*\right)\neq H_3'\left(m^*\right)+\mathrm{H}_1'\left(m^*\right)\mathrm{H}_2\left(m^*\right)$.

Proof. For $m=m^*$ then the latent equation of the system (2) for E_3 is expressed as

$$\left(\lambda^2+\mathrm{H}_2\right)\left(\lambda+\mathrm{H}_1\right)=0.$$

We can obtain latent values from above equation are $\lambda=i\sqrt{\mathrm{H}_2}, \lambda=-i\sqrt{\mathrm{H}_2}, \lambda=-\mathrm{H}_1$. Of these three latent values, two that are entirely imaginary and one that is entirely negative. Taking m into account as bifurcation parameter allows us to alter the roots in the following way:

$$\lambda_1(m)=\Upsilon_1(m)+i\Upsilon_2(m),\quad \lambda_2(m)=\Upsilon_1(m)-i\Upsilon_2(m),\quad \lambda_3(m)=-\mathrm{H}_1. \tag{6}$$

We can solve above equation by substituting $\lambda_1(m)=\Upsilon_1(m)+i\Upsilon_2(m)$ first, differentiate it with respect to m, and then separating the real and imaginary parts.

Then, we have,

$$G_1(m)\Upsilon_1'(m)-G_2(m)\Upsilon_2'(m)+G_3=0$$

$$G_2(m)\Upsilon_1'(m)-G_1(m)\Upsilon_2'(m)+G_4=0 \tag{7}$$

Where $G_1(m)=3\left(\Upsilon_1^2-\Upsilon_2^2\right)+2H_1\Upsilon_1+H_2$, $G_2(m)=6\Upsilon_1\Upsilon_2+2H_1\Upsilon_2$, $G_3(m)=H_1'\left(\Upsilon_1^2-\Upsilon_2^2\right)+H_2'\Upsilon_1+H_3'$,

$G_4(m)=2H_1'\Upsilon_1\Upsilon_2+H_2'\Upsilon_2$

Observing that $\Upsilon_1\left(m^*\right)=0, \Upsilon_2\left(m^*\right)=\sqrt{\mathrm{H}_2\left(m^*\right)}$

we get

$G_1\left(m^*\right)=-2H_2\left(m^*\right), G_2\left(m^*\right)=2H_1\left(m^*\right)\sqrt{\mathrm{H}_2\left(m^*\right)}, G_3\left(m^*\right)=H_3'\left(m^*\right)-H_1'\left(m^*\right)H_2\left(m^*\right)$ and

$$G_4\left(m^*\right)=H_2'\left(m^*\right)\sqrt{\mathrm{H}_2\left(m^*\right)}. \tag{8}$$

Now

$$\frac{d}{dm}\left(\mathrm{Re}\,\lambda(m)\right)G_1\left(m^*\right)\Big|_{m=m^*}=\frac{G_2\left(m^*\right)G_4\left(m^*\right)+G_1\left(m^*\right)G_3\left(m^*\right)}{G_1\left(m^*\right)^2+G_2\left(m^*\right)^2}$$

$$=\frac{2H_1\left(m^*\right)\sqrt{\mathrm{H}_2\left(m^*\right)}H_2'\left(m^*\right)\sqrt{\mathrm{H}_2\left(m^*\right)}+\left(-2\mathrm{H}_2\left(m^*\right)\right)\left(H_3'\left(m^*\right)-H_1'\left(m^*\right)H_2'\left(m^*\right)\right)}{\left(-2\mathrm{H}_2\left(m^*\right)\right)^2+\left(2H_1\left(m^*\right)\sqrt{\mathrm{H}_2\left(m^*\right)}\right)^2}$$

$$=\frac{H_1\left(m^*\right)H_2'\left(m^*\right)-H_3'\left(m^*\right)+\mathrm{H}_1'\left(m^*\right)\mathrm{H}_2\left(m^*\right)}{2\left(\mathrm{H}_2\left(m^*\right)+\left(H_1\left(m^*\right)\right)^2\right)}\neq 0$$ if $H_1\left(m^*\right)H_2'\left(m^*\right)\neq H_3'\left(m^*\right)+\mathrm{H}_1'\left(m^*\right)\mathrm{H}_2\left(m^*\right)$ and

$\lambda_3\left(m^*\right)=-H_1\left(m^*\right)\neq 0$.

Hence, when $m = m^*$, the system shows Hopf bifurcation.

Remarks: Likewise, θ_1 may also be considered a bifurcating parameter.

6 Numerical Simulations

No analytical study is ever considered complete without some kind of numerical confirmation of the findings. This section presents a computer simulation of some of the system's solutions. In addition to verifying our analytical conclusions, these numerical solutions are quite useful in practice. The study focuses on critical characteristics such as prey refuge rate m and disease transmission rate θ_1. We analyses System (2) using various values of the aforementioned parameters.

Example 1. Consider the set of parameter values as $P_1 = 0.1095; \theta_1 = 0.5405; A_1 = 0.7789;$ $A_2 = 1.3684; B_1 = 0.5716; B_2 = 1.2784; M_1 = 0.0676; M_2 = 0.2365; C_1 = 0.0646; C_2 = 0.1445$ .The requirement of Theorem 4 is satisfied by this parameter selection; therefore, $x* = 0.672, y* = 0.3221, z* = 0.2408$. We begin by analyzing how the system (2) dynamically responds to changes in the prey refuge amount m. If we take $m = 0.2$ system exhibits stale solution (Fig.1a) and increasing m value to 0.8 then the system exhibits unstable solution (Fig. 1b).The bifurcation illustrations for the prey-predator population in relation to m for the interval [0.1, 0.9] is shown in Fig. 2. The complicated dynamical characteristic in our suggested model (2), which extends from the stable focus to chaos, can be observed in the bifurcation diagram in relation to Fig. 2. The system then experiences a Hopf bifurcation with m acting as the bifurcation parameter, as deduced from Theorem 7. System (2) experiences a Hopf bifurcation of periodic solution around $m = m^* = 0.25$, as can be easily seen if we take $m = 0.25$.

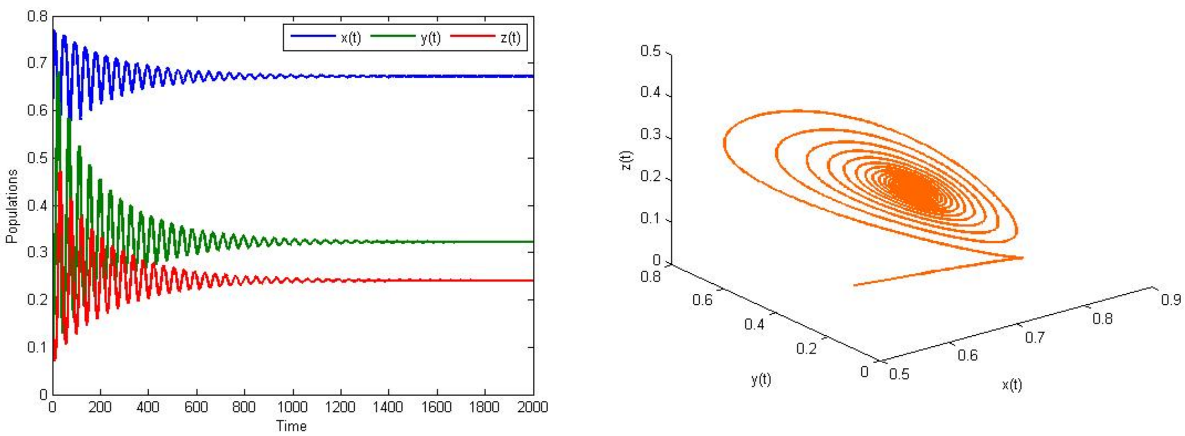


Fig. 1a Time trajectories and phase portrait of the system (2) for $m = 0.2$

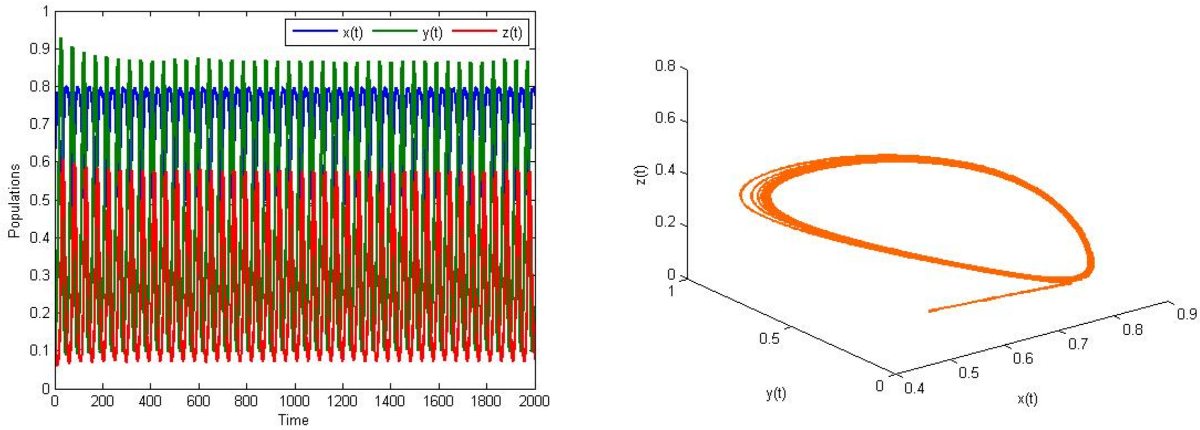
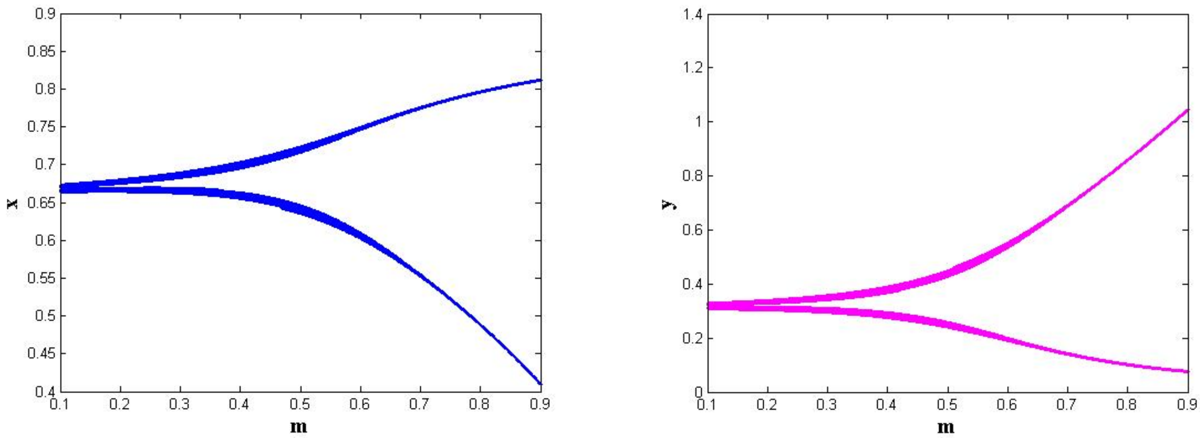


Fig. 1b Time trajectories and phase portrait of the system (2) for $m = 0.8$

Example 2. Now we introduce the rate of infection θ_1 and taking the same parameter values as Example1. If we take $\theta_1 = 0.18270$ system exhibits stale solution (Fig. 3a) and increasing θ_1 value to 0.32270 then the system exhibits unstable solution (Fig. 3b). For the interval [0.09, 0.5], Fig. 4 shows the bifurcation diagram illustrating the prey-predator population with respect to θ_1. The complicated dynamical characteristic in our suggested model (2) is shown in the bifurcation diagram with respect to m in Fig. 4, which goes from the stable focus to chaos. Then, under the influence of θ_1, the system experiences a Hopf bifurcation. The Hopf bifurcation of the periodic solution in system (2) happens at $\theta_1 = \theta_1^* = 0.19$ if we set $\theta_1 = 0.19$.



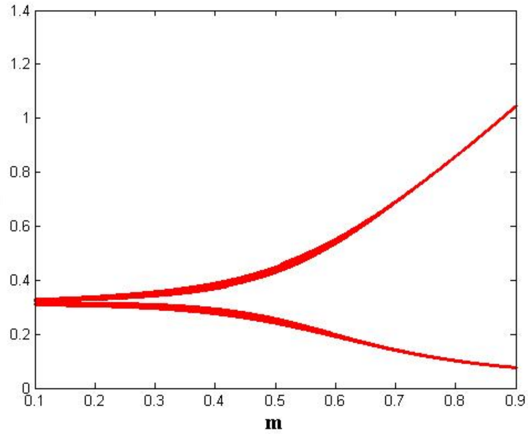


Fig. 2 The bifurcation illustration of System (2) in relation to m , while the remaining values remain the same as shown in Fig. 1.

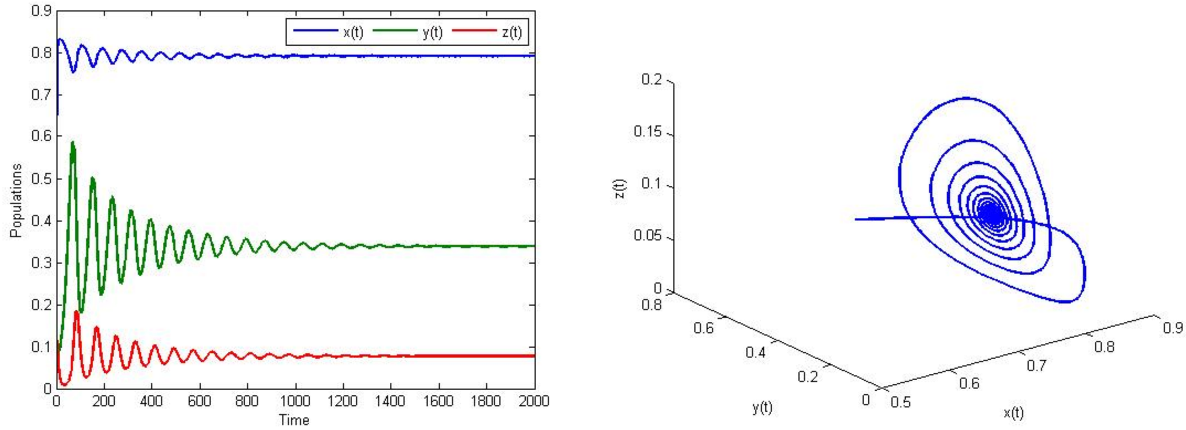


Fig. 3a Time trajectories and phase portrait of the system (2) for $\theta_1 = 0.18270$

7 Discussion

Our study differs from previous ones in that it includes prey infection, which has a nonlinear prevalence level, an integrated functional response for vulnerable and contaminated prey populations, and improved Leslie-Gower predator dynamics. The system's dynamics are enhanced and the system becomes more realistic compared to previous models with the inclusion of these extra ecological components.

Using an eco-epidemiological model including three species, this study examines how the refuge effect affects the prey population. When the prey shelter influence is added to the prey population, the eco-epidemiological model's dynamical behaviour changes significantly. Our numerical simulation validates the stabilising impact on the prey hideout. By generating bifurcation diagrams, we were able to confirm it. More importantly, the infection rate (θ_1) is also critical. A rising amount of θ_1 may transform a stable system into an oscillating one via Hopf bifurcation; the numerical simulation shows that θ_1 has both stabilising and destabilising effects.

The system's complicated and rich dynamics are shown by both analytical and numerical simulation findings. In future studies, environmental factors like extra food, delay, Allee effect (Venkataiah et al., 2024a) and a development of territorial and spatiotemporal structure may be combined with vertical and horizontal spread of diseases to the predator population.

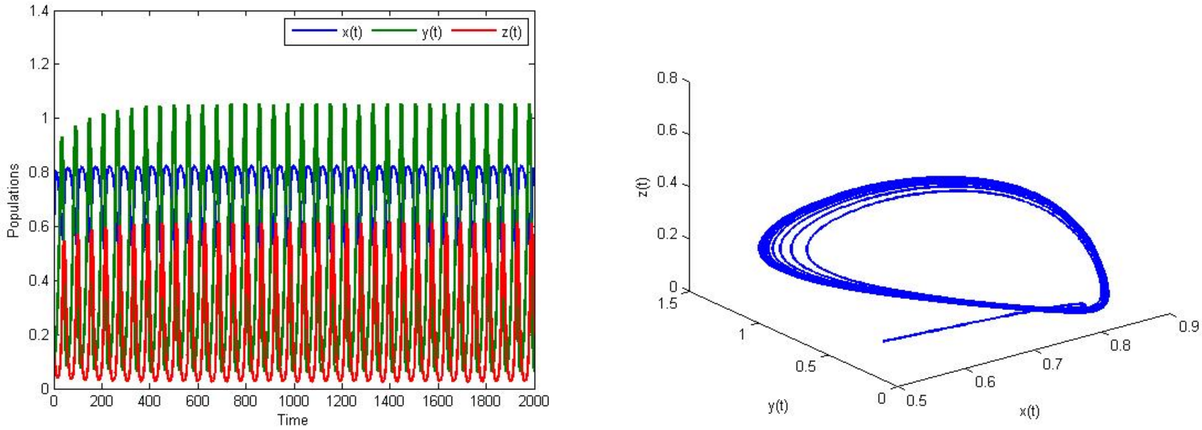


Fig. 3b Time trajectories and phase portrait of the system (2) for $\theta_1 = 0.32270$

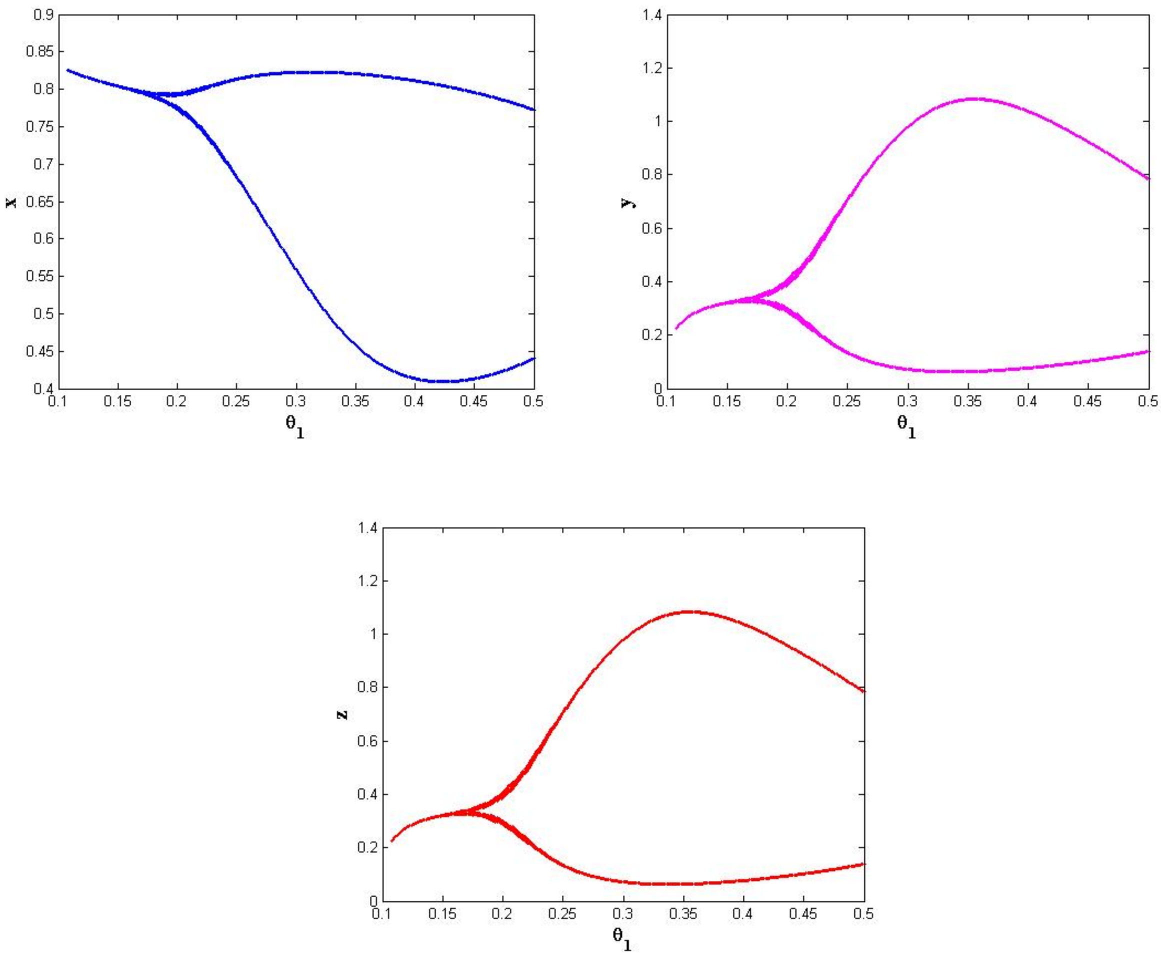


Fig. 4 The bifurcation illustration of System (2) in relation to θ_1, while the remaining values remain the same as shown in Fig.1

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